SUBSPACE SUPERCYCLICITY OF THE TUPLES OF OPERATORS

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Abstract: In this paper, we investigate subspace supercyclicity for tuples of operators and we give conditions under which a tuple of operators satisfying the subspace supercyclicity criterion.

AMS Subject Classification: 47B37, 47B33

Key Words: tuple of operators, subspace-supercyclicity, subspace-hypercyclicity, supercyclicity criterion

1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space \(X\).

**Definition 1.1.** Let \(\mathcal{T} = (T_1, T_2, ..., T_n)\) be an n-tuple of operators acting on a separable infinite dimensional Banach space \(X\) over \(\mathbb{C}\) and let \(M\) be a nonzero subspace of \(X\). We will let \(\mathcal{F} = \{T_1^{k_1}T_2^{k_2}...T_n^{k_n} : k_i \geq 0, i = 1, ..., n\}\) be the semigroup generated by \(\mathcal{T}\). For \(x \in X\), the orbit of \(x\) under the tuple \(\mathcal{T}\) is the set \(\text{Orb}(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}\). A vector \(x\) is called a \(M\)-supercyclic \((M\)-hypercyclic\) vector for \(\mathcal{T}\) if \(\mathcal{COrb}(\mathcal{T}, x) \cap M \ (\text{Orb}(\mathcal{T}, x) \cap M)\) is dense in \(M\) and in this case the tuple \(\mathcal{T}\) is called \(M\)-supercyclic \((M\)-hypercyclic\). The
set of all $M$-supercyclic vectors of $\mathcal{T}$ is denoted by $SC(\mathcal{T}, M)$. Also, for all $k \geq 2$, by $T_d^{(k)}$ we will refer to the set of all $k$ copies of an element of $\mathcal{F}$, i.e. $T_d^{(k)} = \{S_1 \oplus \ldots \oplus S_k : S_1 = \ldots = S_k \in \mathcal{F}\}$. We say that $T_d^{(k)}$ is subspace-supercyclic, with respect to $M$, provided there exist $x_1, \ldots, x_k \in X$ such that $\mathcal{C}\{W(x_1 \oplus \ldots \oplus x_k) : W \in T_d^{(k)}\} \cap M$ is dense in the $k$ copies of $M, M \oplus \ldots \oplus M$.

Suprisingly, there are something that does not happen for single operators. For example, supercyclic tuples can arise in finite dimensional, and there are operators that have somewhere dense orbits that are not everywhere dense. Also, we note that there are subspace-supercyclic operators that are not supercyclic. For some topics we refer to [1]-[2].

2. Main Results

In this section, we investigate an equivalent condition for subspace-supercyclicity criterion for tuples of operators.

**Theorem 2.1.** Let $M$ be a nonzero subspace of separable infinite dimensional Banach space $X$ and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a tuple of operators $T_1, T_2, \ldots, T_n$. Then $\mathcal{T}$ is $M$-supercyclic if and only if for any two nonempty relatively open sets $U$ and $V$ in $M$, there exist $m_i \geq 1$ for $i = 1, \ldots, n$ and and $\lambda \in \mathcal{C}\{0\}$ such that $\lambda T_1^{m_1} \ldots T_n^{m_n} (U) \cap V$ is nonempty and $T_1^{m_1} \ldots T_n^{m_n} M \subset M$.

**Proof.** Suppose that $x$ is a $M$-supercyclic vector for $\mathcal{T}$, hence $COrb(\mathcal{T}, x) \cap M$ is dense in $M$ and so there exist $\lambda_1 \in \mathcal{C}\{0\}$ and $m_i \in \mathbb{N}$; $i = 1, \ldots, n$ such that $\lambda_1 T_1^{m_1} \ldots T_n^{m_n} x \in U$. Let $y = \lambda_1 T_1^{m_1} \ldots T_n^{m_n} x$. Since $COrb(\mathcal{T}, x) \cap M$ is also dense in $M$, thus $\lambda T_1^{m_1} \ldots T_n^{m_n} y \in V$ for some $\lambda \in \mathcal{C}\{0\}$ and $m_i \in \mathbb{N}$; $i = 1, \ldots, n$, hence $\lambda T_1^{m_1} \ldots T_n^{m_n} (U) \cap V$ is nonempty. To show that $T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} M \subset M$, let $x \in M$ and let $W$ be a relatively open nonempty subset of $\lambda^{-1} T_1^{m_1} \ldots T_n^{m_n} (V) \cap U$ in $M$. Note that

$$\lambda T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} W \subset V \cap \lambda T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} U \subset V \subset M.$$ 

Hence $T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} W \subset M$. If $x_0 \in W$, then there exists $r > 0$ small enough such that $x_0 + r x \in W$, since $W$ is relatively open. Thus

$$T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} (x_0 + r x) = T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} x_0 + r T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} x \in M,$$

which implies that $T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} x \in M$. Thus $T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} M \subset M$. For the converse, fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of relatively open balls in $M$ with
Hence for large rational radii, and centers in a countable dense subset of $M$. By the continuity of the operators $T_1, T_2, \ldots, T_n$, the sets

$$G_k = \bigcup \{ \lambda T_1^{-m_1} \cdots T_n^{-m_n}(B_k) \cap M : m_i \geq 0; i = 1, \ldots, n, \lambda \in \mathbb{C} \}$$

are relatively open in $M$. Since for any two nonempty relatively open sets $U$ and $V$ in $M$, there exist $m_i \geq 1; i = 1, \ldots, n$, and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda T_1^{m_1} \cdots T_n^{m_n}(U) \cap V$ contains a relatively open nonempty subset of $M$, thus one can see that $SC(T)$ is exactly equal to $\bigcap \{ G_k : k \in \mathbb{N} \}$ that is dense in $M$. \hfill \Box

**Theorem 2.2.** (Subspace-Supercyclicity Criterion for Tuples) Let $T = (T_1, T_2, \ldots, T_n)$ be a tuple of continuous operators acting on a separable infinite dimensional Banach space $X$ and $M$ be a nonzero closed subspace of $X$. Suppose that there exist two dense subsets $Y$ and $Z$ in $M$, and strictly increasing sequences of positive integers $\{ m_j \}$ for $i = 1, \ldots, n$, and a sequence of mappings $S_j : Z \to M$ such that:

1. For every $z \in Z$, $T_1^{m_j(1)} \cdots T_n^{m_j(n)} S_j z \to z$,
2. $\| T_1^{m_j(1)} \cdots T_n^{m_j(n)} y \| \| S_j z \| \to 0$ for all $y \in Y$ and all $z \in Z$,
3. $M$ is invariant subspace for $T_1^{m_j(1)} \cdots T_n^{m_j(n)}$ for all $j$.

Then $T$ is subspace-supercyclic with respect to $M$.

**Proof.** Let $U$ and $V$ be two nonempty relatively open subsets of $M$. Choose $v \in Y \cap V$ and $u \in Z \cap U$. Since $U$ and $V$ are relatively open, there exists $\epsilon > 0$ such that $B(v, \epsilon) \cap M \subset V$ and $B(u, \epsilon) \cap M \subset U$. By the hypothesis we have:

(i). $T_1^{m_j(1)} \cdots T_n^{m_j(n)} S_j u \to u$,  
(ii). $\| T_1^{m_j(1)} \cdots T_n^{m_j(n)} v \| \| S_j u \| \to 0$,
(iii). $M$ is invariant subspace for $T_1^{m_j(1)} \cdots T_n^{m_j(n)}$ for all $j$.

Hence for large $j$, we can find $\lambda_j \in \mathbb{C} \setminus \{0\}$ such that

$$\| \lambda_j T_1^{m_j(1)} \cdots T_n^{m_j(n)} v \| < \epsilon/2, \| \lambda_j^{-1} S_j u \| < \epsilon, \| T_1^{m_j(1)} \cdots T_n^{m_j(n)} S_j u - u \| < \epsilon/2.$$

Note that since $v \in M$ and $S_j u \in M$, we have $v + \lambda_j^{-1} S_j u \in M$ and $\| (v + \lambda_j^{-1} S_j u) - v \| = \| \lambda_j^{-1} S_j u \| < \epsilon$. Hence $v + S_j u \in V$. Also we note that $T_1^{m_j(1)} \cdots T_n^{m_j(n)} (v + \lambda_j^{-1} S_j u) \in M$ and $\| \lambda_j T_1^{m_j(1)} \cdots T_n^{m_j(n)} (v + \lambda_j^{-1} S_j u) - u \| < \epsilon$.

Thus $\lambda_j T_1^{m_j(1)} \cdots T_n^{m_j(n)} (v + \lambda_j^{-1} S_j u) \in B(u, \epsilon) \cap M$ and so $\lambda_j T_1^{m_j(1)} \cdots T_n^{m_j(n)} (v + \lambda_j^{-1} S_j u) \in U$. Hence $v + \lambda_j^{-1} S_j u \in \lambda_j^{-1} T_1^{-m_j(1)} \cdots T_n^{-m_j(n)} (U) \cap V$ and so by Theorem 2.1, $T$ is subspace-supercyclic with respect to $M$. \hfill \Box
Theorem 2.3. Let $M$ be a nonzero subspace of a separable infinite dimensional Banach space $X$ and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. If $\mathcal{T}^{(2)}_d$ is $M \oplus M$-supercyclic, then $\mathcal{T}$ satisfies the Subspace-Supercyclicity Criterion with respect to $M$.

Proof. Suppose that $\mathcal{T}^{(2)}_d$ is $M \oplus M$-supercyclic and let $(x, y)$ be a $M \oplus M$-supercyclic vector for $\mathcal{T}^{(2)}_d$. In particular $x$ and $y$ are supercyclic vectors for $\mathcal{T}$. For all $k \in \mathbb{N}$, put $U_k = B(0, \frac{1}{k}) \cap M$. Then there exist $\{m_{j_i}\}_j \subset \mathbb{N}$ for $i = 1, ..., n$ and $\lambda_j \in \mathbb{C}$ such that

$$\lambda_j(T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}} \oplus T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}})(x, y) \in U_j \oplus (x + U_j),$$

$$T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}} \oplus T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}} M \oplus M \subset M \oplus M.$$  

Therefore, $\lambda_jT_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}x \in U_j$, $\lambda_jT_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}y \in x + U_j$, and $T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}M \subset M$ for all $j \in \mathbb{N}$. This implies that

$$\lambda_jT_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}x \rightarrow 0, \lambda_jT_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}y \rightarrow x.$$ 

Let $Y = Z = \mathbb{C}Orb(\mathcal{T}, x)$ which is dense in $M$. Also for all $j \in \mathbb{N}, \lambda \in \mathbb{C}$ and $k_{r_i} \in \mathbb{N}$ for all $i = 1, ..., n$, define $S_j(\lambda T_1^{k_{r_1}}...T_n^{k_{r_n}}x) = \lambda \lambda_jT_1^{k_{r_1}}...T_n^{k_{r_n}}y$. Note that $T_1^{m_{j_1}}...T_n^{m_{j_n}}S_j(\lambda T_1^{k_{r_1}}...T_n^{k_{r_n}}x) = \lambda T_1^{k_{r_1}}...T_n^{k_{r_n}}(\lambda T_1^{m_{j_1}}...T_n^{m_{j_n}}y)$ which tends to $\lambda T_1^{k_{r_1}}T_2^{k_{r_2}}...T_n^{k_{r_n}}x$ as $j \rightarrow \infty$. So $T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}S_jz \rightarrow z$ for all $z \in Z$. Also for all $\lambda, w \in \mathbb{C}$ and $t_{s_i}, k_{r_i} \in \mathbb{N}$ for $i = 1, ..., n$, clearly we can see that $||T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}(\lambda T_1^{k_{r_1}}T_2^{k_{r_2}}...T_n^{k_{r_n}}x)|| = ||S_j(w T_1^{k_{r_1}}T_2^{k_{r_2}}...T_n^{k_{r_n}}x)|| \rightarrow 0$ as $j \rightarrow \infty$. Thus for all $y \in Y$ and $z \in Z$, we get $||T_1^{m_{j_1}}T_2^{m_{j_2}}...T_n^{m_{j_n}}y|| = ||S_jz|| \rightarrow 0$ and so $\mathcal{T}$ satisfies the $M$-Supercyclicity Criterion. 

References
