International Journal of Pure and Applied Mathematics
Volume 112 No. 3 2017, 469-480
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v112i3.2
ijpam.eu

EXISTENCE OF WEAK SOLUTIONS FOR A FUNCTIONAL INTEGRAL INCLUSION

Ahmed M.A. El-Sayed[1], Nesreen F.M. El-Haddad[2 §]
[1]Faculty of Science
Alexandria University
Alexandria, EGYPT
[2]Faculty of Science
Damanhour University
Behera, EGYPT

Abstract: In this paper we study the existence of weak solution $x \in C[I, E]$ for the nonlinear functional integral inclusion

$$x(t) \in F\left(t, \int_0^t g(s, x(m(s)))ds\right), \quad t \in [0, T]$$

in the reflexive Banach space E under the assumption that the set-valued function F satisfy Lipschitz condition.

AMS Subject Classification: 45N05, 46T20, 47J05
Key Words: set-valued function, weak solution, functional inclusion, Lipschitz condition

1. Introduction

Let $I = [0, T]$, and let $L^1(I)$ be the class of all Lebesgue integrable functions defined on the interval I. Let E be a reflexive Banach space with norm $\|.\|$ and dual E^*.

Denote $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm $\|x\|_C = \sup \|x(t)\|_E$.

Received: August 28, 2016
Revised: December 1, 2016
Published: February 7, 2017

[§]Correspondence author

Consider the functional integral inclusion

$$x(t) \in F\left(t, \int_0^t g(s, x(m(s)))ds\right), \qquad t \in [0, T], \tag{1}$$

where $F : I \times E \to P(E)$ is a nonlinear set-valued mapping, and $P(E)$ denote the family of nonempty subsets of the Banach space E.

Here we study the existence of weak solution $x \in C[I, E]$ of the functional integral inclusion (1) in the reflexive Banach space E, under the assumption that the set-valued function F satisfy Lipschitz condition.

Indeed a set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [8]-[10] and [12]-[13]).

The existence of weak solutions of the integral equations were studied by a number of authors such as (see for instance[1]-[2], [5] and [15]-[16]).

2. Preliminaries

Here, we present some auxiliary results that will be needed in this work.

Let E be a Banach space and let $x : I \to E$, then:

(1) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^{*}$, $\phi(x(.))$ is continuous (measurable) at t_0 .

(2) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequence in E to weakly convergent sequence in E .

If x is weakly continuous on I , then x is strongly measurable and hence weakly measurable (see[4] and[7]). Note that in reflexive Banach spaces weakly measurable functions are pettis integrable (see[7] and [11] for the definition) if and only if $\phi(x(.))$ is Lebesgue integrable on I for every $\phi \in E^{*}$.

Now we state a fixed point theorem and some propositions which will be used in the sequal (see[14]).

Theorem 1 (O'Regan Fixed Point Theorem)**.** *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space $(C[0, T], E)$ and let $A : Q \to Q$ be a weakly sequentially continuous and assume that $AQ(t)$ is relatively weakly compact in E for each $t \in [0, T]$. Then A has a fixed point in the set Q.*

Proposition 2. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 3. *Let E be a normed space with $y \neq 0$. Then there exists a $\phi \in E^{*}$ with $\|\phi\| = 1$ and $\|y\| = \phi(y)$.*

Definition 4 (see [6])**.** A set-valued map F from $I \times E$ to the family of all nonempty closed subsets of E is called Lipschitzian if there exists $L > 0$ such that for all $t_1, t_2 \in I$ and all $x_1, x_2 \in E$, we have

$$H(F(t_1, x_1), F(t_2, x_2)) \leq L(|t_1 - t_2| + \|x_1 - x_2\|) \tag{2}$$

where $H(A, B)$ is the Hausdorff metric between the two subsets $A, B \in I \times E$.

Denote $S_F = Lip(I, E)$ be the set of all Lipschitz selections of F.

3. Existence of Weak Solution

In this section, we present our main result by proving the existence of weak solution $x \in C[I, E]$ of the functional integral inclusion (1) in the reflexive Banach space E, under the assumption that the set-valued function F satisfy Lipschitz condition.

3.1. Coupled System Approach

Consider now the functional integral inclusion (1) under the following assumptions:

(H1) The set $F(t, x)$ is compact and convex for all $(t, x) \in I \times E$.

(H2) The set-valued map F is Lipschitzian with a Lipschitz constant $L > 0$.

(H3) The set of all Lipschitz selections S_F is nonempty.

(H4) $g(t, .)$ is weakly sequentially continuous for each $t \in I$.

(H5) $g(., x)$ is weakly measurable on I for every $x \in E$.

(H6) There exists a function $a \in L^1[0, T]$ and a constant $b > 0$ such that

$$\|g(t, x)\| \leq |a(t)| + b\|x\|, \quad \forall t \in I$$

(H7) $m : [0, T] \to [0, T]$ is continuous.

Remark 5. From assumptions (H2) and (H3), there exists $f \in S_F$ such that

$$\|f(t_2, x) - f(t_1, y)\| \leq L(|t_2 - t_1| + \|x - y\|),$$

and

$$x(t) = f(t, \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T]. \tag{3}$$

Then the solution of the functional integral equation (3), if it exists, is a solution of the functional integral inclusion (1).

Now let

$$y(t) = \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T], \tag{4}$$

then from (3) we have

$$x(t) = f(t, y(t)), \quad t \in [0, T]. \tag{5}$$

Then the functional integral equation (3) is equivalent to the coupled system (4) and (5).

Consider now the coupled system (4) and (5).

Now, we study the existence of a weak solution of the functional integral equation (3), which is a solution of the functional integral inclusion (1), by getting the weak solution of the coupled system (4) and (5).

Definition 6. By a weak solution of the coupled system (4) and (5) we mean the ordered pair of functions (x, y), $x, y \in C[I, E]$ such that

$$\phi(x(t)) = \phi(f(t, y(t))), \quad t \in [0, T]$$

$$\phi(y(t)) = \int_0^t \phi(g(s, x(m(s))))ds, \quad t \in [0, T]$$

for all $\phi \in E^{\square}$.

Now let X be the class of all ordered pair $U = (x, y)$, $x, y \in C[I, E]$, with norm

$$\|(x, y)\|_X = \|x\| + \|y\|, \quad x, y \in C[I, E].$$

Now for the existence of a weak continuous solution of the coupled system (4) and (5) we have the following theorem.

Theorem 7. *Let the assumptions (H1)-(H7) be satisfied. Then the coupled system (4) and (5) has at least one weak continuous solution.*

Proof. Let

$$\begin{aligned} U(t) &= (x(t), y(t)) \\ &= (f(t, y(t)), \int_0^t g(s, x(m(s)))ds), \quad t \in [0, T] \end{aligned}$$

Let A be any operator defined by

$$AU(t) = A(x(t), y(t)) = (A_1 y(t), A_2 x(t))$$

where

$$A_1 y(t) = f(t, y(t)), \quad t \in [0, T]$$

$$A_2 x(t) = \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T]$$

Let the set Q_r defined by

$$Q_r = \{U = (x, y) \in X : x, y \in C[I, E], \quad \|y\| \le r_1, \ \|x\| \le r_2, \quad r = r_1 + r_2\}.$$

Let $U = (x, y) \in Q_r$ be an arbitrary ordered pair, then we have from proposition 3

$$\begin{aligned} \|A_1 y(t)\| &= \phi(A_1 y(t)) \\ &= \phi(f(t, y(t))) \\ &= \|f(t, y(s))\| \\ &\le L\|y\| + sup|f(t, 0)| \\ &\le L\|y\| + M_1, \end{aligned}$$

where $M_1 = sup|f(t, 0)|$. Then

$$\|A_1 y(t)\| \le Lr_1 + M_1 = r_1, \quad where \quad r_1 = \frac{M_1}{1 - L}.$$

And

$$\begin{aligned} \|A_2 x(t)\| &= \phi(A_2 x(t)) \\ &= \int_0^t \phi(g(s, x(m(s))))ds \\ &= \int_0^t \|g(s, x(m(s)))\|ds \\ &\le \int_0^t \{|a(s)| + b\|x(m(s))\|\}ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t |a(s)|ds + b\int_0^t \|x(m(s))\|ds \\
&\leq M_2 + br_2T.
\end{aligned}$$

Therefore

$$\|A_2x(t)\| \leq M_2 + br_2T = r_2, \quad where \quad r_2 = \frac{M_2}{1-bT}.$$

Now

$$\begin{aligned}
\|AU(t)\|_X &= \|A_1y(t)\| + \|A_2x(t)\| \\
&\leq M_1T + Lr_1 + M_2 + br_2T \\
&= r.
\end{aligned}$$

Then

$$\|AU\|_X \leq r.$$

Hence, $AU \in Q_r$, which proves that $AQ_r \subset Q_r$, i.e. $A : Q_r \to Q_r$, and the class of functions $\{AQ_r\}$ is uniformly bounded.

Now Q_r is nonempty, closed, convex and uniformly bounded.

As a consequence of proposition 2, then $\{AQ_r\}$ is relatively weakly compact.

Now, we shall prove that $A : X \to X$.

Let $t_1, t_2 \in I$, $\;t_1 < t_2$ (without loss of generality assume that $\|AU(t_2) - AU(t_1)\| \neq 0$), then firstly we have

$$\begin{aligned}
\|A_1y(t_2) - A_1y(t_1)\| &= \|f(t_2, y(t_2)) - f(t_1, y(t_1))\| \\
&\leq L(|t_2 - t_1| + \|y(t_2) - y(t_1)\|)
\end{aligned}$$

and

$$\begin{aligned}
\|A_2x(t_2) - A_2x(t_1)\| &= \phi(A_2x(t_2) - A_2x(t_1)) \\
&= \int_{t_1}^{t_2} \phi(g(s, x(m(s))))ds \\
&= \int_{t_1}^{t_2} \|(g(s, x(m(s))))\|ds \\
&\leq \int_{t_1}^{t_2} \{|a(s)| + b\|x(m(s))\|\}ds \\
&\leq \int_{t_1}^{t_2} |a(s)|ds + b\int_{t_1}^{t_2} \|x(m(s))\|\}ds
\end{aligned}$$

$$\leq \int_{t_1}^{t_2} |a(s)|ds + br_2|t_2 - t_1|.$$

Then

$$\begin{aligned} \|AU(t_2) - AU(t_1)\|_X &= \|(A_1y(t_2), A_2x(t_2)) - (A_1y(t_1), A_2x(t_1))\| \\ &= \|((A_1y(t_2) - A_1y(t_1)), (A_2x(t_2) - A_2x(t_1)))\| \\ &= \|A_1y(t_2) - A_1y(t_1)\| + \|A_2x(t_2) - A_2x(t_1)\|. \end{aligned}$$

Which proves that $A : X \to X$.

Finally, we want to prove that A is weakly sequentially continuous.

Let $\{U_n\}$ be a sequence in Q_r converges weakly to $U \ \ \forall t \in I$, then we have the two sequences $\{y_n\}, \ \{x_n\}$, such that $\{y_n\}$ converges strongly to y and $\{x_n\}$ converges weakly to x, i.e. $y_n(t) \to y, \ x_n(t) \rightharpoonup x, \quad \forall t \in I$.

Since $g(t, x(m(t)))$ is weakly sequentially continuous in x, then $g(t, x_n(m(t)))$ converges weakly to $g(t, x(m(t)))$.

Thus $\phi(g(t, x_n(m(t))))$ converges strongly to $\phi(g(t, x(m(t))))$, and $\phi(f(t, y_n(t)))$ converges strongly to $\phi(f(t, y(t)))$.

Also,

$$\|g(t, x_n(m(s)))\| \leq |a(t)| + b\|x_n\|$$

By applying Lebesgue dominated convergence theorem for Pettis integral, then we get

$$\begin{aligned} \phi(A_1y_n(t)) &= \phi(f(t, y_n(t))) \\ &= \|f(t, y_n(t))\| \\ &\to \|f(t, y(t))\|, \quad \forall \ \phi \in E^{\square}, \quad t \in I \end{aligned}$$

i.e. $\phi(A_1y_n(t)) \to \phi(A_1y(t))$, and then

$$\|A_1y_n(t)\| \to \|A_1y(t)\|$$

and

$$\begin{aligned} \phi(\int_0^t g(s, x_n(m(s)))ds) &= \int_0^t \phi(g(s, x_n(m(s))))ds \\ &= \int_0^t \|g(s, x_n(m(s)))\|ds \\ &\to \int_0^t \|g(s, x(m(s)))\|ds, \quad \forall \ \phi \in E^{\square}, \quad t \in I \end{aligned}$$

i.e. $\phi(A_2x_n(t)) \to \phi(A_2x(t))$, and then

$$\|A_2x_n(t)\| \to \|A_2x(t)\|.$$

Therefore,

$$\begin{aligned}
\|AU_n(t)\|_X &= \|A(x_n(t), y_n(t))\| \\
&= \|(A_1y_n(t), A_2x_n(t))\| \\
&= \|A_1y_n(t)\| + \|A_2x_n(t)\| \\
&\to \|A_1y(t)\| + \|A_2x(t)\| \\
&\to \|(A_1y(t), A_2x(t))\| \\
&\to \|AU(t)\|_X
\end{aligned}$$

Hence, A is weakly sequentially continuous (i.e. $AU_n(t) \to AU(t), \quad \forall t \in I$ weakly).

Since all conditions of O'Regan theorem are satisfied, then the operator A has at least one fixed point $U \in Q_r$, and then the coupled system (4) and (5) has at least one weak solution $(x, y) \in X$, then there exists at least one weak solution $x \in C[I, E]$ of the functional integral equation (3).

Consequently, there exists at least one weak continuous solution $x \in C[I, E]$ of the functional integral inclusion (1). □

3.2. Functional Integral Inclusion Approach

Definition 8. By a solution of the functional integral inclusion (1) we mean the function $x \in C[I, E]$ satisfying (1).

Now for the existence of weak solution $x \in C[I, E]$ of the functional integral inclusion (1) we have the following theorem.

Theorem 9. *Let the assumptions (H1)-(H7) be satisfied, then there exists at least one weak solution $x \in C[I, E]$ of the functional integral inclusion (1).*

Proof. Let the set-valued function F satisfy the assumptions (H1)-(H3), then there exists $f \in S_F$ such that

$$\|f(t_2, x) - f(t_1, y)\| \le L(|t_2 - t_1| + \|x - y\|),$$

And f satisfy the functional integral equation (3)

Define the operator A by

$$Ax(t) = f(t, \int_0^t g(s, x(m(s)))ds), \quad t \in I$$

Let the set Q_r defined by

$$Q_r = \{x \in C[I, E], \quad \|x\| \leq r\}.$$

Let $x \in Q_r$ be arbitrary, then we have from proposition 3

$$\begin{aligned}
\|Ax(t)\| &= \phi(Ax(t)) \\
&= \phi f(t, \int_0^t g(s, x(m(s)))ds) \\
&= \|f(t, \int_0^t g(s, x(m(s)))ds)\| \\
&\leq L\|\int_0^t g(s, x(m(s)))ds\| + sup|f(t, 0)| \\
&\leq L\int_0^t \|g(s, x(m(s)))\|ds + sup|f(t, 0)| \\
&\leq L\int_0^t \{|a(s)| + b\|x\|\}ds + sup|f(t, 0)| \\
&\leq L\{\int_0^t |a(s)|ds + b\int_0^t \|x\|ds\} + sup|f(t, 0)| \\
&\leq L\{M_2 + bTr\} + M_1
\end{aligned}$$

where $M_1 = sup|f(t, 0)|$, $M_2 = \int_0^t |a(t)|ds$.

Therefore

$$\|Ax(t)\| \leq LM_2 + LbTr + M_1 = r, \quad where \quad r = \frac{LM_2 + M_1}{1 - LbT}.$$

Then

$$\|Ax(t)\| \leq r.$$

Hence, $Ax \in Q_r$, which proves that $AQ_r \subset Q_r$, i.e. $A : Q_r \to Q_r$, and the class of functions $\{AQ_r\}$ is uniformly bounded.

Now Q_r is nonempty, closed, convex and uniformly bounded.

As a consequence of proposition 1, then $\{AQ_r\}$ is relatively weakly compact.

Now, we shall prove that $A : C[I, E] \to C[I, E]$.

Let $t_1, t_2 \in I$, $t_1 < t_2$ (without loss of generality assume that $\|Ax(t_2) - Ax(t_1)\| \neq 0$), then we have

$$\|Ax(t_2) - Ax(t_1)\| = \|f(t_2, \int_0^{t_2} g(s, x(m(s)))ds) - f(t_1, \int_0^{t_1} g(s, x(m(s)))ds)\|$$

$$\begin{aligned}
&\leq L(|t_2 - t_1| + \| \int_0^{t_2} g(s, x(m(s)))ds - \int_0^{t_1} g(s, x(m(s)))ds\|) \\
&\leq L(|t_2 - t_1| + \| \int_{t_1}^{t_2} g(s, x(m(s)))ds\|) \\
&\leq L(|t_2 - t_1| + \int_{t_1}^{t_2} \|g(s, x(m(s)))\|ds) \\
&\leq L(|t_2 - t_1| + \int_{t_1}^{t_2} \{|a(s)| + b\|x\|\}ds) \\
&\leq L(|t_2 - t_1| + \int_{t_1}^{t_2} |a(s)|ds + b\int_{t_1}^{t_2} \|x\|ds) \\
&\leq L(|t_2 - t_1| + \int_{t_1}^{t_2} |a(s)|ds + br|t_2 - t_1|).
\end{aligned}$$

Then

$$\|Ax(t_2) - Ax(t_1)\| \quad \leq \quad L(|t_2 - t_1| + \int_{t_1}^{t_2} |a(t)|ds + br|t_2 - t_1|).$$

Which proves that $A : C[I, E] \to C[I, E]$.

Finally, we want to prove that A is weakly sequentially continuous.

Let $\{x_n\}$ be a sequence in Q_r converges weakly to $x \quad \forall t \in I$, i.e. $x_n(t) \rightharpoonup x, \quad \forall t \in I$.

Since $g(t, x(m(t)))$ is weakly sequentially continuous in x, then $g(t, x_n(m(t)))$ converges weakly to $g(t, x(m(t)))$.

Thus $\phi(g(t, x_n(m(t))))$ converges strongly to $\phi(g(t, x(m(t))))$.

Also,

$$\|g(t, x_n(m(s)))\| \leq |a(t)| + b\|x_n\|$$

Then by applying Lebesgue dominated convergence theorem for Pettis integral, we get

$$\begin{aligned}
\phi(Ax_n(t)) \quad &= \quad \phi(f(t, \int_0^t g(s, x_n(m(s)))ds)) \\
&= \quad \|f(t, \int_0^t g(s, x_n(m(s)))ds)\| \\
&\to \quad \|f(t, \int_0^t g(s, x(m(s)))ds)\|, \quad \forall\, \phi \in E^{*}, \quad t \in I
\end{aligned}$$

i.e. $\phi(Ax_n(t)) \to \phi(Ax(t))$, and then

$$\|Ax_n(t)\| \to \|Ax(t)\|$$

Hence, A is weakly sequentially continuous (i.e. $Ax_n(t) \to Ax(t), \quad \forall t \in I$ weakly).

Since all conditions of O'Regan theorem are satisfied, then the operator A has at least one fixed point $x \in Q_r$, and then there exists at least one weak solution $x \in C[I, E]$ of the functional integral equation (3).

Consequently, there exists at least one weak solution $x \in C[I, E]$ of the functional integral inclusion (1). □

References

[1] D. Bugajewski, On the existence of weak solutions of integrl equations in Banach spaces, *Comment. Math. Univ. Carolin*, **35** (1994),35-41.

[2] M. Cichon, I. Kubiaczyk, Existence theorem for the Hammerstien integral equation, *Discuss. Math. Diff. Incl. Control and optimization*, **16**(1996), 171-177.

[3] T.Cardinali, K. Nikodem and F. Papalini, Some results on stability and characterization of k-convexity of set-valued functions, *Ann. Polon, Math.* **58** (1993), 185-192.

[4] N. Dunford, J. T. Schwartz, *Linear operators*, Interscience, Wiley, New York, (1958).

[5] A. M. A. EL-Sayed, H. H. G. Hashem, Weak maximal and minimal solutions for Hammerstien and Urysohn integral equations in reflexive Banach spaces, In: *Differential equation and control processes*, **4**(2008), 50-62.

[6] A. M. A. El-sayed and A. G. Ibrahim, Multivalued fractional differential equations, In: *Applied Mathematics and Computation*, **68** (1995), 15-25.

[7] E. Hille, R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31,Amer. Math. Soc., Providence, R. I. (1957).

[8] K.Nikodem, On quadratic set-valued functions, *Publ. Math. Debrecen*, **30** (1984), 297-301.

[9] K. Nikodem, On jensen's functional equation for set-valued functions, *Rad.Math*, **3** (1987), 23-33.

[10] K. Nikodem, Set-valued solutions of the pexider functional equations, *Funkcial.Ekvac*, **31** (1988), 227-231.

[11] B. J. Pettis, On integration in vector spacecs, *Trans. Amer. Math. Soc.*, **44** (1938), 277-304.

[12] D.Popa, Functional inclusions on square symmetric grupoids and Hyers-Ulam stability, *Mathematical Inequal & Appl.*, **7** (2004), 419-428.

[13] D.Popa, Aproperty of a functional inclusion connected with Hyers-Ulam stability. *J. Math. Inequal*, **4** (2009), 591-598.

[14] D. O'Regan, Fixed point theory for wealy sequentially continuous mapping, *Math. Comput. Modeling*, **27** (1998), 1-14.

[15] D. O'Regan, Integral equations in reflexive Banach spaces and weak topologies, *Amer. Mah. Soc.*, **2** (1996), 607-614.

[16] H.A.H. Salem, A. M. A. EL-Sayed, Fractional order integral equations in reflexive Banach spaces, *Math. Slovaca.*, **2** (2005), 169-181.