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GENERALIZED TWO-HYBRID ONE-STEP IMPLICIT THIRD DERIVATIVE BLOCK METHOD FOR THE DIRECT SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: In this article, a general two-hybrid one-step implicit third derivative block method is developed for the direct solution of the second order initial value problems through interpolation and collocation approach. To derive this method, the approximate basis function is interpolated at the values $\{x_n, x_{n+r}\}$ while its second and third derivatives are collocated at all points $\{x_n, x_{n+r}, x_{n+s}, x_{n+1}\}$ in the given interval. The new developed method produces better accuracy if compared to the existing methods when solving the same problems.

AMS Subject Classification: 35J25, 35Q60, 65L06, 65L05

Key Words: hybrid block method, second order ordinary differential equation, direct solution, third derivative, interpolation and collocation

1. Introduction

In this article, we are interested in solving the following second order ordinary differential equation

$$y'' = f(x, y, y'), \quad y(a) = y_0\,, \ \ y'(a) = y_1\,, \quad a \le x \le b. \tag{1}$$

There are many methods available for approximating (1). One of the methods

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is called a block method which initially developed by [2] to provide starting values for predictor-corrector schemes. The usage of this method was further extended by [11] and [15] to get better approximation for solving (1).

However, block methods are sometimes confined by zero-stability barrier (see [16]). Hence, several mathematicians introduced hybrid block methods to overcome this drawback of block methods. In hybrid block methods step and off-step points are combined to form a single block for solving ODEs, see [5],[9],[10].

In order to enhance the accuracy of the solution, [13] suggested second derivative methods which can solve stiff ODEs. For the same reason, [12] also proposed a Simpson's type second derivative method to approximate the solution of first order stiff ODEs system.

Based on the previous work mentioned above, we attempt to develop a new generalized two-hybrid one-step third derivative implicit method for solving second order ODEs directly using interpolation and collocation approach which can improve the accuracy.

2. Development of the Method

The following power series is used as a basis function for an approximate solution to (1)

$$y(x) = \sum_{j=0}^{2v+u-1} a_j \left(\frac{x - x_n}{h}\right)^j, \qquad (2)$$

where u and v are the number of interpolation and collocation points respectively. Then, the second and third derivatives are

$$y''(x) = \sum_{j=2}^{2v+u-1} \frac{a_j j!}{h^2 (j-2)!} \left(\frac{x - x_n}{h}\right)^{j-2} = f(x, y, y'), \qquad (3)$$

$$y'''(x) = \sum_{j=3}^{2v+u-1} \frac{a_j j!}{h^3 (j-3)!} \left(\frac{x - x_n}{h}\right)^{j-3} = g(x, y, y'). \qquad (4)$$

Interpolating (2) at $x_{n+\hat{u}}$, for $\hat{u} = \{0, r\}$ and collocating (3) and (4) at all points $x_{n+\hat{v}}$, for $\hat{v} = \{0, r, s, 1\}$ where $\{0 < r < s < 1\}$, and then combining the resulted equations gives a system of equations in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & r & r^2 & r^3 & r^4 & \cdots & r^9 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6r}{h^2} & \frac{12r^2}{h^2} & \cdots & \frac{9!r^7}{7!h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2} & \cdots & \frac{9!s^7}{7!h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \cdots & \frac{9!}{7!h^2} \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24r}{h^3} & \cdots & \frac{9!r^6}{6!h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s}{h^3} & \cdots & \frac{9!s^6}{6!h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \cdots & \frac{9!}{6!h^3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+r} \\ f_n \\ f_{n+r} \\ f_{n+s} \\ f_{n+1} \\ g_n \\ g_{n+r} \\ g_{n+s} \\ g_{n+1} \end{bmatrix}. \tag{5}$$

Solving the unknown coefficients $a_j's$ in (5) using matrix manipulation and substituting them back into Equation (2) yields

$$\begin{aligned} y(x) &= \alpha_0 y_n + \alpha_r y_{n+r} \\ &+ h^2[\beta_r f_{n+r} + \beta_s f_{n+s} + \sum_{i=0}^{1} \beta_i f_{n+i}] + h^3[\gamma_r g_{n+r} + \gamma_s g_{n+s} + \sum_{i=0}^{1} \gamma_i g_{n+i}], \end{aligned} \tag{6}$$

where $n = 0, 1, 2, ..., N-1$, $h = x_n - x_{n-1}$ is the constant step size for the partition π_N of the interval $[a, b]$ which is given by $\pi_N = [a = x_0 < x_1 < ... < x_{N-1} < x_N = b]$, $\alpha_0, \alpha_r, \beta_0, \beta_r, \beta_s, \beta_1, \gamma_0, \gamma_r, \gamma_s$ and γ_1 are undetermined constants listed in Appendix I. Differentiating (6) with respect to x produces

$$\begin{aligned} y'(x) = \frac{\partial}{\partial x}\alpha_0 y_n + \frac{\partial}{\partial x}\alpha_r y_{n+r} + h^2[\frac{\partial}{\partial x}\beta_r f_{n+r} + \frac{\partial}{\partial x}\beta_s f_{n+s} + \sum_{i=0}^{1}\frac{\partial}{\partial x}\beta_i f_{n+i}] \\ + h^3[\frac{\partial}{\partial x}\gamma_r g_{n+r} + \frac{\partial}{\partial x}\gamma_s g_{n+s} + \sum_{i=0}^{1}\frac{\partial}{\partial x}\gamma_i g_{n+i}]. \end{aligned} \tag{7}$$

Evaluating (6) at the non interpolating points $\{x_{n+s}, x_{n+1}\}$ and (7) at all points $x_{n+i},\ i = \{0, r, s, 1\}$ produces the following general equations in block form

$$A^{(0)} Y_{m+1} = A^{(1)} Y_m + \sum_{i=0}^{1} B^{(i)} F_{m+i} + \sum_{i=0}^{1} D^{(i)} G_{m+i}, \tag{8}$$

where

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & rh \\ 0 & 0 & 1 & 0 & 0 & sh \\ 0 & 0 & 1 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{16}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & B_{26}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & B_{36}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & B_{46}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & B_{56}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & B_{66}^{(0)} \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} B_{11}^{(1)} & B_{12}^{(1)} & B_{13}^{(1)} \\ B_{21}^{(1)} & B_{22}^{(1)} & B_{23}^{(1)} \\ B_{31}^{(1)} & B_{32}^{(1)} & B_{33}^{(1)} \\ B_{41}^{(1)} & B_{42}^{(1)} & B_{43}^{(1)} \\ B_{51}^{(1)} & B_{52}^{(1)} & B_{53}^{(1)} \\ B_{61}^{(1)} & B_{62}^{(1)} & B_{63}^{(1)} \end{bmatrix},$$

$$D^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & D_{16}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & D_{26}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & D_{36}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & D_{46}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & D_{56}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & D_{66}^{(0)} \end{bmatrix}, \quad D^{(1)} = \begin{bmatrix} D_{11}^{(1)} & D_{12}^{(1)} & D_{13}^{(1)} \\ D_{21}^{(1)} & D_{22}^{(1)} & D_{32}^{(1)} \\ D_{31}^{(1)} & D_{32}^{(1)} & D_{33}^{(1)} \\ D_{41}^{(1)} & D_{42}^{(1)} & D_{43}^{(1)} \\ D_{51}^{(1)} & D_{52}^{(1)} & D_{53}^{(1)} \\ D_{61}^{(1)} & D_{62}^{(1)} & D_{63}^{(1)} \end{bmatrix}.$$

$$\begin{aligned} Y_{m+1} &= \left[y_{n+r}, y_{n+s}, y_{n+1}, y'_{n+r}, y'_{n+s}, y'_{n+1}\right]^T, \\ Y_m &= \left[y_{n-s}, y_{n-r}, y_n, y'_{n-s}, y'_{n-r}, y'_n\right]^T, \\ F_m &= \left[f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n\right]^T, \\ F_{m+1} &= \left[f_{n+r}, f_{n+s}, f_{n+1}\right]^T, \\ G_m &= \left[g_{n-5}, g_{n-4}, g_{n-2}, g_{n-1}, g_n\right]^T, \\ G_{m+1} &= \left[g_{n+r}, g_{n+s}, g_{n+1}\right]^T. \end{aligned}$$

Here, the non-zero entries of $B^{(0)}, B^{(1)}, D^{(0)}$ and $D^{(1)}$ are listed in Appendix II.

3. Analysis of the Method

3.1. Zero Stability

Definition 1. The hybrid block method formula (8) is said to be zero stable if no root (R_z) of the first characteristic equation $\rho(R)$ has modulus greater than one i.e $\mid R_z \mid \leq 1$ and if $R_z = 1$ then the multiplicity of R_z must not exceed two .

To show that the roots of the first characteristic equation satisfies the prior definition we assume that $\{r, s\} \in (0,1)$ and hence

$$\rho(R) = det([RA^{(0)} - A^{(1)}]) = 0$$

$$\rho(R) = \left| \begin{bmatrix} R & 0 & & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & rh \\ 0 & 0 & 1 & 0 & 0 & sh \\ 0 & 0 & 1 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0,$$

which implies

$$R^4(R-1)^2 = 0,$$

whose solutions are

$$R_1 = R_2 = R_3 = R_4 = 0, \quad R_5 = R_6 = 1.$$

Therefore, the developed method is zero stable.

3.2. Order of the Method

The linear operator $\hat{L}$ associated with the hybrid block methods formula (8) according to [3] and [4] is said to be of order p if

$$\hat{L}\{y(x); h\} = A^{(0)}Y_m - A^{(1)}Y_{m+1} - \sum_{i=0}^{1} B^{(i)}F_{m+i} - \sum_{i=0}^{1} D^{(i)}G_{m+i},$$

expanding in Taylor series and combining like terms

$$\hat{L}\{y(x); h\} = \sum_{i=0}^{\infty} C_i h^i y^{(i)} = 0, \tag{9}$$

where

$$C_0 = C_1 = ... = C_{p+1} = 0 \text{ and } C_{p+2} \neq 0.$$

The term C_{p+2} is called the error constant and the local truncation error is given by :

$$t_{n+k} = C_{p+2} y^{p+2} h^{p+2}(x_n) + O(h^{p+3}).$$

Equation (9) can be expressed as shown below

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where the values E_1, E_2, E_3, E_4, E_5 and E_6 are listed in Appendix III. Comparing the coefficients of y^i and h^i produces $C_0 = C_1 = ... = C_9 = 0$ with vector of error constants

$$C_{10} =$$
$$\mathbf{K} \begin{bmatrix} r^6 s^2 (3r^4 - 10r^3 s^2 - 10r^3 + 9r^2 s^3 + 36r^2 s^2 + 9r^2 - 36rs^3 - 36rs^2 + 42s^2) \\ s^7 (9r^2 s^4 - 36r^2 s^3 + 42r^2 s - 10rs^5 + 36rs^4 - 36rs^3 + 3s^5 - 10s^4 + 9s^3) \\ (42r^2 s^4 - 36r^2 s^3 + 9r^2 s - 36rs^4 + 36rs^3 - 10rs + 9s^3 - 10s^2 + 3) \\ 2r^5 s^2 (5r^4 - 15r^3 s^2 - 15r^3 + 12r^2 s^3 + 48r^2 s^2 + 12r^2 - 42rs^3 - 42rs^2 + 42s^2) \\ 2s^6 (12r^2 s^4 - 42r^2 s^3 + 42r^2 s - 15rs^5 + 48rs^4 - 42rs^3 + 5s^5 - 15s^4 + 12s^3) \\ (84r^2 s^4 - 84r^2 s^3 + 24r^2 s - 84rs^4 + 96rs^3 - 30rs + 24s^3 - 30s^2 + 10), \end{bmatrix},$$

where $\mathbf{K} = \frac{1}{101606400}$, which implies that the order p of this method is 8.

3.3. Consistency

Definition 2. A block method is said to be consistent if its order p is greater than one.

Since the order $p = 8$ of the hybrid block method from the above analysis that is greater than one hence the consistency property is satisfied.

3.4. Convergence

Theorem 3 (Henrici, 1962)**.** *Consistency and zero stability are sufficient conditions for a linear multi-step method to be convergent.*

The hybrid block formula (8) is convergent since it fulfils both the consistency and zero stability conditions.

4. Numerical Examples

In this section the accuracy of the general two hybrid one-step implicit hybrid block formula (8) with order 8 is examined on three test problems, with step size $h = \frac{1}{100}$. The computed results are then compared with the latest methods.

Problem 1. $f(x, y, y') = 100y$, $y(0) = 1$, $y'(0) = -10$.
Exact Solution. $y = e^{-10x}$, with $h = \frac{1}{100}$.
Source. [6].

X value	Exact solution	Computed solution	Error in new method	Error for [6]
0.01	0.904837418035959520	0.904837418035959520	0.000000(+00)	7.360845(-11)
0.02	0.818730753077981820	0.818730753078006130	2.431388(-14)	2.854670(-10)
0.03	0.740818220681717880	0.740818220681788930	7.105427(-14)	6.246097(-10)
0.04	0.670320046035639330	0.670320046035777770	1.384448(-13)	1.082590(-9)
0.05	0.606530659712633420	0.606530659712859130	2.257083(-13)	1.551924(-9)
0.06	0.548811636094026390	0.548811636094358010	3.316236(-13)	2.332691(-9)
0.07	0.496585303791409470	0.496585303791865050	4.555800(-13)	3.118856(-9)
0.08	0.449328964117221560	0.449328964117819030	5.974665(-13)	4.011705(-9)
0.09	0.406569659740599170	0.406569659741356670	7.575052(-13)	5.012991(-9)
0.10	0.367879441171442330	0.367879441172378530	9.361956(-13)	6.126237(-9)
0.11	0.332871083698079610	0.332871083699213700	1.134093(-12)	7.356706(-9)
0.12	0.301194211912202190	0.301194211913554670	1.352474(-12)	8.711392(-9)

Table 1: Comparison of the proposed method with [6].

Problem 2. $f(x, y, z) = x(y')^2$, $y(0) = 1$, $y'(0) = \frac{1}{2}$.
Exact Solution. $y = 1 + ln(\frac{2+x}{2-x})$, with $h = \frac{1}{100}$.
Source. [7].

X value	Exact solution	Computed solution	Error in new method	Error for [7]
0.10	1.05004172927849140	1.05004172927849070	6.661338(-16)	1.1102(-15)
0.20	1.10033534773107560	1.10033534773107470	8.881784(-16)	1.9318(-14)
0.30	1.15114043593646700	1.15114043593646590	1.110223(-15)	1.1169(-13)
0.40	1.20273255405408230	1.20273255405408190	4.440892(-16)	4.1034(-13)
0.50	1.25541281188299550	1.25541281188299390	1.554312(-15)	1.1482(-12)
0.60	1.30951960420311190	1.30951960420310760	4.218847(-15)	2.7629(-12)
0.70	1.36544375427139640	1.36544375427138460	1.176836(-14)	6.1566(-12)
0.80	1.42364893019360220	1.42364893019357660	2.553513(-14)	1.2952(-11)
0.90	1.48470027859405200	1.48470027859399840	5.351275(-14)	2.6431(-11)
1.00	1.54930614433405540	1.54930614433394000	1.154632(-13)	5.4045(-11)

Table 2: Comparison of the proposed method with [7].

Problem 3. $f(x, y, y') = y'$, $y(0) = 0$, $y'(1) = -1$.
Exact Solution. $y = 1 - e^x$, with $h = \frac{1}{100}$.
Source. [8].

X value	Exact solution	Computed solution	Error in new method	Error for [8]
0.10	-0.105170918075647710	-0.105170918075647630	8.326673(-17)	2.2360(-13)
0.21	-0.233678059956743400	-0.233678059956743120	2.775558(-16)	1.6425(-12)
0.30	-0.349858807576003180	-0.349858807576002630	5.551115(-16)	3.4625(-12)
0.40	-0.491824697641270570	-0.491824697641269630	9.436896(-16)	8.6628(-12)
0.50	-0.648721270700128640	-0.648721270700126530	2.109424(-15)	1.1338(-11)
0.60	-0.822118800390509550	-0.822118800390506330	3.219647(-15)	2.0317(-11)
0.70	-1.013752707470477500	-1.013752707470473100	4.440892(-15)	3.2476(-11)
0.80	-1.225540928492468800	-1.225540928492462800	5.995204(-15)	4.5463(-11)
0.90	-1.459603111156951200	-1.459603111156943400	7.771561(-15)	6.1781(-11)
1.00	-1.718281828459047300	-1.718281828459036700	1.065814(-14)	8.2113(-11)

Table 3: Comparison of the proposed method with [8].

5. Conclusion

A general two hybrid one-step block method of uniform order 8 has been developed for the direct solution of general second order ODEs. The developed method is tested on three problems.

Numerical analysis shows that the developed method is consistent and zero stable which implies its convergence.

Besides having excellent properties of the numerical method, the numerical results reveals that the new method has perform better than the existing methods.

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Appendix I

$$\alpha_0 = \frac{(x_n - x + hr)}{(hr)}, \quad \alpha_r = \frac{(x - x_n)}{(hr)},$$

$$\beta_0 = \big((((x-x_n)^2/2 - ((x-x_n)^4(3r^2s^2+4r^2s+3r^2+4rs^2+4rs+3s^2))/ (12h^2r^2s^2) + ((x-x_n)^9(r+s+rs))/(36h^7r^3s^3) + ((x-x_n)^5(r^3s^3+ 4r^3s^2+4r^3s+r^3+4r^2s^3+8r^2s^2+4r^2s+4rs^3+4rs^2+s^3))/(10h^3r^3s^3) +(hr(x-x_n)(-10r^5s-10r^5+36r^4s^2+53r^4s+36r^4-36r^3s^3-108r^3s^2 -108r^3s-36r^3+90r^2s^3+24r^2s^2+90r^2s+168rs^3+168rs^2-882s^3))/ (2520s^3) - ((x-x_n)^8(4r^2s+4r^2+4rs^2+11rs+4r+4s^2+4s))/(56h^6r^3 s^3) - ((x-x_n)^6(4r^3s^2+7r^3s+4r^3+4r^2s^3+20r^2s^2+20r^2s+4r^2+7rs^3 +20rs^2+7rs+4s^3+4s^2))/(30h^4r^3s^3) + ((x-x_n)^7(r^3s+r^3+4r^2s^2+8r^2 s+4r^2+rs^3+8rs^2+8rs+r+s^3+4s^2+s))/(21h^5r^3s^3)\big),$$

$$\beta_r = \big((((x-x_n)^8(7r^3+7r^2s+7r^2-8rs^2-13rs-8r+4s^2+4s))/(56h^6r^3(r -s)^3(r-1)^3) - (hr(x-x_n)(105r^6-385r^5s-385r^5+468r^4s^2+1457r^4s +468r^4-180r^3s^3-1836r^3s^2-1836r^3s-180r^3+720r^2s^3+2418r^2s^2+ 720r^2s-966rs^3-966rs^2+378s^3))/(2520(r-s)^3(r-1)^3) - ((x-x_n)^6 (-7r^3s^2-28r^3s-7r^3+5r^2s^3+13r^2s^2+13r^2s+5r^2+5rs^3+4rs^2+5rs -4s^3-4s^2))/(30h^4r^3(r-s)^3(r-1)^3) + ((x-x_n)^9(2r-s+2rs-3r^2))/ (36h^7r^3(r-s)^3(r-1)^3) + ((x-x_n)^7(-7r^3s-7r^3+2r^2s^2-2r^2s+2r^2+ 2rs^3+7rs^2+7rs+2r-s^3-4s^2-s))/(21h^5r^3(r-s)^3(r-1)^3) - (s(x- x_n)^5(7r^3s+7r^3-5r^2s^2-7r^2s-5r^2+rs^2+rs+s^2))/(10h^3r^3(r-s)^3(r -1)^3) - (s^2(x-x_n)^4(5r-3s+5rs-7r^2))/(12h^2r^2(r-s)^3(r-1)^3)))\big),$$

$$\beta_s = \big(((((x-x_n)^7(-2r^3s+r^3-2r^2s^2-7r^2s+4r^2+7rs^3+2rs^2-7rs+r+ 7s^3-2s^2-2s))/(21h^5s^3(r-s)^3(s-1)^3) + ((x-x_n)^6(5r^3s^2+5r^3s-4r^3 -7r^2s^3+13r^2s^2+4r^2s-4r^2-28rs^3+13rs^2+5rs-7s^3+5s^2))/(30h^4s^3 (r-s)^3(s-1)^3) + ((x-x_n)^9(r-2s-2rs+3s^2))/(36h^7s^3(r-s)^3(s-1)^3) -((x-x_n)^8(-8r^2s+4r^2+7rs^2-13rs+4r+7s^3+7s^2-8s))/(56h^6s^3(r -s)^3(s-1)^3) - (hr^5(x-x_n)(-20r^4s+10r^4+75r^3s^2+25r^3s-36r^3-63r^2 s^3-243r^2s^2+108r^2s+36r^2+252rs^3+138rs^2-198rs-294s^3+210s^2)) /(2520s^3(r-s)^3(s-1)^3) + (r(x-x_n)^5(-5r^2s^2+r^2s+r^2+7rs^3-7rs^2+ rs+7s^3-5s^2))/(10h^3s^3(r-s)^3(s-1)^3) - (r^2.(x-x_n)^4(3r-5s-5rs+$$

$$7s^2))/(12h^2s^2(r-s)^3(s-1)^3)))\big),$$

$$\beta_1 = \big((((x-x_n)^8(4r^2s-8r^2+4rs^2-13rs+7r-8s^2+7s+7))/(56h^6 (r-1)^3(s-1)^3)-((x-x_n)^7(r^3s-2r^3+4r^2s^2-7r^2s-2r^2+rs^3 -7rs^2+2rs+7r-2s^3-2s^2+7s))/(21h^5(r-1)^3(s-1)^3)-((x- x_n)^6(-4r^3s^2+5r^3s+5r^3-4r^2s^3+4r^2s^2+13r^2s-7r^2+5rs^3+13 rs^2-28rs+5s^3-7s^2))/(30h^4(r-1)^3(s-1)^3)+((x-x_n)^9(2r+2s -rs-3))/(36h^7(r-1)^3(s-1)^3)+(hr^5(x-x_n)(10r^4s-20r^4-36r^3 s^2+25r^3s+75r^3+36r^2s^3+108r^2s^2-243r^2s-63r^2-198rs^3+138 rs^2+252rs+210s^3-294s^2))/(2520(r-1)^3(s-1)^3)-(rs(x-x_n)^5 (r^2s^2+r^2s-5r^2+rs^2-7rs+7r-5s^2+7s))/(10h^3(r-1)^3(s-1)^3) -(r^2s^2(x-x_n)^4(5r+5s-3rs-7))/(12h^2(r-1)^3(s-1)^3))\big),$$

$$\gamma_0 = \big((((x-x_n)(x_n-x+hr)(18h^7r^6s-5h^7r^7+18h^7r^6-18h^7r^5s^2-72h^7 r^5s-18h^7r^5+84h^7r^4s^2+84h^7r^4s-126h^7r^3s^2-5h^6r^6x+5h^6r^6x_n+ 18h^6r^5sx-18h^6r^5sx_n+18h^6r^5x-18h^6r^5x_n-18h^6r^4s^2x+18h^6r^4s^2 x_n-72h^6r^4sx+72h^6r^4sx_n-18h^6r^4x+18h^6r^4x_n+84h^6r^3s^2x-84h^6 r^3s^2x_n+84h^6r^3sx-84h^6r^3sx_n-126h^6r^2s^2x+126h^6r^2s^2x_n-5h^5r^5x^2 +10h^5r^5xx_n-5h^5r^5x_n^2+18h^5r^4sx^2-36h^5r^4sxx_n+18h^5r^4sx_n^2+18h^5 r^4x^2-36h^5r^4xx_n+18h^5r^4x_n^2-18h^5r^3s^2x^2+36h^5r^3s^2xx_n-18h^5r^3s^2 x_n^2-72h^5r^3sx^2+144h^5r^3sxx_n-72h^5r^3sx_n^2-18h^5r^3x^2+36h^5r^3xx_n- 18h^5r^3x_n^2+84h^5r^2s^2x^2-168h^5r^2s^2xx_n+84h^5r^2s^2x_n^2+84h^5r^2sx^2-168 h^5r^2sxx_n+84h^5r^2sx_n^2+294h^5rs^2x^2-588h^5rs^2xx_n+294h^5rs^2x_n^2-5h^4 r^4x^3+15h^4r^4x^2x_n-15h^4r^4xx_n^2+5h^4r^4x_n^3+18h^4r^3sx^3-54h^4r^3sx^2x_n +54h^4r^3sxx_n^2-18h^4r^3sx_n^3+18h^4r^3x^3-54h^4r^3x^2x_n+54h^4r^3xx_n^2-18h^4 r^3x_n^3-18h^4r^2s^2x^3+54h^4r^2s^2x^2x_n-54h^4r^2s^2xx_n^2+18h^4r^2s^2x_n^3-72h^4 r^2sx^3+216h^4r^2sx^2x_n-216h^4r^2sxx_n^2+72h^4r^2sx_n^3-18h^4r^2x^3+54h^4r^2 x^2x_n-54h^4r^2xx_n^2+18h^4r^2x_n^3-336h^4rs^2x^3+1008h^4rs^2x^2x_n-1008h^4r s^2xx_n^2+336h^4rs^2x_n^3-336h^4rsx^3+1008h^4rsx^2x_n-1008h^4rsxx_n^2+336h^4 rsx_n^3-126h^4s^2x^3+378h^4s^2x^2x_n-378h^4s^2xx_n^2+126h^4s^2x_n^3-5h^3r^3x^4+ 20h^3r^3x^3x_n-30h^3r^3x^2x_n^2+20h^3r^3xx_n^3-5h^3r^3x_n^4+18h^3r^2sx^4-72h^3r^2sx^3 x_n+108h^3r^2sx^2x_n^2-72h^3r^2sxx_n^3+18h^3r^2sx_n^4+18h^3r^2x^4-72h^3r^2x^3x_n+ 108h^3r^2x^2x_n^2-72h^3r^2xx_n^3+18h^3r^2x_n^4+108h^3rs^2x^4-432h^3rs^2x^3x_n+648 h^3rs^2x^2x_n^2-432h^3rs^2xx_n^3+108h^3rs^2x_n^4+432h^3rsx^4-1728h^3rsx^3x_n+$$

$$2592h^3rsx^2x_n^2-1728h^3rsxx_n^3+432h^3rsx_n^4+108h^3rx^4-432h^3rx^3x_n +648h^3rx^2x_n^2-432h^3rxx_n^3+108h^3rx_n^4+168h^3s^2x^4-672h^3s^2x^3x_n+1008 h^3s^2x^2x_n^2-672h^3s^2xx_n^3+168h^3s^2x_n^4+168h^3sx^4-672h^3sx^3x_n+1008h^3sx^2 x_n^2-672h^3sxx_n^3+168h^3sx_n^4-5h^2r^2x^5+25h^2r^2x^4x_n-50h^2r^2x^3x_n^2+50h^2 r^2x^2x_n^3-25h^2r^2xx_n^4+5h^2r^2x_n^5-150h^2rsx^5+750h^2rsx^4x_n-1500h^2rsx^3 x_n^2+1500h^2rsx^2x_n^3-750h^2rsxx_n^4+150h^2rsx_n^5-150h^2rx^5+750h^2rx^4x_n- 1500h^2rx^3x_n^2+1500h^2rx^2x_n^3-750h^2rxx_n^4+150h^2rx_n^5-60h^2s^2x^5+300h^2s^2 x^4x_n-600h^2s^2x^3x_n^2+600h^2s^2x^2x_n^3-300h^2s^2xx_n^4+60h^2s^2x_n^5-240h^2sx^5+ 1200h^2sx^4x_n-2400h^2sx^3x_n^2+2400h^2sx^2x_n^3-1200h^2sxx_n^4+240h^2sx_n^5-60 h^2x^5+300h^2x^4x_n-600h^2x^3x_n^2+600h^2x^2x_n^3-300h^2xx_n^4+60h^2x_n^5+55hrx^6 -330hrx^5x_n+825hrx^4x_n^2-1100hrx^3x_n^3+825hrx^2x_n^4-330hrxx_n^5+55hrx_n^6 +90hsx^6-540hsx^5x_n+1350hsx^4x_n^2-1800hsx^3x_n^3+1350hsx^2x_n^4-540hsx x_n^5+90hsx_n^6+90hx^6-540hx^5x_n+1350hx^4x_n^2-1800hx^3x_n^3+1350hx^2x_n^4- 540hxx_n^5+90hx_n^6-35x^7+245x^6x_n-735x^5x_n^2+1225x^4x_n^3-1225x^3x_n^4+735 x^2x_n^5-245xx_n^6+35x_n^7))/(2520h^6r^2s^2)\big),$$

$$\gamma_r=\big(((x-x_n)^9/(72h^6r^2(r-s)^2(r-1)^2)-((x-x_n)^6(r^2s^2+4r^2s+r^2-rs^3- 2rs^2+rs-2s^3-2s^2))/(30h^3r^2(r-s)^3(r-1)^2)-(s^2(x-x_n)^4)/(12hr(r -s)^2(r-1)^2)+(h^2r^2(x-x_n)(5r^4-15r^3s-15r^3+12r^2s^2+48r^2s+12r^2 -42rs^2-42rs+42s^2))/(1260(r-s)^2(r-1)^2)-((x-x_n)^8(r^2+rs+2r -2s^2-2s))/(56h^5r^2(r-s)^3(r-1)^2)+((x-x_n)^7(2r^2s+2r^2-rs^2+2rs +r-s^3-4s^2-s))/(42h^4r^2(r-s)^3(r-1)^2)-(s(x-x_n)^5(-2r^2s-2r^2 +2rs^2+rs+s^2))/(20h^2r^2(r-s)^3(r-1)^2))\big),$$

$$\gamma_s=\big(((r^2s(x-x_n)^4(r-1))/(12h(s-1)(rs-s^2)^2(r+s-rs-1))+((x-x_n)^6 (hr^9s^3+hr^9s^2-2hr^9s-hr^8s^4-3hr^8s^2+4hr^8s-hr^7s^4-3hr^7s^3+4hr^7s^2 +8hr^6s^4-4hr^6s^2-4hr^6s-8hr^5s^4+3hr^5s^3+3hr^5s^2+2hr^5s+hr^4s^4-h r^4s^2+hr^3s^4-hr^3s^3))/(30h^4r^3s^2(r-s)^2(r-1)^2(s-1)^2(rs-s^2)(r+s-rs -1))-((x-x_n)^8(2hr^8s^2-2hr^8s-hr^7s^3-3hr^7s^2+4hr^7s-hr^6s^4+2hr^6 s^3-hr^6s^2+3hr^5s^4+hr^5s^2-4hr^5s-3hr^4s^4-2hr^4s^3+3hr^4s^2+2hr^4s+ hr^3s^4+hr^3s^3-2hr^3s^2))/(56h^6r^3s(r-s)(r-1)^2(s-1)(r+s-rs-1)(r^2s^2 -r^2s^3+2rs^4-2rs^3-s^5+s^4))+((x-x_n)(5h^3r^{13}s^2-5h^3r^{13}s-14h^3r^{12}s^3 -19h^3r^{12}s^2+33h^3r^{12}s+9h^3r^{11}s^4+87h^3r^{11}s^3-9h^3r^{11}s^2-87h^3r^{11}s-63h^3$$

$$r^{10}s^4-201h^3r^{10}s^3+151h^3r^{10}s^2+113h^3r^{10}s+177h^3r^9s^4+179h^3r^9s^3-284$$
$$h^3r^9s^2-72h^3r^9s-243h^3r^8s^4+9h^3r^8s^3+216h^3r^8s^2+18h^3r^8s+162h^3r^7s^4$$
$$-102h^3r^7s^3-60h^3r^7s^2-42h^3r^6s^4+42h^3r^6s^3))/(2520hrs^2(r-s)^2$$
$$(r-1)^2(s-1)^2(rs-s^2)(r+s-rs-1))+((x-x_n)^5(2hr^9s^3-hr^9s^2-$$
$$hr^9s-2hr^8s^4-3hr^8s^3+2hr^8s^2+3hr^8s+4hr^7s^4-hr^7s^3-3hr^7s+hr^6$$
$$s^3-2hr^6s^2+hr^6s-4hr^5s^4+3hr^5s^3+hr^5s^2+2hr^4s^4-2hr^4s^3))/(20h^3$$
$$r^3s^2(r-s)^2(r-1)(s-1)(rs-s^2)(r+s-rs-1)^2)+((x-x_n)^9(hr^7s^2$$
$$-hr^7s-hr^6s^3-2hr^6s^2+3hr^6s+3hr^5s^3-3hr^5s-3hr^4s^3+2hr^4s^2+h$$
$$r^4s+hr^3s^3-hr^3s^2))/(72h^7r^3s^2(r-s)^2(r-1)(s-1)(rs-s^2)(r^2s^2-2r^2$$
$$s+r^2-2rs^2+4rs-2r+s^2-2s+1))-((x-x_n)^7(hr^9s^2-hr^9s+hr^8$$
$$s^3-hr^8s-2hr^7s^4-3hr^7s^3-3hr^7s^2+8hr^7s+4hr^6s^4+4hr^6s^3-8hr^6s$$
$$-4hr^5s^3+3hr^5s^2+hr^5s-4hr^4s^4+3hr^4s^3+hr^4s+2hr^3s^4-hr^3s^3-h$$
$$r^3s^2))/(42h^5rs^2(r-s)(r-1)(s-1)^2(rs-s^2)(r+s-rs-1)(r^2s-r^3s-$$
$$r^3+r^4))))\big),$$

$$\gamma_1=\big((((x-x_n)(5h^3r^{13}s^4-5h^3r^{13}s^3-33h^3r^{12}s^5+19h^3r^{12}s^4+14h^3r^{12}s^3+$$
$$87h^3r^{11}s^6+9h^3r^{11}s^5-87h^3r^{11}s^4-9h^3r^{11}s^3-113h^3r^{10}s^7-151h^3r^{10}s^6$$
$$+201h^3r^{10}s^5+63h^3r^{10}s^4+72h^3r^9s^8+284h^3r^9s^7-179h^3r^9s^6-177h^3r^9$$
$$s^5-18h^3r^8s^9-216h^3r^8s^8-9h^3r^8s^7+243h^3r^8s^6+60h^3r^7s^9+102h^3r^7$$
$$s^8-162h^3r^7s^7-42h^3r^6s^9+42h^3r^6s^8))/(2520hrs^2(r-s)^2(r-1)^2(s-$$
$$1)^2(rs-s^2)(r+s-rs-1))-((x-x_n)^4(hr^8s^4-hr^8s^5+3hr^7s^6-2h$$
$$r^7s^5-hr^7s^4-3hr^6s^7+3hr^6s^5+hr^5s^8+2hr^5s^7-3hr^5s^6-hr^4s^8+hr^4$$
$$s^7))/(12h^2r^2(r-s)(r-1)^2(s-1)^2(rs-s^2)^2(r+s-rs-1))-((x-x_n)^6$$
$$(hr^9s^4-2hr^9s^5+hr^9s^3+4hr^8s^6-3hr^8s^5-hr^8s^3+4hr^7s^6-3hr^7s^5-h$$
$$r^7s^4-4hr^6s^8-4hr^6s^7+8hr^6s^5+2hr^5s^9+3hr^5s^8+3hr^5s^7-8hr^5s^6-h$$
$$r^4s^9+hr^4s^7-hr^3s^9+hr^3s^8))/(30h^4r^3s^2(r-s)^2(r-1)^2(s-1)^2(rs-s^2)$$
$$(r+s-rs-1))-((x-x_n)^5(2hr^9s^4-hr^9s^5-hr^9s^6+3hr^8s^7+2hr^8s^6-$$
$$3hr^8s^5-2hr^8s^4-3hr^7s^8-hr^7s^6+4hr^7s^5+hr^6s^9-2hr^6s^8+hr^6s^7+h$$
$$r^5s^9+3hr^5s^8-4hr^5s^7-2hr^4s^9+2hr^4s^8))/20h^3r^3s^2(r-s)^2(r-1)(s-1)$$
$$(rs-s^2)(r+s-rs-1)^2)-((x-x_n)^9(hr^7s^3-hr^7s^4+3hr^6s^5-2hr^6s^4-$$
$$hr^6s^3-3hr^5s^6+3hr^5s^4+hr^4s^7+2hr^4s^6-3hr^4s^5-hr^3s^7+hr^3s^6))/(72$$
$$h^7r^3s^2(r-s)^2(r-1)(s-1)(rs-s^2)(r^2s^2-2r^2s+r^2-2rs^2+4rs-2r+$$
$$s^2-2s+1))-(s^2(r-s)^2(x-x_n)^8(2r+2s+1))/(56h^5(r-1)(r+s-rs$$

$-1)(r^2s^2-r^2s^3+2rs^4-2rs^3-s^5+s^4))+((x-x_n)^7(hr^9s^3-hr^9s^4-hr^8$
$s^5+hr^8s^3+8hr^7s^6-3hr^7s^5-3hr^7s^4-2hr^7s^3-8hr^6s^7+4hr^6s^5+4hr^6s^4$
$+hr^5s^8+3hr^5s^7-4hr^5s^6+hr^4s^9+3hr^4s^7-4hr^4s^6-hr^3s^9-hr^3s^8+2hr^3$
$s^7))/(42h^5rs^2(r-s)(r-1)(s-1)^2(rs-s^2)(r+s-rs-1)(r^2s-r^3s-r^3$
$+r^4)))\big)\,.$

Appendex II

$$B_{16}^{(0)}=-\frac{h^2r^2}{2520s^3}\big(-10r^5s-10r^5+36r^4s^2+53r^4s+36r^4-36r^3s^3-108r^3s^2-108r^3s-36r^3+90r^2s^3+24r^2s^2+90r^2s+168rs^3+168rs^2-882s^3\big)\,,$$

$$B_{26}^{(0)}=-\frac{h^2s^2}{2520r^3}\big(-36r^3s^3+90r^3s^2+168r^3s-882r^3+36r^2s^4-108r^2s^3+24r^2s^2+168r^2s-10rs^5+53rs^4-108rs^3+90rs^2-10s^5+36s^4-36s^3\big)\,,$$

$$B_{36}^{(0)}=-\frac{h^2}{2520r^3s^3}\big(-882r^3s^3+168r^3s^2+90r^3s-36r^3+168r^2s^3+24r^2s^2-108r^2s+36r^2+90rs^3-108rs^2+53rs-10r-36s^3+36s^2-10s\big)\,,$$

$$B_{46}^{(0)}=-\frac{hr}{420s^3}\big(-5r^5s-5r^5+16r^4s^2+23r^4s+16r^4-14r^3s^3-40r^3s^2-40r^3s-14r^3+28r^2s^3+28r^2s+56rs^3+56rs^2-210s^3\big)\,,$$

$$B_{56}^{(0)}=\frac{hs}{420r^3}\big(14r^3s^3-28r^3s^2-56r^3s+210r^3-16r^2s^4+40r^2s^3-56r^2s+5rs^5-23rs^4+40rs^3-28rs^2+5s^5-16s^4+14s^3\big)\,,$$

$$B_{66}^{(0)}=\frac{h}{420r^3s^3}\big(210r^3s^3-56r^3s^2-28r^3s+14r^3-56r^2s^3+40r^2s-16r^2-28rs^3+40rs^2-23rs+5r+14s^3-16s^2+5s)\big)\,.$$

$$B_{11}^{(1)}=-\frac{h^2r^2}{2520(r-s)^3(r-1)^3}\big(105r^6-385r^5s-385r^5+468r^4s^2+1457r^4s+468r^4-180r^3s^3-1836r^3s^2-1836r^3s-180r^3+720r^2s^3+2418r^2s^2+720r^2s-966rs^3-966rs^2+378s^3\big)\,,$$

$$B_{12}^{(1)}=\frac{h^2r^6}{2520s^3(r-s)^3(s-1)^3}\big(-20r^4s+10r^4+75r^3s^2+25r^3s-36r^3-63r^2s^3$$

$$-243r^2s^2+108r^2s+36r^2+252rs^3+138rs^2-198rs-294s^3+210s^2\big),$$

$$B_{13}^{(1)}=-\frac{h^2r^6}{2520(r-1)^3(s-1)^3}\big(10r^4s-20r^4-36r^3s^2+25r^3s+75r^3+36r^2s^3$$
$$+108r^2s^2-243r^2s-63r^2-198rs^3+138rs^2+252rs+210s^3-294s^2\big),$$

$$B_{21}^{(1)}=-\frac{h^2s^6}{2520r^3(r-s)^3(r-1)^3)}\big(-63r^3s^2+252r^3s-294r^3+75r^2s^3-243r^2s^2$$
$$+138r^2s+210r^2-20rs^4+25rs^3+108rs^2-198rs+10s^4-36s^3+36s^2\big),$$

$$B_{22}^{(1)}=\frac{-h^2s^2}{2520(r-s)^3(s-1)^3}\big(-180r^3s^3+720r^3s^2-966r^3s+378r^3+468r^2s^4-$$
$$1836r^2s^3+105s^6+2418r^2s^2-966r^2s-385rs^5+1457rs^4-1836rs^3+$$
$$720rs^2-385s^5+468s^4-180s^3\big),$$

$$B_{23}^{(1)}=-\frac{h^2s^6}{2520(r-1)^3(s-1)^3}\big(36r^3s^2-198r^3s+210r^3-36r^2s^3+108r^2s^2+$$
$$138r^2s-294r^2+10rs^4+25rs^3-243rs^2+252rs-20s^4+75s^3-63s^2\big),$$

$$B_{31}^{(1)}=-\frac{h^2}{2520r^3(r-s)^3(r-1)^3}\big(-294r^3s^2+252r^3s-63r^3+210r^2s^3+138r^2s^2$$
$$-243r^2s+75r^2-198rs^3+108rs^2+25rs-20r+36s^3-36s^2+10s\big),$$

$$B_{32}^{(1)}=\frac{h^2}{2520s^3(r-s)^3(s-1)^3}\big(210r^3s^2-198r^3s+36r^3-294r^2s^3+138r^2s^2$$
$$+108r^2s-36r^2+252rs^3-243rs^2+25rs+10r-63s^3+75s^2-20s\big),$$

$$B_{33}^{(1)}=-\frac{h^2}{2520(r-1)^3(s-1)^3}\big(-378r^3s^3+966r^3s^2-720r^3s+180r^3+966r^2s^3$$
$$-2418r^2s^2+1836r^2s-468r^2-720rs^3+1836rs^2-1457rs+385r+180$$
$$s^3-468s^2+385s-105\big),$$

$$B_{41}^{(1)}=\frac{hr}{420(r-s)^3(r-1)^3}\big(105r^6-350r^5s-350r^5+388r^4s^2+1187r^4s+388r^4$$
$$-140r^3s^3-1342r^3s^2-1342r^3s-140r^3+490r^2s^3+1554r^2s^2+490r^2s-$$
$$574rs^3-574rs^2+210s^3\big),$$

$$B_{42}^{(1)}=\frac{hr^5}{420s^3(r-s)^3(s-1)^3}\big(-10r^4s+5r^4+35r^3s^2+10r^3s-16r^3-28r^2s^3-$$
$$98r^2s^2+46r^2s+14r^2+98rs^3+42rs^2-70rs-98s^3+70s^2\big),$$

$$B_{43}^{(1)}=-\frac{hr^5}{420(r-1)^3(s-1)^3}\big(5r^4s-10r^4-16r^3s^2+10r^3s+35r^3+14r^2s^3$$

$$+46r^2s^2-98r^2s-28r^2-70rs^3+42rs^2+98rs+70s^3-98s^2\big),$$

$$B_{51}^{(1)}=-\frac{hs^5}{420r^3(r-s)^3(r-1)^3}\big(-28r^3s^2+98r^3s-98r^3+35r^2s^3-98r^2s^2+ 42r^2s+70r^2-10rs^4+10rs^3+46rs^2-70rs+5s^4-16s^3+14s^2\big),$$

$$B_{52}^{(1)}=\frac{-hs}{420(r-s)^3(s-1)^3}\big(-140r^3s^3+490r^3s^2-574r^3s+210r^3+388r^2s^4 -1342r^2s^3+105s^6+1554r^2s^2-574r^2s-350rs^5+1187rs^4-1342rs^3 +490rs^2-350s^5+388s^4-140s^3\big),$$

$$B_{53}^{(1)}=-\frac{hs^5}{420(r-1)^3(s-1)^3}\big(14r^3s^2-70r^3s+70r^3-16r^2s^3+46r^2s^2+42r^2s -98r^2+5rs^4+10rs^3-98rs^2+98rs-10s^4+35s^3-28s^2\big),$$

$$B_{61}^{(1)}=-\frac{h}{420r^3(r-s)^3(r-1)^3}\big(-98r^3s^2+98r^3s-28r^3+70r^2s^3+42r^2s^2- 98r^2s+35r^2-70rs^3+46rs^2+10rs-10r+14s^3-16s^2+5s\big),$$

$$B_{62}^{(1)}=\frac{h}{420s^3(r-s)^3(s-1)^3}\big(70r^3s^2-70r^3s+14r^3-98r^2s^3+42r^2s^2+ 46r^2s-16r^2+98rs^3-98rs^2+10rs+5r-28s^3+35s^2-10s\big),$$

$$B_{63}^{(1)}=-\frac{h}{420(r-1)^3(s-1)^3}\big(-210r^3s^3+574r^3s^2-490r^3s+140r^3+574r^2 s^3-1554r^2s^2+1342r^2s-388r^2-490rs^3+1342rs^2-1187rs+350 r+140s^3-388s^2+350s-105\big).$$

$$D_{16}^{(0)}=\frac{h^3r^3}{2520s^2}\big(5r^4-18r^3s-18r^3+18r^2s^2+72r^2s+18r^2-84rs^2-84rs +126s^2\big),$$

$$D_{26}^{(0)}=\frac{h^3s^3}{2520r^2}\big(18r^2s^2-84r^2s+126r^2-18rs^3+72rs^2-84rs+5s^4-18s^3 +18s^2\big),$$

$$D_{36}^{(0)}=\frac{h^3}{2520r^2s^2}\big(126r^2s^2-84r^2s+18r^2-84rs^2+72rs-18r+18s^2-18s +5\big),$$

$$D_{46}^{(0)}=\frac{h^2r^2}{840s^2}\big(5r^4-16r^3s-16r^3+14r^2s^2+56r^2s+14r^2-56rs^2-56rs +70s^2\big),$$

$$D_{56}^{(0)} =\frac{h^2s^2}{840r^2}\left(14r^2s^2 - 56r^2s + 70r^2 - 16rs^3 + 56rs^2 - 56rs + 5s^4 - 16s^3 \right.$$
$$\left. +14s^2\right),$$
$$D_{66}^{(0)} =\frac{h^2}{840r^2s^2}\left(70r^2s^2 - 56r^2s + 14r^2 - 56rs^2 + 56rs - 16r + 14s^2 - 16s + 5\right).$$
$$D_{11}^{(1)} = -\frac{h^3r^3}{1260(r-s)^2(r-1)^2}\left(5r^4 - 15r^3s - 15r^3 + 12r^2s^2 + 48r^2s + 12r^2 - \right.$$
$$\left. 42rs^2 - 42rs + 42s^2\right),$$
$$D_{12}^{(1)} =\frac{h^3r^6}{2520s^2(r-s)^2(s-1)^2}\left(18r - 42s + 36rs - 9r^2s - 18r^2 + 5r^3\right),$$
$$D_{13}^{(1)} =\frac{h^3r^6}{2520(r-1)^2(s-1)^2}\left(5r^3 - 18r^2s - 9r^2 + 18rs^2 + 36rs - 42s^2\right),$$
$$D_{21}^{(1)} = -\frac{h^3s^6}{2520r^2(r-s)^2(r-1)^2}\left(42r - 18s - 36rs + 9rs^2 + 18s^2 - 5s^3\right),$$
$$D_{22}^{(1)} = -\frac{h^3s^3}{1260(r-s)^2(s-1)^2}\left(12r^2s^2 - 42r^2s + 42r^2 - 15rs^3 + 48rs^2 - \right.$$
$$\left. 42rs + 5s^4 - 15s^3 + 12s^2\right),$$
$$D_{23}^{(1)} =\frac{h^3s^6}{2520(r-1)^2(s-1)^2}\left(18r^2s - 42r^2 - 18rs^2 + 36rs + 5s^3 - 9s^2\right),$$
$$D_{31}^{(1)} = -\frac{h^3}{2520r^2(r-s)^2(r-1)^2}\left(9r + 18s - 36rs + 42rs^2 - 18s^2 - 5\right),$$
$$D_{32}^{(1)} = -\frac{h^3}{2520s^2(r-s)^2(s-1)^2}\left(18r + 9s - 36rs + 42r^2s - 18r^2 - 5\right),$$
$$D_{33}^{(1)} = -\frac{h^3}{1260(r-1)^2(s-1)^2}\left(42r^2s^2 - 42r^2s + 12r^2 - 42rs^2 + 48rs - 15r \right.$$
$$\left. +12s^2 - 15s + 5\right),$$
$$D_{41}^{(1)} = -\frac{h^2r^2}{840(r-s)^2(r-1)^2}\left(15r^4 - 40r^3s - 40r^3 + 28r^2s^2 + 112r^2s + 28r^2 \right.$$
$$\left. -84rs^2 - 84rs + 70s^2\right),$$
$$D_{42}^{(1)} =\frac{h^2r^5}{840s^2(r-s)^2(s-1)^2}\left(14r - 28s + 28rs - 8r^2s - 16r^2 + 5r^3\right),$$
$$D_{43}^{(1)} =\frac{h^2r^5}{840(r-1)^2(s-1)^2}\left(5r^3 - 16r^2s - 8r^2 + 14rs^2 + 28rs - 28s^2\right),$$

$$D_{51}^{(1)} = -\frac{h^2 s^5}{840 r^2 (r-s)^2 (r-1)^2}\left(28r - 14s - 28rs + 8rs^2 + 16s^2 - 5s^3\right)$$

$$D_{52}^{(1)} = -\frac{h^2 s^2}{840 (r-s)^2 (s-1)^2}\left(28r^2 s^2 - 84r^2 s + 70r^2 - 40rs^3 + 112rs^2 - \right.$$
$$\left. 84rs + 15s^4 - 40s^3 + 28s^2\right),$$

$$D_{53}^{(1)} = \frac{h^2 s^5}{840 (r-1)^2 (s-1)^2}\left(14r^2 s - 28r^2 - 16rs^2 + 28rs + 5s^3 - 8s^2\right),$$

$$D_{61}^{(1)} = -\frac{h^2}{840 r^2 (r-s)^2 (r-1)^2}\left(8r + 16s - 28rs + 28rs^2 - 14s^2 - 5\right),$$

$$D_{62}^{(1)} = -\frac{h^2}{840 s^2 (r-s)^2 (s-1)^2}\left(16r + 8s - 28rs + 28r^2 s - 14r^2 - 5\right),$$

$$D_{63}^{(1)} = -\frac{h^2}{840 (r-1)^2 (s-1)^2}\left(70r^2 s^2 - 84r^2 s + 28r^2 - 84rs^2 + 112rs - \right.$$
$$\left. 40r + 28s^2 - 40s + 15\right).$$

Appindex III

$$E_1 = \Big(\textstyle\sum_{i=0}^{\infty}(\frac{h^i r^i}{i!})y_n^{(i)} - y_n - hry'_n - \frac{1}{2520s^3}(h^2 y_n^{(2)} r^2 (882s^3 + 10r^5(1+s)$$
$$-168rs^2(1+s) + 36r^3(1+s)^3 - 6r^2 s(15 + 4s + 15s^2) - r^4(36 +$$
$$53s + 36s^2))) - \textstyle\sum_{i=0}^{\infty} \frac{h^{i+2} y_n^{(i+2)}}{i!}[\frac{1}{2520(-1+r)^3(r-s)^3)}(r)^i (r^2(105r^6 + 378$$
$$s^3 - 385r^5(1+s) - 966rs^2(1+s) + 6r^2 s(120 + 403s + 120s^2) + r^4$$
$$(468 + 1457s + 468s^2) - 36r^3(5 + 51s + 51s^2 + 5s^3)) - \tfrac{1}{(2520(r-s)^3(-1+s)^3 s^3))}$$
$$(r^6(10r^4(-1+2s) + 42s^2(-5+7s) + r^3(36 - 25s - 75s^2) - 6rs(-33$$
$$+23s + 42s^2) + 9r^2(-4 - 12s + 27s^2 + 7s^3))(s)^i - \tfrac{1}{(2520(-1+r)^3(-1+s)^3)}$$
$$(r^6(10r^4(-2+s) + 42s^2(-7+5s) + r^3(75 + 25s - 36s^2) - 6rs(-42$$
$$-23s + 33s^2) + 9r^2(-7 - 27s + 12s^2 + 4s^3))] - \tfrac{1}{(2520s^2)}(h^3 y_n^{(3)} r^3 (5r^4 +$$
$$126s^2 - 18r^3(1+s) - 84rs(1+s) + 18r^2(1 + 4s + s^2))) + \textstyle\sum_{i=0}^{\infty} \frac{h^{i+3} y_n^{(i+3)}}{i!}$$
$$[\tfrac{1}{(1260(-1+r)^2(r-s)^2))} r^i ((r^3(5r^4 + 42s^2 - 15r^3(1+s) - 42rs(1+s) + 12r^2$$
$$(1 + 4s + s^2))) - \tfrac{1}{(2520(r-s)^2(-1+s)^2 s^2)} s^i ((r^6(5r^3 - 42s - 9r^2(2+s) + 18r$$
$$(1+2s)))) - \tfrac{1}{(2520(-1+r)^2(-1+s)^2)}((r^6(5r^3 - 42s^2 + 18rs(2+s) - 9r^2(1+2s)))\Big),$$

$$E_2 = \Big[\sum_{i=0}^{\infty}(\tfrac{h^i s^i}{i!})y_n^{(i)} - y_n - hsy_n' - \tfrac{1}{(2520r^3)}(h^2 y_n^{(2)} s^2(2s^3(18-18s+5s^2) - 12r^2 s$$
$$(14+2s-9s^2+3s^3) + 6r^3(147-28s-15s^2+6s^3) + rs^2(-90+108s-$$
$$53s^2+10s^3))) - \sum_{i=0}^{\infty}\tfrac{h^{i+2}y_n^{(i+2)}}{i!}[\tfrac{1}{(2520(-1+r)^3 r^3(r-s)^3)}(r)^i((s^6(21r^3(14-12s$$
$$+3s^2) - 2s^2(18-18s+5s^2) + rs(198-108s-25s^2+20s^3) - 3r^2(70+46s$$
$$-81s^2+25s^3)))) + \tfrac{1}{(2520(r-s)^3(-1+s)^3)}(s)^i((s^2(s^3(180-468s+385s^2-105s^3)$$
$$+6r^3(-63+161s-120s^2+30s^3) - 6r^2 s(-161+403s-306s^2+78s^3) + rs^2$$
$$(-720+1836s-1457s^2+385s^3)))) - \tfrac{1}{(2520(-1+r)^3(-1+s)^3))}(((s^6(s^2(-63+75s$$
$$-20s^2) + 6r^3(35-33s+6s^2) - 6r^2(49-23s-18s^2+6s^3) + rs(252-243s$$
$$+25s^2+10s^3))))] - [\tfrac{1}{(2520r^2)}(h^3 y_n^{(3)}(s^3(6r^2(21-14s+3s^2) - 6rs(14-12s+3s^2)$$
$$+s^2(18-18s+5s^2))))] - \sum_{i=0}^{\infty}\tfrac{h^{i+3}y_n^{(i+3)}}{840i!}[\tfrac{1}{(2520(-1+r)^2 r^2(r-s)^2)}r^i((s^6(r(-42+36s$$
$$-9s^2) + s(18-18s+5s^2)))) - \tfrac{1}{(1260(r-s)^2(-1+s)^2))}s^i(((s^3(6r^2(7-7s+2s^2) - 3rs$$
$$(14-16s+5s^2) + s^2(12-15s+5s^2))))\tfrac{(s^6(-18r(-2+s)s+6r^2(-7+3s)+s^2(-9+5s)))}{(2520(-1+r)^2(-1+s)^2)}]\Big],$$

$$E_3 = \Big(\sum_{i=0}^{\infty}(\tfrac{h^i}{i!})y_n^{(i)} - y_n - hy_n' - \tfrac{1}{2520r^3 s^3}(h^2 y_n^{(2)}(2s(5-18s+18s^2) + r(10-53s$$
$$+108s^2-90s^3) - 12r^2(3-9s+2s^2+14s^3) + 6r^3(6-15s-28s^2+147s^3))$$
$$-\sum_{i=0}^{\infty}\tfrac{h^{i+2}y_n^{(i+2)}}{i!}[\tfrac{1}{(2520(-1+r)^3 r^3(r-s)^3)}r^i((21r^3(3-12s+14s^2) - 2s(5-18s$$
$$+18s^2) - 3r^2(25-81s+46s^2+70s^3) + r(20-25s-108s^2+198s^3)))+$$
$$\tfrac{1}{(2520(r-s)^3(-1+s)^3 s^3)}s^i((s(-20+75s-63s^2) + 6r^3(6-33s+35s^2) - 6r^2(6-18s$$
$$-23s^2+49s^3) + r(10+25s-243s^2+252s^3))) + \tfrac{1}{(2520(-1+r)^3(-1+s)^3)}((105$$
$$-385s+468s^2-180s^3+6r^3(-30+120s-161s^2+63s^3) - 6r^2(-78+306s$$
$$-403s^2+161s^3) + r(-385+1457s-1836s^2+720s^3)))] - \tfrac{1}{(2520r^2 s^2)}(h^3 y_n^{(3)}(5$$
$$-18s+18s^2-6r(3-12s+14s^2) + 6r^2(3-14s+21s^2))) - \sum_{i=0}^{\infty}\tfrac{h^{i+3}y_n^{(i+3)}}{i!}$$
$$[\tfrac{1}{(2520(-1+r)^2 r^2(r-s)^2)}r^i((5-18s+18s^2+r(-9+36s-42s^2)))+$$
$$\tfrac{1}{(2520(r-s)^2(-1+s)^2 s^2)}s^i((5-9s+18r(-1+2s) - 6r^2(-3+7s)))$$
$$\tfrac{1}{(1260(-1+r)^2(-1+s)^2)}((-5+15s-12s^2-6r^2(2-7s+7s^2) + 3r(5-16s+14s^2))))]\Big),$$

$$E_4 = \Big(\sum_{i=0}^{\infty}(\tfrac{h^i r^i}{i!})y_n^{(i+1)} - y_n' - \tfrac{1}{(420s^3)}(hy_n^{(2)}r(210s^3+5r^5(1+s) - 56rs^2(1+s)$$
$$-r^4(16+23s+16s^2) - 28r^2(s+s^3) + 2r^3(7+20s+20s^2+7s^3)))-$$
$$\sum_{i=0}^{\infty}\tfrac{h^{i+1}y_n^{(i+2)}}{i!}[\tfrac{1}{(420(-1+r)^3(r-s)^3)}r^i(r(105r^6+210s^3-350r^5(1+s) - 574rs^2$$

$$(1+s)+14r^2s(35+111s+35s^2)+r^4(388+1187s+388s^2)-2r^3(70+671s$$
$$+671s^2+70s^3)))+\frac{1}{(420(r-s)^3(-1+s)^3s^3))}s^i(-((r^5(5r^4(-1+2s)+14s^2(-5+7s)$$
$$+r^3(16-10s-35s^2)-14rs(-5+3s+7s^2)+2r^2(-7-23s+49s^2+14s^3))))$$
$$+\frac{1}{(420(-1+r)^3(-1+s)^3))}(-r^5(5r^4(-2+s)+14s^2(-7+5s)+r^3(35+10s-16s^2)$$
$$+14rs(7+3s-5s^2)+2r^2(-14-49s+23s^2+7s^3)))]-\frac{1}{(840s^2)}(h^2y_n^{(3)}(r^2(5r^4$$
$$+70s^2-16r^3(1+s)-56rs(1+s)+14r^2(1+4s+s^2))))+\sum_{i=0}^{\infty}\frac{h^{i+2}y_n^{(i+3)}}{i!}$$
$$[\frac{1}{(840(-1+r)^2(r-s)^2))}r^i(-((r^2(15r^4+70s^2-40r^3(1+s)-84rs(1+s)+28r^2$$
$$(1+4s+s^2))))+\frac{1}{(840(r-s)^2(-1+s)^2s^2)}s^i((r^5(5r^3-28s-8r^2(2+s)+14r(1+2s))))$$
$$+\frac{1}{(840(-1+r)^2(-1+s)^2)}((r^5(5r^3-28s^2+14rs(2+s)-8r^2(1+2s)))))]\Big),$$

$$E_5=\Big(\sum_{i=0}^{\infty}(\frac{h^is^i}{i!})y_n^{(i+1)}-y'_n-\frac{1}{(420r^3))}(hy_n^{(2)}s(s^3(14-16s+5s^2)+14r^3(15-4s-$$
$$2s^2+s^3)-8r^2s(7-5s^2+2s^3)+rs^2(-28+40s-23s^2+5s^3)))-\sum_{i=0}^{\infty}\frac{h^{i+1}y_n^{(i+2)}}{i!}$$
$$[\frac{1}{(420(-1+r)^3r^3(r-s)^3)}r^i((s^5(s^2(-14+16s-5s^2)+14r^3(7-7s+2s^2)-7r^2$$
$$(10+6s-14s^2+5s^3)+2rs(35-23s-5s^2+5s^3))))+\frac{1}{(420(r-s)^3(-1+s)^3))}$$
$$s^i(s(s^3(140-388s+350s^2-105s^3)+14r^3(-15+41s-35s^2+10s^3)-$$
$$2r^2s(-287+777s-671s^2+194s^3)+rs^2(-490+1342s-1187s^2+350$$
$$s^3)))+\frac{1}{(420(-1+r)^3(-1+s)^3))}(-s^5(s^2(-28+35s-10s^2)+14r^3(5-5s+s^2)$$
$$+r^2(-98+42s+46s^2-16s^3)+rs(98-98s+10s^2+5s^3)))]-\frac{1}{(840r^2)}$$
$$(h^2y_n^{(3)}(s^2(14r^2(5-4s+s^2)-8rs(7-7s+2s^2)+s^2(14-16s+5s^2))))$$
$$+\sum_{i=0}^{\infty}\frac{h^{i+2}y_n^{(i+3)}}{i!}[\frac{1}{(840(-1+r)^2r^2(r-s)^2)}r^i((s^5(-4r(7-7s+2s^2)+s(14-16s$$
$$+5s^2))))+\frac{1}{(840(r-s)^2(-1+s)^2))}s^i(-((s^2(14r^2(5-6s+2s^2)-4rs(21-28s+$$
$$10s^2)+s^2(28-40s+15s^2))))+\frac{(s^5(14r^2(-2+s)-4rs(-7+4s)+s^2(-8+5s)))}{(840(-1+r)^2(-1+s)^2)}]\Big),$$

$$E_6=\Big(\sum_{i=0}^{\infty}(\frac{h^i}{i!})y_n^{(i+1)}-y'_n-\frac{1}{(420r^3s^3))}(hy_n^{(2)}(s(5-16s+14s^2)+r(5-23s+40s^2$$
$$-28s^3)-8r^2(2-5s+7s^3)+14r^3(1-2s-4s^2+15s^3)))-\sum_{i=0}^{\infty}\frac{h^{i+1}y_n^{(i+2)}}{i!}$$
$$[\frac{1}{(420(-1+r)^3r^3(r-s)^3)}r^i((s(-5+16s-14s^2)+14r^3(2-7s+7s^2)-7r^2(5-$$
$$14s+6s^2+10s^3)+2r(5-5s-23s^2+35s^3)))+\frac{1}{(420(r-s)^3(-1+s)^3s^3)}s^i((s(-10$$
$$+35s-28s^2)+14r^3(1-5s+5s^2)+r^2(-16+46s+42s^2-98s^3)+r(5+10s$$
$$-98s^2+98s^3)))+\frac{1}{(420(-1+r)^3(-1+s)^3))}((105-350s+388s^2-140s^3+14r^3(-10$$
$$+35s-41s^2+15s^3)-2r^2(-194+671s-777s^2+287s^3)+r(-350+1187s$$

$$-1342s^2 + 490s^3)))] - \frac{1}{(840r^2s^2)}(h^{i+2}y_n^{(3)}(5 - 16s + 14s^2 + 14r^2(1 - 4s + 5s^2)$$
$$-8r(2 - 7s + 7s^2))) - \sum_{i=0}^{\infty} \frac{h^{i+2}y_n^{(i+3)}}{i!}[\frac{1}{(840(-1+r)^2r^2(r-s)^2)}r^i((5 - 16s + 14s^2 - 4r$$
$$(2 - 7s + 7s^2))) + \frac{1}{(840(r-s)^2(-1+s)^2s^2)}s^i((5 + r^2(14 - 28s) - 8s + 4r(-4 + 7s)))$$
$$+\frac{1}{(840(-1+r)^2(-1+s)^2)}((-15 + 40s - 28s^2 - 14r^2(2 - 6s + 5s^2) + 4r(10 - 28s+$$
$$21s^2)))\big) .$$