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A NEW PROOF OF THE NAGATA-SMIRNOV METRIZATION THEOREM

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Abstract: In this note, I present a new, elementary proof of the Nagata-Smirnov metrization theorem.

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The Nagata-Smirnov metrization theorem in topology characterizes the metrizability of a topological space (see [3],[5] and [6]). There exist several equivalent formulations of this theorem in the literature, but are rather complicated. In this note, based on Rudin's proof of Stone's result on paracompactness (see [4]), I offer a simple and elementary proof of this fundamental theorem. For the terminology used here, see, e.g., books [1] and [2].

Theorem. *A topological space (X, τ) is metrizable if and only if it is regular and has a σ-locally finite base.*

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Proof. Let (X, τ) be a metrizable space and let d be a metric for τ. Given $m \in \mathbb{N}$, let the open covering $\mathcal{C}_m = \{B_d(t, \frac{1}{m}) | t \in X\}$. By Well-Ordering Principle, there exists a well-ordered set I which serves as an index set for the elements of X, so we may write $\mathcal{C}_m = \mathcal{C}^i_m$. Fix an $m \in \mathbb{N}$. For $i \in I$, $n \in \mathbb{N}$ define, inductively over n

$$D_{m,n,i} = \{x \in X | B(x, \frac{3}{2^n}) \subset \mathcal{C}^i_m, x \notin \bigcup_{j<i} \mathcal{C}^j_m \cup \bigcup_{j \in I, k<n} V_{m,k,j}, \tag{1}$$

$$V_{m,n,i} = \bigcup_{m \in \mathbb{N}} (\bigcup_{x \in D_{m,n,i}} B(x, \frac{1}{2^n})). \tag{2}$$

Let

$$\mathcal{V}_\mu = \bigcup_{m=1}^{\mu} \bigcup_{n=1}^{\mu} \bigcup_{i \in I} V_{m,n,i}. \tag{3}$$

We prove that $(\mathcal{V}_\mu)_{\mu \in \mathbb{N}}$ is a σ-locally finite base. Fix $m^*, n^* \in \mathbb{N}$. We first show that the family $\{V_{m^*,n^*,i} | i \in I\}$ is locally finite. For $x \in X$ let $i^* = \min\{i \in I | x \in \mathcal{C}^i_{m^*}\}$ and choose $n \in \mathbb{N}$ such that $B(x, \frac{3}{2^n}) \subset \mathcal{C}^{i^*}_{m^*}$. Then, by (1), either $x \in V_{m^*,k,j}$ for some $j \in I$ and $k < n$ or we have $x \in V_{m^*,n,i^*}$. Therefore,

$$\{V_{m^*,n,i} | n \in \mathbb{N}, i \in I\} \tag{4}$$

covers X. Let $x \in X$ and define $\widehat{i} = \min\{i \in I | x \in \bigcup_{n \in \mathbb{N}} V_{m^*,n,i}\}$. Then, we can choose $n, k \in \mathbb{N}$ such that $B(x, \frac{1}{2^k}) \subset V_{m^*,n,\widehat{i}}$. We prove that $B(x, \frac{1}{2^{n+k}})$ intersects $V_{m^*,n^*,i}$ for at most one $i \in I$. We have two cases to consider: (α) $n^* \geq n + k$; (β) $n^* < n + k$. To prove case (α) suppose that $y \in D_{m^*,n^*,i}$. Since $n^* > n$, (1) implies that $y \notin V_{m^*,n,\widehat{i}}$. Therefore, $B(x, \frac{1}{2^k}) \subset V_{m^*,n,\widehat{i}}$ implies that $d(x, y) \geq \frac{1}{2^k}$. Now, $z \in B(x, \frac{1}{2^{n+k}}) \cap B(x, \frac{1}{2^{n^*}})$ would imply $d(x, y) \leq d(x, z) + d(z, y) < \frac{1}{2^{n+k}} + \frac{1}{2^{n^*}} \leq \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}$ which is a contradiction. Thus, $B(x, \frac{1}{2^{n+k}}) \cap B(y, \frac{1}{2^{n^*}}) = \emptyset$. It follows $B(x, \frac{1}{2^{n+k}}) \cap V_{m^*,n^*,i} = \emptyset$. To prove (β), let $y \in V_{m^*,n^*,i}$. Since $x \in V_{m^*,n,\widehat{i}}$ there are x', y' such that $x \in B(x', \frac{1}{2^{n^*}}) \subset V_{m^*,n,\widehat{i}}$, $y \in B(y', \frac{1}{2^n}) \subset V_{m^*,n^*,i}$, $B(x', \frac{3}{2^{n^*}}) \subset \mathcal{C}^{\widehat{i}}_{m^*}$ and $B(y', \frac{3}{2^n}) \subset \mathcal{C}^i_{m^*}$. Suppose that $\widehat{i} < i$. Then, by (1), $y' \notin \mathcal{C}^{\widehat{i}}_{m^*}$.

This implies $d(x',y') \geq \frac{3}{2^{n^*}}$ and with the triangle inequality we have $\frac{3}{2^{n^*}} \leq d(x',y') \leq d(x',x) + d(x,y) + d(y,y') < d(x,y) + \frac{1}{2^{n^*}} + \frac{1}{2^n}$. If $n^* > n$, we have nothing to prove because of (α). Suppose that $n > n^*$. Then, we have $d(x,y) > \frac{1}{2^{n^*}} > \frac{1}{2^{n+k}}$. Similarly, if $i < \widehat{i}$ we have that $d(x,y) = d(y,x) > \frac{1}{2^n} > \frac{1}{2^{n+k}}$. Therefore, $B(x, \frac{1}{2^{n+k}}) \cap V_{m^*,n^*,j} = \emptyset$ whenever $j \neq \widehat{i}$. It follows that $B(x, \frac{1}{2^{n+k}})$ intersects at most $V_{m^*,n^*,\widehat{i}}$. Therefore, for each $m, n \in \mathbb{N}$ the family $\{V_{m,n,i} | i \in I\}$ is locally finite. Since $\mathcal{V}_\mu$ is the finite union of locally finite families is locally finite. To prove that $(\mathcal{V}_\mu)_{\mu\in\mathbb{N}}$ is a σ-locally finite base it remains to show that $\bigcup_{\mu\in\mathbb{N}} \mathcal{V}_\mu$ is a base for τ. We prove that given $O \in \tau$ and $x \in O$, there is an element $V_{m,n,i}$ containing x and contained in O. Choose $m \in \mathbb{N}$ so that $\mathcal{C}^j_m = B(x, \frac{1}{m}) \subset O$. By (4) for $m^* = 5m$ we have that $\{V_{5m,n,i} | m, n \in \mathbb{N}, i \in I\}$ is a covering of X. Then, $x \in V_{5m,n,i} \subseteq \mathcal{C}^i_{5m} = B(t, \frac{1}{5m})$ for some $n \in \mathbb{N}, i \in I$ and $t \in X$. Suppose that $\gamma \in V_{5m,n,i}$. Then, since $V_{5m,n,i}$ has diameter at most $\frac{2}{5m}$ we have $d(x,\gamma) \leq d(x,t) + d(t,\gamma) < \frac{2}{5m} + \frac{2}{5m} = \frac{4}{5m} < \frac{1}{m}$. It follows that $x \in V_{5m,n,i} \subset B(x, \frac{1}{m}) \subset O$.

To prove the converse, suppose that (X,τ) admits a σ-locally finite base $\widetilde{\mathcal{V}} = \{\mathcal{V}_n | n \in \mathbb{N}\}$. We can assume that each $\mathcal{V}_n$ contains X and the empty set. Fix $n \in \mathbb{N}$. For each $x \in X$, let $M^n_x = \bigcap\{A | x \in A, A \in \mathcal{V}_n\}$, $N^n_x = \bigcap\{X \setminus cl_\tau A | x \in X \setminus cl_\tau A \text{ and } A \in \mathcal{V}_n\}$ and $U_n = \{M^n_x \cap N^n_x \times M^n_x \cap N^n_x | x \in X\}$ ($cl_\tau A$ denotes the closure of A with respect to τ). The family $(U_n)_{n\in\mathbb{N}}$ is nested since for all $m, n \in \mathbb{N}$, $\mathcal{V}_m \subseteq \mathcal{V}_n$ implies $U_m \supseteq U_n$. Since $\widetilde{\mathcal{V}}$ is a σ-locally finite base, M^n_x, N^n_x are open and for each $x \in X$, $U_n(x)$ is a neighborhood system at x which generates τ. Let $x \in X$ and $n \in \mathbb{N}$. Since (X,τ) is regular and $(U_n)_{n\in\mathbb{N}}$ is nested, we may choose $m \in \mathbb{N}$ ($m > n$) and $B \in \mathcal{V}_m$ so that $x \in U_m(x) \subseteq B \subseteq cl_\tau B \subseteq U_n(x)$. Then, we have $U^2_m(x) = U_m(U_m(x)) = \bigcup_{y\in X} \{M^m_y \cap N^m_y | M^m_y \cap N^m_y \cap U_m(x) \neq \emptyset\}$. Let $y \in U_n(x)$. Then, $M^m_y \subseteq U_n(x)$. If $y \notin U_n(x)$, then $y \in X \setminus cl_\tau B$ and $N^m_y \cap B = \emptyset$. It follows that $N^m_y \cap U_m(x) = \emptyset$ which implies that $U^2_m(x) \subseteq U_n(x)$. Inductively we define a sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ of neighborhnets of X as follows: $\mathcal{W}_1 = U^2_1$ and for each $n \in \mathbb{N}$

$$\mathcal{W}_{n+1} = \bigcup\{U_m(x) \times U_m(x) | m \geq n+1, x \in X \text{ and } U^3_m(x) \times U^3_m(x) \subseteq \mathcal{W}_n\}.$$

It is easy to check that $\mathcal{W}_{n+1}^2 \subseteq \mathcal{W}_n$. It follows that $(\mathcal{W}_n)_{n\in\mathbb{N}}$ is a base for a uniformity which is compatible with τ ($\mathcal{W}_n \subseteq U_n^2$). We define the functions f and d on $X \times X$ by setting: $f(x,y) = inf\{\frac{1}{n}|n \in \mathbb{N}$ and $y \in \mathcal{W}_n^2(x)\}$ and $d(x,y) = \inf\{\sum_{i=0}^{n} f(x_i, x_{i+1})|n \in \mathbb{N}, x_i \in X$ for each i, $x_0 = x$ and $x_{n+1} = y\}$. Clearly, by definition, d is a metric on X. It remains to prove that $\tau_d = \tau$. We first prove that $d(x,y) \leq f(x,y) \leq 2^n d(x,y)$ for each $x, y \in X$. The first inequality is evident. To see the second one, we first show that

$$f(x,y) \leq 2(f(x,z) + f(z,y)) \text{ for all } x,z,y \in X. \tag{5}$$

The inequality clearly holds in case $\max(f(x,z), f(z,y)) = 1$. Suppose that $\max(f(x,z), f(z,y)) \leq \frac{1}{2}$. Then, there exists $n \in \mathbb{N}$ such that $\frac{1}{2^{n+1}} < \max(f(x,z),$ $f(z,y)) = \frac{1}{2^n}$. Without loss of generality, we assume that $d(x,z) = \frac{1}{2^\mu} \leq \frac{1}{2^n}$ and $d(z,y) = \frac{1}{2^n}$. It follows that $z \in \mathcal{W}_\mu^2(x) \subseteq \mathcal{W}_n^2(x)$ $(n < \mu)$ and $y \in \mathcal{W}_n^2(z)$. Hence, $y \in \mathcal{W}_n^4(x) \subseteq \mathcal{W}_{n-1}^2(x)$. Therefore, $f(x,y) \leq \frac{1}{2^{n-1}} = \frac{2}{2^n} = 2\max(f(x,z), f(z,y)) \leq 2(f(x,z) + f(z,y))$. We now prove that for all $x_0, x_1, ..., x_{n+1} \in X$ we have

$$f(x_0, x_{n+1}) \leq 2^n\{\sum_{i=0}^{n} f(x_i, x_{i+1})|n \in \mathbb{N}, x_i \in X\}. \tag{6}$$

For $n = 1$, the inequality reduces to $f(x_0, x_2) \leq 2f(x_0, x_1) + 2f(x_1, x_2)$ and this holds by (5). Assume that the result has been proved for $n = k$. Then for all $x_0, x_1, ..., x_{k+1} \in X$ we have

$$f(x_0, x_k) \leq 2^k\{\sum_{i=0}^{k} f(x_i, x_{i+1})|k \in \mathbb{N}, x_i \in X\}. \tag{7}$$

But then, by using (5) we have $f(x_0, x_{k+1}) \leq 2f(x_0, x_k) + 2f(x_k, x_{k+1}) \leq 2^{k+1}\{\sum_{i=0}^{n} f(x_i, x_{k+1})|k \in \mathbb{N}, x_i \in X\} + 2f(x_k, x_{k+1}) \leq 2^{k+1}\{\sum_{i=0}^{k+1} f(x_i, x_{k+1})|k \in \mathbb{N}, x_i \in X\}$. Therefore, (6) holds. So we have that $f(x,y) \leq 2^n d(x,y)$ as required. To prove that $\tau \subseteq \tau_d$, let $y \in B_d(x, \frac{1}{2^{2n}})$. Then we have that $f(x,y) \leq$

$2^n d(x,y) \le 2^n \frac{1}{2^{2n}}$, in other words, that $y \in \mathcal{W}_n^2(x) \subseteq \mathcal{W}_{n-1}(x)$. We have shown that $B_d(x, \frac{1}{2^{2n}}) \subset \mathcal{W}_{n-1}(x)$. To prove that $\tau_d \subseteq \tau$, let $y \in \mathcal{W}_n^2(x)$. Then we have $d(x,y) \le f(x,y) \le \frac{1}{2^n} < \frac{1}{2^{n-1}}$ It follows that $\mathcal{W}_n(x) \subseteq \mathcal{W}_n^2(x) \subset B_d(x, \frac{1}{2^{n-1}})$ which completes the proof.

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