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FULLY IDEMPOTENT Γ-SEMIRING

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Abstract: In this paper we prove that fifteen classes of Γ− semirings coincide with the class of Γ-semiring named in the title. Then we characterize each such Γ-semiring by the property that each ideal is intersection of those prime ideals which contain it.

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1. Introduction

Nowadays semiring is a well known algebraic structure. It was introduced by Vandiver[8] as a generalization of ring. In algebra there is an another generalization of ring which is known as Γ−ring. It was introduced by N. Nobusawa[4]. M. M K. Rao[5] introduced the notion of Γ-semiring, which was a generalization of semiring and also a generalization of Γ−ring and hence a generalization of ring. Suppose (S,+) and (Γ,+) are two additive commutative monoids. If there exits a mapping S×Γ×S → S, (with (m,γ, n)→ mγn ∈ S), satisfying the following conditions: for all m, n, p ∈ S and for all γ, $\mu \in \Gamma$, m$\gamma(n+p) = m\gamma$n+mγp; $(m + n)\gamma p = m\gamma p + n\gamma$p; $m(\gamma+\mu)n = m\gamma$n+mμn; m$\gamma(n\mu p) = (m\gamma n)\mu p$; m$\gamma$0=0$\gamma$m=0, 0 is the zero element of S and mθn=0, θ is the zero element of Γ then S is called a Γ-semiring. If A and B are subsets of a Γ-semiring S and $\Delta \subseteq \Gamma$, we denote by $A\Delta B$, the subset of S consisting of all finite sums of the form $\sum_i a_i\alpha_i b_i$ where $a_i \in A, b_i \in B$ and $\alpha_i \in \Delta$. An additive submonoid I of

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a Γ-semiring S is called a left(right) ideal of S if SΓI⊆I (IΓS⊆I). If I is both a left ideal and right ideal then I is called ideal of S. A proper ideal P of S is said to be prime if for any two ideals H and K of S, HΓK⊆P implies either H⊆P or K⊆P. P is said to be semiprime if for any ideal A of S, AΓA⊆P implies A⊆P. An ideal I of a Γ-semiring S is called a k-ideal if $s_1+s_2 \in$I, $s_1 \in$S, $s_2 \in$I imply $s_1 \in$I. A proper ideal I of a Γ-semiring is said to be irreducible if for ideals H and K of S, $I = H \cap K$ implies that $I = H$ or $I = K$. A proper ideal I of a Γ-semiring is said to be strongly irreducible if for ideals H and K of S, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$. Clearly a strongly irreducible ideal is irreducible and a prime ideal is strongly irreducible. For a proper ideal A of a Γ-semiring S the Γ− congruence on S, denoted by ρ_A, defined as $s\rho_A s'$ if and only if $s + a_1 = s' + a_2$ for some $a_1, a_2 \in$A, is called the Bourne Γ− congruence[1] on S defined by the ideal A. We denote the Bourne Γ−congruence(ρ_A) class of an element r of S by r/ρ_A or simply r/A and denote the set of all such Γ−congruence classes of the elements of the Γ-semiring S by S/ρ_A or simply S/A. It should be noted here that for any $s \in S$ and for any ideal of S, s/A is not necessarily equal to $s + A = \{s + a : a \in A\}$ but surely contains it. For any k-ideal A of Γ-semiring S if the Bourne Γ−congruence ρ_A defined by A, is proper i.e., $0/A \neq S$ then we define on S/A the following operations: $s/A+t/A = (s+t)/A$ and $s/A\alpha t/A = (s\alpha t)/A$ for all $\alpha \in \Gamma$.

The operator semirings of a Γ-semiring, defined by Dutta and Sardar [2], turned out to be a very effective tool in transferring the notion of semirings to Γ-semirings. The relation ρ on F, defined by $\sum_{i=1}^{m}(x_i, \alpha_i)\rho\sum_{j=1}^{n}(y_j, \beta_j)$ if and only if $\sum_{i=1}^{m} x_i\alpha_i a = \sum_{j=1}^{n} y_j\beta_j a$ for all $a \in$S (m, n$\in Z^+$) is a congruence on F. Congruence class containing $\sum_{i=1}^{m}(x_i, \alpha_i)$ is denoted by $\sum_{i=1}^{m}[x_i, \alpha_i]$. Then F/$\rho$ is an additive commutative semigroup. Now F/ρ forms a semiring with the multiplication defined by

$$(\sum_{i=1}^{m}[x_i, \alpha_i])\sum_{j=1}^{n}[y_j, \beta_j]) = \sum_{i,j}[x_i\alpha_i y_j, \beta_j].$$

This semiring is denoted by L and called the left operator semiring of the Γ-semiring S. Dually the right operator semiring R of the Γ-semiring S has been defined where R=$\{\sum_{i=1}^{m}[\alpha_i, x_i] : \alpha_i \in \Gamma, x_i \in$ S$, i = 1, 2, ...m; m \in$ Z$^+\}$ and the multiplication on R is defined as $(\sum_{i=1}^{m}[\alpha_i, x_i])(\sum_{j=1}^{n}[\beta_j, y_j]) = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$. If there exists an element $\sum_{i=1}^{m}[e_i, \delta_i]$ ∈L $(\sum_{j=1}^{n}[\gamma_j, x_j] \in R)$ such that $\sum_{i=1}^{m} e_i\delta_i$a=a $(\sum_{j=1}^{n} a\gamma_j x_j = a)$ for all a∈S then S is said to have the left unity $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively right unity $\sum_{j=1}^{n}[\gamma_j, x_j]$). We refer the readers [1], [2], [3], [7] for more preliminaries.

2. Fully Idempotent Γ-Semiring

An ideal I of a Γ-semiring S is said to be idempotent[3] if $I\Gamma I = I$. A Γ-semiring S is said to be fully idempotent if every ideal is idempotent. A Γ-semiring S is said to be regular[3] if $a \in a\Gamma S\Gamma a$ for all $a \in S$ where $a\Gamma S\Gamma a$ denotes the finite sum of the elements of the form $a\gamma s\gamma a$, $s \in S$.

Proposition 1. *If a Γ-semiring S is regular then it is fully idempotent.*

Proof. Let I be an ideal of the regular Γ-semiring S. If $I = \{0\}$ or S then $I\Gamma I = I$. Let I be a nontrivial ideal of S. Then $I\Gamma I \subseteq I\Gamma S \subseteq I$. Now let $a \in I$. Then $a \in a\Gamma S\Gamma a \subseteq I\Gamma S\Gamma a \subseteq I\Gamma a \subseteq I\Gamma I$. So $I\Gamma I = I$. Hence S is fully idempotent. □

A Γ-semiring S is said to be 0-simple[2] if $S\Gamma S \neq \{0\}$ and S has no ideals other than $\{0\}$ and S.

Proposition 2. *If a 0-simple Γ-semiring S is fully idempotent.*

Proof. Let S be a 0-simple Γ-semiring. Then it has only two ideals $\{0\}$ and S. Now $\{0\}\Gamma\{0\}$ and $S\Gamma S = S$. Hence S is fully idempotent. □

Proposition 3. *[3] A commutative fully idempotent Γ-semiring S is regular.*

Note: Proposition 1 and Proposition 3 can be combined as follows: "A commutative Γ-semiring S is regular if and only if S is fully idempotent."

Notation: Let U and V be two subsets of a Γ-semiring S. Then we put $(U : V)$=$\{s \in S : v\alpha s \in U$ for all $v \in V$ and $\alpha \in \Gamma\}$ and $(U : V)'$=$\{s \in S : s\alpha v \in U$ for all $v \in V$ and $\alpha \in \Gamma\}$.

Proposition 4. *If U and V be two subsets of a* Γ*-semiring S and* $(U : V)$ *and* $(U : V)'$ *are nonempty then they are ideals of S.*

A proper ideal I of a Γ-semiring S is said to be nilpotent if $(I\Gamma)^{n-1}I = \{0\}$ for some positive integer n. A Γ-semiring S is said to be semiprime if $\{0\}$ is a semiprime ideal.

Proposition 5. *A* Γ*-semiring S is semiprime if and only if no nonzero ideal is nilpotent.*

Proof. Let S be a semiprime Γ-semiring. If possible let I be a nonzero nilpotent ideal of S. Then for some positive integer n, $(I\Gamma)^{n-1}I = \{0\}$. Since $\{0\}$ is a semiprime ideal ideal and $(I\Gamma)^m I$ is an ideal of S, this implies that I =$\{0\}$. This is a contradiction. Hence S can not have any no nonzero nilpotent ideal.

Conversely, let I be an ideal of S such that $I\Gamma I$=$\{0\}$. Since S has no nonzero nilpotent ideal, this implies that I =$\{0\}$. Thus $\{0\}$ is a semiprime ideal of S. Hence S is semiprime Γ-semiring. □

Proposition 6. *Every Bourne factor* Γ*-semiring* S/I *of a fully idempotent* Γ*-semiring S is a fully idempotent.*

Proof. Let I be a k-ideal of S. Let P/I be an ideal of S/I.Then P is an ideal of S. So $P\Gamma P = P$. Now $(P/I)\Gamma(P/I) = (P\Gamma P)/I = (P/I)$. Hence S/I is a fully idempotent Γ-semiring. □

Proposition 7. *A fully idempotent* Γ*-semiring S has no nonzero nilpotent ideal.*

Proof. If possible let I be a nonzero nilpotent ideal of S with $(I\Gamma)^{n-1}I = \{0\}$ for some positive integer n. Since S is flly idempotent, this implies that $I = \{0\}$. This is a contradiction. Hence S has no nonzero nilpotent ideal. □

3. Main Result

Theorem 8. *Let S be a* Γ*-semiring with unities. Then following assertions on S are equivalent:*

1. S is fully idempotent. 2. For each pair of ideals I, J *of S,* $I \cap J = I\Gamma J$*.*

3. For each pair of ideals I, J *of S,* $(I : J) \cap J = J \cap I$*.*

4. For each pair of ideals I, J *of S,* $(I : J)' \cap J = J \cap I$*.*

5. If $K \subseteq J$ *then* $(I : J) \cap K = I \cap K$ *where I, J, K are ideals of S.*

6. If $K \subseteq J$ then $(I:J)' \cap K = I \cap K$ where I, J, K are ideals of S.

7. For each right ideal A and ideal I of S, $A \cap I \subseteq I\Gamma A$.

8. For each left ideal B and ideal I of S, $I \cap B \subseteq B\Gamma I$.

9. For each right ideal A and ideal I of S, if $A \subseteq I$ then $A \subseteq I\Gamma A$.

10. For each left ideal B and ideal I of S, if $B \subseteq I$ then $B \subseteq B\Gamma I$.

11. For each right ideal A and ideal I of S $(A:I) \cap I \subseteq I \cap A$.

12. For each left ideal B and ideal I of S $(B:I)' \cap I \subseteq B \cap I$.

13. If $A \subseteq I$ then $(A:I) \cap I \subseteq A$ where A is a right ideal and I is an ideal of S.

14. If $B \subseteq I$ then $(B:I)' \cap I \subseteq A$ where B is a left ideal and I is an ideal of S.

15. Every Bourne factor Γ-semiring of S is a semiprime Γ-semiring.

Proof. (1)⇒(2) Clearly for any two ideals I, J of S $I\Gamma J \subseteq I \cap J$. Again by (1) $I \cap J = (I \cap J)\Gamma(I \cap J) \subseteq I\Gamma J$. Hence (2) follows.

(2)⇒(1) Putting $J = I$ we get the result.

(2)⇒(3) Let I and J be two ideals of S. Two cases may arise. Case(i) $(I:J) = \Phi$. Then for any $x \in S$ there exists $j \in J$ and $\alpha \in \Gamma$ such that $j\alpha x \notin I$. This implies that $I \cap J = \Phi$(otherwise $x \in I \cap J$ would imply $J\Gamma x \subseteq J\Gamma I \subseteq S\Gamma I \subseteq I$ whence $j\alpha x \in I$ for all $j \in J$ and $\alpha \in \Gamma$) and hence (3)follows. Case(ii) $(I:J) \neq \Phi$. Then $(I:J)$ is an ideal of S (cf. Proposition 4). So by (2), $J \cap (I:J) = J\Gamma(I:J) \subseteq J \cap I$ $[J\Gamma(I:J) \subseteq J\Gamma S \subseteq J$ and for $j \in J, \alpha \in \Gamma, x \in (I:J), j\alpha x \in I$. So $J\Gamma(I:J) \subseteq I]$. Conversely, let $x \in J \cap I$. Then $J\Gamma x \subseteq J\Gamma I \subseteq S\Gamma I \subseteq I$. So $x \in (I:J)$. Hence $x \in J \cap (I:J)$ whence $J \cap I \subseteq J \cap (I:J)$. Hence $(I:J) \cap J = J \cap I$.

(2)⇒(4) Similar as above.

(3)⇒(2) Let I, J be two ideals of S. Clearly $I\Gamma J \subseteq I \cap J$. Let $x \in I \cap J$. Since $I\Gamma J$ is an ideal of S, this implies that $i\alpha x \in I\Gamma J$ for all $i \in I, \alpha \in \Gamma$. So $x \in (I\Gamma J : J) \cap I$ (by assumption $x \in I$)$= I \cap (I\Gamma J)$ [by (3)]$= I\Gamma J$ (since $I\Gamma j \subseteq I\Gamma S \subseteq I$). Hence $I \cap J \subseteq I\Gamma J$ and so $I \cap J = I\Gamma J$.

(4)⇒(2) Similar as above.

(3)⇒(5) Let I, J, K be three ideals of S with $K \subseteq J$. Let $x \in (I:J) \cap K$ and $x \in K \subseteq J$. So $x \in (I:J) \cap J \Rightarrow x \in K \cap (I:J) \cap J$. Hence $(I:J) \cap K \subseteq K \cap (I:J) \cap J = K \cap J \cap I$[by (3)]$=K \cap J$. Conversely, let $x \in K \cap I$. Then $x \in K$ and $x \in I$. Since for $j \in J, \alpha \in \Gamma, j\alpha x \in J\Gamma x \subseteq J\Gamma I \subseteq S\Gamma I \subseteq I$. Therefore$x \in (I:J)$ and so $x \in (I:J) \cap K$. Hence $K \cap I \subseteq (I:J) \cap K$. Thus $(I:J) \cap K = K \cap I$.

(5)⇒(3) Putting $K = J$ we get the result.

Considering the symmetry of (3) and (4), (5) and(6) we have that the statements from (1) to (6) are equivalent.

(9)$\Rightarrow$(1) Let I be an ideal of S. Then by (9), $I \subseteq I\Gamma I$ (taking $A = I$). Hence $I\Gamma I = I$. Thus (1) follows.

Consequently, (9) implies all the staments from (1) to (6).

(3)$\Rightarrow$(7) Let A be a right ideal and I be an ideal of S. Let $x \in I \cap R$. Now $I\Gamma x \subseteq I\Gamma A$. This implies that $i\alpha x \in I\Gamma A$ for all $i \in I, \alpha \in \Gamma$ hence $x \in (I\Gamma A : I)$. Thus $I \cap A \subseteq (I\Gamma A : I) \cap I = I \cap I\Gamma A$ [since A is an right ideal of S, $I\Gamma A$ is an ideal of S and by using (3)]$= I\Gamma A$ [since $I\Gamma A \subseteq I$]. Hence (7) is proved.

(7)$\Rightarrow$(9) is obvious.

Considering the symmetry of (7) and (8), (9) and(10) we have that the statements from (1) to (10) are equivalent.

(13)$\Rightarrow$(9) Let I be an ideal and A be a right ideal of S with $A \subseteq I$. Let $x \in A$, then $x \in I$. Now for $i \in I, \alpha \in \Gamma, i\alpha x \in I\Gamma A$. This implies that $x \in (I\Gamma A : I)$. Hence $A = A \cap I \subseteq (I\Gamma A : I) \subseteq I\Gamma A$ [since A is a right ideal of S, $I\Gamma A$ is also a right ideal of S and by using (13)]. Thus (9) is proved.

(7) $\Rightarrow$ (11) Let I be an ideal and A be a right ideal of S. Then $(A : I)$ is a right ideal of S and so by (7),$(A : I) \cap I \subseteq I\Gamma(A : I)$. Let $i \in I, \alpha \in \Gamma, x \in (A : I)$. Then $i\alpha I\alpha x \in A$. Also $i\alpha x \in A$. Also $i\alpha x \in I\Gamma x \subseteq I\Gamma S \subseteq I$. Hence $I\Gamma(A : I) \subseteq I \cap A$ and so $(A : I) \cap I \subseteq I \cap A$. Thus (11) is proved. (11)$\Rightarrow$ (13) is obvious.

Considering the symmetries of (11) and (12), (13) and (14), the first fourteen statements have proved to be equivalent.

(1) $\Rightarrow$ (15) Let S be a fully idempotent Γ-semiring and S/P be a Bourne factor Γ-semiring of S. Then by proposition 6, S/P is a fully idempotent Γ-semiring. So by Proposition 7, S/P has no non-zero nilpotent ideal. hence by proposition 5, S/P is a semiprime Γ-semiring.

(15)$\Rightarrow$ (1) Let the Γ-semiring S be not fully idempotent. Then there exists an ideal P such that $P\Gamma P \neq P$. Then the Bourne factor Γ-semiring $S/(P\Gamma P)$ is not semiprime. This is a contradiction to (15). Hence S is fully idempotent. □

Remark: Latice of ideals of a Γ-semiring is not, in general, distributive, or even modular. This follows from the Theorem 6[2] But the following theorem shows that lattice of ideals of a fully idempotent Γ-semiring is a complete Brouwerian and hence a distributive lattice.

Here we recall that (i) A lattice Λ is called Brouwerian if for any $a, b \in \Lambda$, the set of all $x \in \Lambda$ satisfying $a \wedge x \leq b$ contains the greatest element c. (ii) In a Brouwerian lattice meet is distributive over arbitrary join.

Theorem 9. *Let S be a fully idempotent Γ-semiring unities. Then the lattice ideal Λ_S of S is a complete Brouwerian lattice.*

Proof. Clearly Λ_S is complete lattice under the sum and intersection of ideals. Let B and C be two ideals of S. Then by Zorn's lemma, there is an ideal M of S which is maximal in the family of ideals X satisfying $B \cap X \subseteq C$. Let I be an ideal of S with $B \cap I \subseteq C$. Since S ius fully idempotent, by Theorem 8, $B\Gamma I \subseteq C$. Let $b \in B, \alpha \in \gamma, i + m \in I + M$. Then $b\alpha(i + m) = b\alpha i + b\alpha m$. Since $B\Gamma I \subseteq C$ and $B\Gamma M \subseteq M$, $b\alpha i + b\alpha m \in C$. Hence $B\Gamma(I + M) \subseteq C$. So by Theorem 8, $B \cap (I + M) \subseteq C$. By the maximality of M, $I + M = M$ and therefore $I \subseteq M$. Hence Λ_S is a Brouwerian lattice □

Corollary 10. *Lattice ideal Λ_S of fully idempotent Γ-semiring S satisfies arbitrary meet distributive law, $P \cap (\sum_\alpha P_\alpha) = \sum_\alpha (P \cap P_\alpha)$*

We know that in a Γ-semiring a prime ideal is strongly irreducible ideal and a strongly irreducible ideal is irreducible. But for a fully idempotent Γ-semiring all the concepts coincide.

Theorem 11. *Let S be a fully idempotent Γ-semiring with unities and P be an ideal of S. Then the following assertion are equivalent:*
(1) P is irreducible.
(2) P is prime.

Proof. Suppose P is irreducible and $I\Gamma J \subseteq P$ where I, J are ideals of S. Since S is fully idempotent, by Theorem 8, $I \cap J \subseteq P$. This implies that $(I \cap J) + P = P$. Hence by the above corollary $(I + P) \cap (J + P) = P$. Then by (1), $I + P = P$ or $J + P = P$. This implies that $I \subseteq P$ or $J \subseteq P$. Hence P is prime.

(2)⇒ is obvious. □

Lemma 12. *Let S be a Γ-semiring with unities and $a(\neq 0) \in S$. Let P be a nonzero proper ideal of S such that $a \notin P$. Then there exists a proper ideal Q of S, maximal in the class of all ideals each of which contains P and does not contain a. Moreover, Q is irreducible.*

Proof. By Zorn's lemma, Q exists. If possible, suppose that Q is not irreducible. Then there exists ideals H, K of S such that $Q = H \cap K$ and both H and K properly contain Q and contain P. Hence by maximality of Q, H, K both contain a. Then $a \in H \cap K$. This implies that $a \in Q$. This is a contradiction. Then $Q = H$ or $Q = K$. Hence Q is irreducible. □

Now we have the following theorem characterizinhg fully idempotent Γ-semiring.

Theorem 13. *Let S be a Γ-semiring with unities and P be an ideal of S. Then the following assertion are equivalent:*

(1) S is fully idempotent.

(2) Each proper ideal of S is the intersection of all prime ideals containing it.

Proof. Let S be fully idempotent. Let P bea proper ideal of S and $\{P_\alpha : \alpha \in \Delta\}$ be a family of prime ideals of S each of which contains P. Clearly $P \subseteq \underset{\alpha\in\Delta}{\cap} P_\alpha \subseteq P$. Suppose $a \notin P$. Then by the Lemma 12, there exists an irreducible ideal Q of S containing P and $a \notin Q$. By Proposition 11, Q is prime ideal of S. Hence Q is a member of the family $\{P_\alpha : \alpha \in \Delta\}$. This implies that $a \notin \underset{\alpha\in\Delta}{\cap} P_\alpha$. Thus (2) is proved.

Conversely, let P be any ideal of S. If $P\Gamma P = P$ the $P = S$ and so P is idempotent. If $P\Gamma P \neq S$ then $P\Gamma P$ is a proper ideal of S. Then it is the intersection of all prime ideals $\{P_\alpha : \alpha \in \Delta\}$ of S where $P \subseteq P_\alpha$ for all $\alpha \in \Delta$. Hence $P\Gamma P = \underset{\alpha\in\Delta}{\cap} P_\alpha \subseteq P_\alpha$ for all $\alpha \in \Delta$. Since P_α is aprime ideal of S for each $\alpha \in \Delta$. This implies that $P \subseteq P_\alpha$ for all $\alpha \in \Delta$. Thus $P \subseteq \underset{\alpha\in\Delta}{\cap} P_\alpha = P\Gamma P$ and so P is idempotent. Hence S is fully idempotent. □

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