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ENTIRE DIRICHLET SERIES WITH MONOTONOUS COEFFICIENTS AND LOGARITHMIC h-MEASURE

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Abstract: Let F be an entire function represented by absolutely convergent for all $z \in \mathbb{C}$ Dirichlet series of the form $F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}$, where a sequence (λ_n) such that $\lambda_n \in \mathbb{R}$ and $(\forall n \geq 0):\ 0 \leq \lambda_n < \beta := \sup\{\lambda_j :\ j \geq 0\} \leq +\infty$. In this paper we find the condition such that the relation $F(x+iy) = (1+o(1))a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}$ holds as $x \to +\infty$ outside some set E of finite logarithmic h-measure (i.e. h-log-meas$(E) := \int_{E\cap[1,+\infty)} h(x)d\ln x < +\infty$) uniformly in $y \in \mathbb{R}$, where h is non-decrease positive continuous function on $[0, +\infty)$.

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1. Introduction

Let $\mathcal{L}$ be the class of positive continuous increasing functions on $[0; +\infty)$ and $\mathcal{L}_+$ the subclass of functions $\Phi \in \mathcal{L}$ such that $\Phi(t) \to +\infty$ $(t \to +\infty)$. By φ we denote inverse function to $\Phi \in \mathcal{L}$.

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Let $\mathcal{D}$ be a class of entire (absolutely convergent in the whole complex plane $\mathbb{C}$) Dirichlet series of the form

$$F(z) = \sum\nolimits_{n=0}^{+\infty} a_n e^{z\lambda_n}, \quad z \in \mathbb{C}, \tag{1}$$

where a sequence (λ_n) such that $\lambda_n \in \mathbb{R}$ $(n \geq 0)$, $\lambda_n \neq \lambda_k$ for any $n \neq k$ and

$$(\forall n \geq 0): \quad 0 \leq \lambda_n < \beta := \sup\{\lambda_j : \ j \geq 0\} \leq +\infty. \tag{2}$$

For $F \in \mathcal{D}$ and $x \in \mathbb{R}$ we denote $M(x, F) = \sup\{|F(x+iy)|\colon y \in \mathbb{R}\}$, $m(x, F) = \inf\{|F(x + iy)|\colon y \in \mathbb{R}\}$, and by $\mu(x, F) = \max\{|a_n|e^{x\lambda_n}\colon n \geq 0\}$, $\nu(x, F) = \max\{n\colon |a_n|e^{x\lambda_n} = \mu(x, F)\}$ the maximal term and central index of series (1), respectively.

By $\mathcal{D}_a$ we denote a subclass of Dirichlet series $F \in \mathcal{D}$ with a fixed sequence $a = (|a_n|)$, $|a_n| \searrow 0$ $(n_0 \leq n \to +\infty)$, and for $\Phi \in \mathcal{L}$ by $\mathcal{D}_a(\Phi)$ subclass of functions $F \in \mathcal{D}_a$ such that $\ln \mu(x, F) \geq x\Phi(x)$ $(x \geq x_0)$. Let $\mu_n := -\ln |a_n|$ $(n \geq 0)$.

In paper [1] one can find such theorem:

Theorem A (O.B. Skaskiv, 1994 [1]). *For every entire function $F \in \mathcal{D}_a$ relation*

$$F(x + iy) = (1 + o(1))a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}} \tag{3}$$

holds as $x \to +\infty$ outside some set E of finite logarithmic measure, i.e.

$$\text{log-meas}(E) := \int_E d\ln x < +\infty,$$

uniformly in $y \in \mathbb{R}$, if and only if

$$\sum\nolimits_{n=n_0}^{+\infty} \frac{1}{\mu_{n+1} - \mu_n} < +\infty. \tag{4}$$

It is easy to see that relation (3) holds for $x \to +\infty$ $(x \notin E)$ uniformly in $y \in \mathbb{R}$, if and only if $M(x, F) \sim \mu(x, F)$ $(x \to +\infty,\ x \notin E)$, hence it follows $M(x, F) \sim m(x, F)$ $(x \to +\infty,\ x \notin E)$.

The finiteness of logarithmic measure of an exceptional set E in Theorem A is the sharp estimate. It follows from such theorem.

Theorem B (Ya.Z. Stasyuk, 2008 [2]). *For every increasing sequence (μ_n), such that condition (4) satisfies and for any function $h \in \mathcal{L}_+$ there exist an entire Dirichlet series $F \in \mathcal{D}_a$ with $|a_n| = \exp\{-\mu_n\}$, a set E and a constant $d > 0$ such that* h-log-meas$(E) := \int_{E\cap[1,+\infty)} h(x)d\ln x = +\infty$ *and* $(\forall x \in E)\colon\ M(x, F) \geq (1 + d)\mu(x, F),\ M(x, F) \geq (1 + d)m(x, F)$.

Due to Theorem B the natural **question** arises: what conditions must satisfy the entire Dirihlet series $F \in \mathcal{D}_a$ in order to relation (3) holds for $x \to +\infty$ outside some set E of finite logarithmic h-measure, i.e. h-log-meas$(E) < +\infty$? In this article we give the answer to the question. Our main result is the following.

Theorem 1. *Let* (μ_n) *be a sequence such that condition* (4) *holds,* $h \in \mathcal{L}_+, \Phi \in \mathcal{L}$ *and* $F \in \mathcal{D}_a(\Phi)$. *If*

$$(\forall\, b > 0)\colon \ \sum_{n=n_0}^{+\infty} h\Big(\varphi(\lambda_n)\cdot\Big(1+\frac{b}{\mu_{n+1}-\mu_n}\Big)\Big)\frac{1}{\mu_{n+1}-\mu_n} < +\infty, \tag{5}$$

then the relation (3) *holds as* $x \to +\infty$ *outside some set* E *of finite logarithmic h-measure uniformly in* $y \in \mathbb{R}$.

The method of proof of Theorem 1 differs from the method of proofs corresponding statements in [1, 3, 4, 5, 6] and is close to the methods of proofs from papers [7, 8].

2. Proof of Theorem 1

We denote $\Delta_0 = 0$ and for $n \geq 1$

$$\Delta_n := \delta\cdot\sum_{j=0}^{n-1}(\mu_{j+1}-\mu_j)\sum_{m=j+1}^{+\infty}\Big(\frac{1}{\mu_m-\mu_{m-1}}+\frac{1}{\mu_{m+1}-\mu_m}\Big), \quad \delta > 0.$$

It is easy to see that

$$\Delta_n \geq n\delta \quad (n \geq 0), \qquad \Delta_n = o(\mu_n) \quad (n \to +\infty). \tag{6}$$

We put $b_n = e^{\lambda_n}$ and consider the Dirichlet series

$$f(s) = \sum\nolimits_{n=0}^{+\infty} b_n\, e^{s\mu_n}, \quad \mu_n = -\ln|a_n|.$$

The condition $\sum_{n=0}^{+\infty} 1/(\mu_{n+1}-\mu_n) < +\infty$ implies $n^2 = o(\mu_n)$ $(n \to +\infty)$ (see [1, 3]), thus $\ln n = o(\mu_n)$ $(n \to +\infty)$. $F \in \mathcal{D}_a$ so $\lim\limits_{n\to+\infty}\dfrac{-\ln|a_n|}{\lambda_n} = +\infty$. Therefore by Valiron's formula for the abscissa of absolute convergence ([9, p.11]) we get

$$\sigma_{\mathrm{abs}}(f) = \varliminf_{n\to+\infty}\frac{-\ln b_n}{\mu_n} = \lim_{n\to+\infty}\Big(\frac{\ln|a_n|}{\lambda_n}\Big)^{-1} = 0.$$

Now we consider Dirichlet series

$$f_q(s)=\sum\nolimits_{n=0}^{+\infty}\frac{b_n}{\alpha_n^q}\ e^{s\mu_n},\quad \alpha_n=e^{\Delta_n},\ q\in\mathbb{R}.$$

From the second relation in (6) and the condition $\sigma_{\rm abs}(f)=0$ it follows that

$$\sigma_{\rm abs}(f_q)=\varliminf_{n\to+\infty}\frac{-\ln b_n+q\Delta_n}{\mu_n}=\sigma_{\rm abs}(f)+q\cdot\lim_{n\to+\infty}\frac{\Delta_n}{\mu_n}=0$$

for any $q\in\mathbb{R}$. Therefore the Dirichlet series of the form

$$f^*(s)=\sum\nolimits_{n=0}^{+\infty}b_ne^{s\mu_n^*},\quad \mu_n^*=\mu_n+\Delta_n,$$

is absolutely convergent in the whole half-plane $\Pi_0:=\{s\colon \text{Re } s<0\}$ and $\sigma_{\rm abs}(f^*)=0$. Indeed, for every fixed $s\in\Pi_0$ we have as $q=-\text{Re } s$

$$\Big|b_ne^{s\mu_n^*}\Big|=\Big|\frac{b_n}{\alpha_n^q}e^{s\mu_n}\Big|\quad(\forall\ n\geq 0).$$

But $\sigma_{\rm abs}(f_q)=0$, thus $\sigma_{\rm abs}(f^*)\geq 0$. In the other hand $b_ne^{s\mu_n^*}\Big|_{s=0}=e^{\lambda_n}\not\to 0$ $(n\to+\infty)$, hence $\sigma_{\rm abs}(f^*)=0$.

From condition (2) (see proof of Lemma 2 in [1, p.121–122]) we get

$$\nu(x,f^*)\to+\infty\quad(x\to-0).$$

We need the following lemma (compare, for example, [7, 8]).

Lemma 2.1. *For all $n\geq 0$ and $k\geq 1$ inequality*

$$\frac{\alpha_n}{\alpha_k}\ e^{\tau_k(\mu_n-\mu_k)}\leq e^{-\delta|n-k|}\tag{7}$$

holds, where $\tau_k:=t_k+\dfrac{\delta}{\mu_k-\mu_{k-1}},\ t_k:=\dfrac{\Delta_{k-1}-\Delta_k}{\mu_k-\mu_{k-1}}.$

Proof of Lemma 2.1. We remark that

$$t_k=-\delta\cdot\sum_{m=k}^{+\infty}\left(\frac{1}{\mu_m-\mu_{m-1}}+\frac{1}{\mu_{m+1}-\mu_m}\right),\tag{8}$$

$$\tau_k=-2\delta\cdot\sum_{m=k+1}^{+\infty}\frac{1}{\mu_m-\mu_{m-1}},\tag{9}$$

$$\tau_{k+1}-\tau_k=\frac{2\delta}{\mu_{k+1}-\mu_k}\quad(k\geq 1).\tag{10}$$

Since $\ln\alpha_n-\ln\alpha_{n-1}=\Delta_n-\Delta_{n-1}=-t_n(\mu_n-\mu_{n-1})$, for $n\geq k+1$ we have

$$\ln\frac{\alpha_n}{\alpha_k}+\tau_k(\mu_n-\mu_k)=-\sum_{j=k+1}^{n}t_j(\mu_j-\mu_{j-1})+\tau_k\sum_{j=k+1}^{n}(\mu_j-\mu_{j-1})=$$

$$= -\sum_{j=k+1}^{n} (t_j - \tau_k)\,(\mu_j - \mu_{j-1}) \le -\sum_{j=k+1}^{n} (t_j - \tau_{j-1})\,(\mu_j - \mu_{j-1}) =$$

$$= -\sum_{j=k+1}^{n} \delta = -(n-k)\cdot\delta.$$

Similarly, for $n \le k-1$ we obtain

$$\ln\frac{\alpha_n}{\alpha_k} + \tau_k(\mu_n - \mu_k) = -\ln\frac{\alpha_k}{\alpha_n} - \tau_k(\mu_k - \mu_n) =$$

$$= \sum_{j=n+1}^{k} t_j(\mu_j - \mu_{j-1}) - \tau_k \sum_{j=n+1}^{k} (\mu_j - \mu_{j-1}) \le$$

$$\le -\sum_{j=n+1}^{k} (\tau_j - t_j)\,(\mu_j - \mu_{j-1}) = -\sum_{j=n+1}^{k} \delta = -(k-n)\cdot\delta$$

and Lemma 2.1 is proved. □

We remark that from definitions of τ_k and t_k (see (9), (8)) and the condition $\sum_{k=0}^{+\infty} 1/(\mu_{k+1} - \mu_k) < +\infty$ it follows that there exists $k_0(\delta)$ such that

$$\tau_k \ge -1 \quad (k \ge k_0(\delta)), \qquad \tau_k < 0 \quad (k \ge 1).$$

Let J be a set of the values of the central index $\nu(\sigma, f^*)$, i.e.

$$J = \big\{k \in \mathbb{N}\colon (\exists\ \sigma < 0)[\nu(\sigma, f^*) = k]\big\}.$$

Denote by (R_k) a sequence of the points of the springs of $\nu(\sigma, f^*)$, enumerate such that $\nu(\sigma, f^*) = k$ for $\sigma \in [R_k, R_{k+1})$ in the case $R_k < R_{k+1}$. Then for $\sigma \in [R_k, R_{k+1})$, $k \in J$ from Lemma 2.1 we have

$$b_n e^{\sigma\mu_n^*} \le b_k e^{\sigma\mu_k^*} \iff \frac{b_n e^{\sigma\mu_n}}{b_k e^{\sigma\mu_k}} \le \frac{\alpha_n^{|\sigma|}}{\alpha_k^{|\sigma|}} \le e^{-|\sigma|\tau_k(\mu_n - \mu_k)} e^{-|\sigma||n-k|\cdot\delta}$$

for all $n \ge 0$. Hence, $\dfrac{b_n e^{(\sigma + |\sigma|\tau_k)\mu_n}}{b_k e^{(\sigma+|\sigma|\tau_k)\mu_k}} \le e^{-|\sigma||n-k|\cdot\delta}$ i.e., for all $k \in J$ and for every $\sigma^* \in \big[R_k(1+|\tau_k|), R_{k+1}(1+|\tau_k|)\big)$

$$\frac{b_n e^{\sigma^*\mu_n}}{b_k e^{\sigma^*\mu_k}} \le \exp\Big\{-\frac{|\sigma^*||n-k|\cdot\delta}{1+|\tau_k|}\Big\}.$$

Thus, as $x = \dfrac{1}{|\sigma^*|} \in \Big[-\dfrac{1}{R_k(1+|\tau_k|)}, -\dfrac{1}{R_{k+1}(1+|\tau_k|)}\Big)$ we get

$$\frac{|a_n|e^{x\lambda_n}}{|a_k|e^{x\lambda_k}} \le \exp\Big\{ -\frac{|n-k|\cdot\delta}{1+|\tau_k|}\Big\}$$

for all $k \in J$, $n \ge 0$. Therefore,

$$\nu(x,F) = k,\ \ \mu(x,F) = |a_k|e^{x\lambda_k},\ \ x \in \Big[-\frac{1}{R_k(1+|\tau_k|)}, -\frac{1}{R_{k+1}(1+|\tau_k|)}\Big). \tag{11}$$

Denote $E^*(\delta) := \bigcup\limits_{k=k_0(\delta)}^{+\infty} \Big[-\dfrac{1}{R_k(1+|\tau_k|)}, -\dfrac{1}{R_{k+1}(1+|\tau_k|)}\Big)$, $E(\delta) := [0,+\infty)\setminus E^*(\delta)$. Then for every $x > 0$, $x \notin E(\delta)$

$$\begin{aligned} &|F(x+iy) - a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}| \le \\ &\le \mu(x,F)\cdot \sum\nolimits_{n\ne\nu(x,F)} \exp\Big\{ -\frac{\delta\cdot|n-\nu(x,F)|}{1+|\tau_{\nu(x,F)}|}\Big\} \le \\ &\le \frac{2e^{-\delta/2}}{1-e^{-\delta/2}}\cdot\mu(x,F) \end{aligned} \tag{12}$$

because $1 + |\tau_{\nu(x,F)}| < 2$ $(x \in E^*(\delta))$. It remains to prove that the logarithmic $h-$measure of a set $E(\delta)$ is finite. Using

$$E(\delta) \subset [0,x_0) \bigcup \Big(\bigcup_{k=k_0(\delta)+1}^{+\infty} \Big[-\frac{1}{R_k(1+|\tau_{k-1}|)}, -\frac{1}{R_k(1+|\tau_k|)}\Big)\Big),$$

$$x_0 = -\frac{1}{R_{k_0(\delta)}(1+|\tau_{k_0(\delta)-1}|)},$$

and equality (10), we obtain

$$\text{h-log-meas}(E \cap [x_0,+\infty)) =$$

$$\begin{aligned} &= \sum_{k=k_0(\delta)+1}^{+\infty} \int_{\Big[-\frac{1}{R_k(1+|\tau_{k-1}|)}, -\frac{1}{R_k(1+|\tau_k|)}\Big)} h(x)d\ln x \le \\ &= \sum_{k=k_0(\delta)+1}^{+\infty} h\left(-\frac{1}{R_k(1+|\tau_k|)}\right) \ln\left(1 + \frac{|\tau_{k-1}|-|\tau_k|}{1+|\tau_k|}\right) \le \\ &\le 2\delta\cdot \sum_{k=k_0(\delta)}^{+\infty} h\left(-\frac{1}{R_{k+1}(1+|\tau_{k+1}|)}\right)\frac{1}{\mu_{k+1}-\mu_k}. \end{aligned} \tag{13}$$

The condition $F \in \mathcal{D}_a(\Phi)$ implies

$$x\Phi(x) \le \ln \mu(x,F) = -\mu_{\nu(x-0,F)} + x\lambda_{\nu(x-0,F)} \le x\lambda_{\nu(x-0,F)}$$

$(x \ge x_1 > 0)$, therefore

$$x \le \varphi(\lambda_{\nu(x-0,F)}) \quad (x \ge x_1 > 0). \tag{14}$$

Denote $\theta_k := -\big(R_{k+1}(1+|\tau_k|)\big)^{-1}$. By (11) we have $\nu(\theta_k - 0) = k$, thus from (10) and (14) it follows

$$-\frac{1}{R_{k+1}(1+|\tau_{k+1}|)} = \theta_k \cdot \frac{1+|\tau_k|}{1+|\tau_{k+1}|} = \theta_k \cdot \Big(1 + \frac{|\tau_k| - |\tau_{k+1}|}{1+|\tau_{k+1}|}\Big) \le$$
$$\le \theta_k \cdot \Big(1 + \frac{2\delta}{\mu_{k+1} - \mu_k}\Big) \le \varphi(\lambda_k) \cdot \Big(1 + \frac{2\delta}{\mu_{k+1} - \mu_k}\Big) \tag{15}$$

for all $k \ge k_1(\delta)$. Using inequality (15) to inequality (13), we get

$$\text{h-log-meas}(E(\delta) \cap [x_0, +\infty)) \le$$
$$\le 2\delta \cdot \sum_{k=k_0(\delta)}^{k_2(\delta)-1} h\Big(-\frac{1}{R_{k+1}(1+|\tau_{k+1}|)}\Big)\frac{1}{\mu_{k+1}-\mu_k} +$$
$$+2\delta \cdot \sum_{k=k_2(\delta)}^{+\infty} h\Big(\varphi(\lambda_k) \cdot \Big(1 + \frac{2\delta}{\mu_{k+1} - \mu_k}\Big)\Big)\frac{1}{\mu_{k+1}-\mu_k} := K(\delta) < +\infty, \tag{16}$$

where $k_2(\delta) = \max\{k_0(\delta), k_1(\delta)\}$. Relation (16) implies that

$$(\forall\ \delta > 0): \quad \text{h-log-meas}(E(\delta) \cap [x, +\infty)) = o(1) \quad (x \to +\infty).$$

We put now $\delta_n = n, \varepsilon_n = 2^{-n}$ $(n \ge 1)$. Then for every $n \ge 1$ there exists $x_n \ge x_0$ such that h-log-meas$(E(\delta_n) \cap [x_n, +\infty)) \le \varepsilon_n$. Without loss of generality we may assume that $x_n < x_{n+1}$ $(n \ge 1)$. Denote $E = \bigcup_{n=1}^{+\infty} \big(E(\delta_n) \cap [x_n; x_{n+1})\big)$. Define a function $\gamma\colon [x_1, +\infty) \to [0, +\infty)$ by equality $\gamma(x) = 4/n$ for $x \in [x_n, x_{n+1})$. Then from inequality (12) it follows

$$|F(x+iy) - a_{\nu(x,F)} e^{(x+iy)\lambda_{\nu(x,F)}}| \le \gamma(x) \cdot \mu(x,F)$$

for all $x \in [x_1, +\infty) \setminus E$ uniformly in $y \in \mathbb{R}$ because $\gamma(x) = 4/n \ge \dfrac{2e^{-\delta_n/2}}{1 - e^{-\delta_n/2}}$ $(n \ge 1)$. But $\gamma(x) \to 0$ $(x \to +\infty)$ and

$$\text{h-log-meas}(E \cap [x_1, +\infty)) \le$$
$$\le \sum\nolimits_{n=1}^{+\infty} \text{h-log-meas}(E(\delta_n) \cap [x_n, x_{n+1})) \le \sum\nolimits_{n=1}^{+\infty} \varepsilon_n = 1.$$

Thus, h-log-meas$(E) < +\infty$.

3. Concluding Remarks

Since $\lambda_n/\mu_n \to 0$ $(n \to +\infty)$, we have $\lambda_n < \mu_n$ for all n large enough. So λ_n one can replace with μ_n in (5).

In the case $\beta = \sup\{\lambda_j : \ j \geq 0\} = +\infty$ condition (5) of Theorem 1 can be written in a simpler form. For example, such theorem from Theorem 1 follows. Let $\Phi \in \mathcal{L}_+$ and $\mathcal{D}_a^*(\Phi) := \bigcup_{\rho>0} \mathcal{D}_a(\Phi_\rho)$, $\Phi_\rho(x) := \rho \cdot \Phi(x\rho)$.

Theorem 2. *Let* (μ_n) *be a sequence such that condition* (4) *holds,* $h \in \mathcal{L}_+, \Phi \in \mathcal{L}_+$ *and* $F \in \mathcal{D}_a^*(\Phi)$. *If*

$$(\forall\ b > 0)\colon \qquad \sum\nolimits_{n=n_0}^{+\infty} \frac{h\left(b\varphi(b\lambda_n)\right)}{\mu_{n+1} - \mu_n} < +\infty, \tag{17}$$

then relation (3) *holds as* $x \to +\infty$ *outside some set* E *of finite logarithmic* h*-measure uniformly in* $y \in \mathbb{R}$.

In the case $\Phi(x) = e^x/x$ we obtain that $\mathcal{D}_a^*(\Phi)$ is the class Dirichlet series of nonzero lower R-order, i.e. $\varliminf_{x\to+\infty} \frac{1}{x} \ln\ln M(x, F) := \rho_R[F] \in (0, +\infty]$ and condition (17) from condition $(\forall\ b > 0)\colon\ \sum_{n=n_0}^{+\infty} h\left(b \ln \lambda_n\right)/(\mu_{n+1} - \mu_n) < +\infty$ follows.

Question 3.1. Is the description of exceptional sets in our Theorems 1 and 2 the best possible?

Question 3.2. Are conditions (5) and (17) in our Theorems 1 and 2 necessary?

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