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ON WEAKLY $w_\tau g$-CLOSED SETS IN ASSOCIATED w-SPACES

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Abstract: In this article, we introduce the notions of weakly $w_\tau g$-closed set and weakly $w_\tau g$-open set which are generalized notions of $w_\tau g$-closed set and $w_\tau g$-open set in associated w-spaces, and study some basic properties of such the notions. In particular, we found that every w-preopen closed set is weakly $w_\tau g$-closed.

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1. Introduction

Siwiec [19] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [8]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [2] and general topological spaces [1]. The notions of weak structure and w-space were investigated in [9]. And we introduced the notion of associated weak structures in [10]. In fact, the set of all g-closed subsets [3] in a topological space is a kind of associated weak structures.

The purpose of this study is to extend the notion of $w_\tau g$-open sets ($w_\tau g$-closed sets) in associated w-spaces. So, we introduce the new notions of weakly

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$w_\tau g$-closed sets and weakly $w_\tau g$-open sets in associated w-spaces, and investigate some basic properties of such notions.

2. Preliminaries

Definition 2.1 ([9]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by $cl(S)$ (resp., $int(S)$). A subset S of X is called a *preopen* set [6] (resp., α-open set [117], *semi-open* [4]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S))), S \subset cl(int(S))$). The complement of a preopen set (resp., α-open set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X.

Moreover, a subset S of X is said to be g-closed [3] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X.

Then the family $GO(X) = \{U \subseteq X : U \text{ is } g\text{-open }\}$, $O(X) = \{U \subseteq X : U \text{ is open }\}$ and $CL(X) = \{F \subseteq X : F \text{ is closed }\}$ are all weak structures on X. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [5] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

Let (X, w_X) be a w-space. For a subset A of X, the *w-closure* of A and the *w-interior* [9] of A are defined as follows:

(1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.

(2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.2 ([9]). *Let (X, w_X) be a w-space and $A \subseteq X$. Then the following things hold:*

(1) If $A \subseteq B$, then $wI(A) \subseteq wI(B)$; $wC(A) \subseteq wC(B)$.

(2) $wI(wI(A)) = wI(A)$; $wC(wC(A)) = wC(A)$.

(3) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.

(4) If A is w-closed (resp., w-open), then $wC(A) = A$ (resp., $wI(A) = A$).

Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w of the power set $P(X)$ is called *an associated weak structure* (simply, w_τ) [10] on X if $\tau \subseteq w$ and w is a weak structure. Then the pair (X, w_τ) is called *an associated w-space* with τ.

Let X be an associated w-space. Then

(1) A is called *a generalized closed set* (simply, g-closed set) [3] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open;

(2) A is called *a w_τ-generalized closed set* (simply, wg-closed set) [12] if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is open;

(3) A is called *a generalized w-closed set* (simply, gw-closed set) [11] if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open.

(4) A is called *a generalized w_τ-closed set* (simply, gw_τ-closed set) [13] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open.

(5) A is called *a weakly generalized w_τ-closed set* (simply, weakly gw_τ-closed set) [16] if $cl(int(A)) \subseteq U$, whenever $A \subseteq U$ and U is w-open.

3. Main Results

Definition 3.1. Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$. Then A is called *a weakly w_τ-generalized closed set* (simply, weakly wg-closed set) if $wC(wI(A)) \subseteq U$, whenever $A \subseteq U$ and U is open.

Then obviously, the next theorem is obtained:

Theorem 3.2. *Let (X, w_τ) be an associated w-space with a topology τ. Then the following things hold:*

(1) Every wg-closed set is weakly wg-closed.

(2) Every gw-closed set is weakly wg-closed.

Proof. (1) Obvious.

(2) For any gw-closed set A in X, let U be an open set in X such that $A \subseteq U$. Then since U is also w-open, we have $wC(wI(A)) \subseteq wC(A) \subseteq U$. Hence, A is weakly wg-closed.

□

In general, the converse of the above theorem is not true as the next example:

Example 3.3. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{b\}, \{b, c\}, X\}$ and a w-structure $w_X = \{\emptyset, \{a\}, \{b\}, \{b, c\}, X\}$ in X. For $A = \{c\}$, since

$wC(A) = \{c, d\}$ and $wI(A) = \emptyset$, easily we can find that A is weakly wg-closed but neither wg-closed nor gw-closed.

Remark 3.4. For a given associated w_τ-space with a topology τ, from the above theorem and example, the following relations are obtained:

$$\begin{array}{ccccccc}
 & \text{weakly } gw_\tau\text{-closed} & \rightarrow & \text{weakly } g\text{-closed} & \rightarrow & \text{weakly } wg\text{-closed} \\
 & \uparrow & & \uparrow & & \uparrow \\
 & gw_\tau\text{-closed} & \rightarrow & g\text{-closed} & \rightarrow & wg\text{-closed} \\
\nearrow & & & & & \\
\text{closed} & & \searrow & & \nearrow & \\
\searrow & & & & & \\
 & w\text{-closed} & \rightarrow & gw\text{-closed} & &
\end{array}$$

Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called

(1) *a w-preopen set* [14] if $A \subseteq wI(wC(A))$. The complement of a w-preopen set is called a *w-preclosed set*;

(2) a *w-regular open set* (resp., *w-regular closed set*) [15] if $A = wI(wC(A))$ (resp., $A = wC(wI(A))$;

Theorem 3.5. *Let (X, w_τ) be an associated w-space with a topology τ. Then the following things hold:*

(1) Every w-preclosed set is weakly wg-closed.

(2) Every w-regular closed set is weakly wg-closed.

Proof. Obvious. □

Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called a *$w\sigma$-open set* (resp., *$w\sigma$-closed set* [15] if $A = wI(A)$ (resp., $A = wC(A)$).

Theorem 3.6. *Let (X, w_τ) be an associated w-space. Then if A is a weakly wg-closed and open set, then A is a $w\sigma$-closed set.*

Proof. Let A be a weakly wg-closed and open set. Then by hypothesis, we have $wC(wI(A)) \subseteq A$. Since A is also w-open, it implies that $wC(wI(A)) = wC(A) \subseteq A \subseteq wC(A)$. So, $A = wC(A)$ and A is a $w\sigma$-closed set. □

Theorem 3.7. *Let (X, w_τ) be an associated w-space. Then A is a weakly wg-closed set if and only if $wC(wI(A)) \cap F = \emptyset$, whenever $A \cap F = \emptyset$ and F is closed.*

Proof. Suppose that A is a weakly wg-closed set. Let F be a closed set such that $A \cap F = \emptyset$. Then $A \subseteq X - F$ and $X - F$ is open, and since A is a weakly wg-closed set, $wC(wI(A)) \subseteq X - F$. So, $wC(wI(A)) \cap F = \emptyset$.

For the converse, let $A \subseteq U$ and U any open set. Put $F = X - U$. Then $A \cap F = \emptyset$ and F is closed. By hypothesis, $wC(wI(A)) \cap F = wC(wI(A)) \cap (X - U) = \emptyset$. It implies that $wC(wI(A)) \subseteq U$. So, A is weakly wg-closed. □

Theorem 3.8. *Let (X, w_τ) be an associated w-space with a topology τ. Then if A is a weakly wg-closed set, then $wC(wI(A)) - A$ contains no any non-empty closed subset.*

Proof. Suppose that there is a closed set F such that $F \subseteq wC(wI(A)) - A$. Then $A \cap F = \emptyset$, and since F is closed, by the above theorem, $wC(wI(A)) \cap F = \emptyset$. It implies $F = \emptyset$ from $F \subseteq wC(wI(A)) - A$.

□

In general, the converse in the above theorem is not true as shown in the next example.

Example 3.9. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{b\}, \{b, c\}, X\}$ and a w-structure $w_X = \{\emptyset, \{a\}, \{b\}, \{b, c\}, X\}$ in X. For $A = \{b\}$, $wI(A) = A$ and $wC(A) = \{b, c, d\}$, and $wC(wI(A)) - A = \{c, d\}$ contains no any non-empty closed subset but A is not weakly wg-closed.

Theorem 3.10. *Let (X, w_τ) be an associated w-space with a topology τ. Then if A is a weakly wg-closed set and $A \subseteq B \subseteq wC(wI(A))$, then B is weakly wg-closed.*

Proof. Let U be any open set such that $B \subseteq U$. Then by Theorem 2.2, $wC(wI(A)) = wC(wI(B))$. Since A is weakly wg-closed and $A \subseteq U$, we have $wC(wI(B)) \subseteq U$. So B is weakly wg-closed. □

Now, we introduce the notion of weakly wg-open set in a given associated w-space:

Definition 3.11. Let (X, w_τ) be an associated w-space and $A \subseteq X$. Then A is called *a weakly wg-open set* (simply, weakly wg-open set) if $X - A$ is weakly wg-closed.

Theorem 3.12. *Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$. Then A is weakly wg-open if and only if $F \subseteq wI(wC(A))$ whenever $F \subseteq A$ and F is closed.*

Proof. From the definition of weakly wg-open set and Theorem 2.2, it is easily obtained. □

Theorem 3.13. *Let (X, w_τ) be an associated w-space. Then if A is weakly wg-open, then $U = X$, whenever $wI(wC(A)) \cup (X - A) \subseteq U$ and U is open.*

Proof. Let U be any open set satisfying $wI(wC(A)) \cup (X - A) \subseteq U$. Then by Theorem 2.2, $X - U \subseteq X - wI(wC(A)) \cap A = wC(wI(X - A)) - (X - A)$, and by Theorem 3.9, the closed set $X - U$ must be empty. So, $U = X$. □

Theorem 3.14. *Let (X, w_τ) be an associated w-space. Then if A is a weakly wg-open set and $wI(wC(A)) \subseteq B \subseteq A$, then B is weakly wg-open.*

Proof. It is similar to the proof of Theorem 3.10. □

Theorem 3.15. *Let (X, w_τ) be an associated w-space. Then if A is a weakly wg-closed set, then $wC(wI(A)) - A$ is weakly wg-open.*

Proof. Let A be a weakly wg-closed set. Then by Theorem 3.8, $\emptyset$ is the only one closed subset of $wC(wI(A)) - A$ and $\emptyset \subseteq wC(wI(wC(wI(A)) - A))$. So, $wC(wI(A)) - A$ is weakly wg-open. □

Theorem 3.16. *Let (X, w_τ) be an associated w-space. Then if A is a weakly wg-open set, then $wI(wC(A)) \cup (X - A)$ is weakly wg-closed.*

Proof. If A is a weakly wg-open set, then by Theorem 3.13, X is the only one open set containing $wI(wC(A)) \cup (X - A)$ and $wC(wI(wI(wC(A)) \cup (X - A)) \subseteq X$. So, $wI(wC(A)) \cup (X - A)$ is weakly wg-closed. □

Remark 3.17. Let (X, w_τ) be an associated w-space with a topology τ.

And we recall that: A is called *a w-preopen set* [14] if $A \subseteq wI(wC(A))$. Then, finally, we have the following relationship:

$$\text{open} \Rightarrow w\text{-open} \Rightarrow w\text{-preopen} \Rightarrow \text{weakly } wg\text{-open}.$$

Theorem 3.18. *Let (X, w_X) be a w-space. Then for $A \subseteq X$, $A \cap wI(wC(A))$ is w-preopen.*

Proof. We show that $A \cap wI(wC(A)) \subseteq wI(wC(A \cap wI(wC(A))))$. For the proof, assume that there is some $x \in A \cap wI(wC(A))$ such that $x \notin wI(wC(A \cap wI(wC(A))))$. First, note that from $x \in A \cap wI(wC(A))$, there exists a w-open set U_x such that $U_x \subseteq wC(A)$, and it implies that $U_x \cap A \neq \emptyset$ and

$U_x \subseteq wI(wC(A))$. Now, since U_x is a w-open set containing x, by assumption, $U_x \nsubseteq wC(A \cap wI(wC(A)))$. So, there exists an element z in U_x and a nonempty w-open set V_z containing z satisfying $V_z \cap (A \cap wI(wC(A))) = \emptyset$. Then $V_z \cap U_x$ is a nonempty w-open set of z and $(V_z \cap U_x) \cap (A \cap wI(wC(A))) = \emptyset$. But it is impossible because $(V_z \cap U_x) \cap A \neq \emptyset$ and $(V_z \cap U_x) \subseteq wI(wC(A))$ by the fact $(V_z \cap U_x) \subseteq U_x \subseteq wC(A)$.

□

Lemma 3.19. *Let (X, w_X) be a w-space and $A \subseteq X$. Then for any w-preopen set $U \subseteq A$, $U \subseteq A \cap wI(wC(A))$.*

Proof. For a w-preopen set $U \subseteq A$, $U \subseteq wI(wC(U)) \subseteq wI(wC(A))$, and so $U \subseteq A \cap wI(wC(A))$.

□

Let (X, w_X) be a w-space. For $A \subseteq X$, the *w-pre-interior* [14] of A, denoted by $wpI(A)$, is defined as:

$wpI(A) = \cup\{U \subseteq X : U \subseteq A, U \text{ is } w\text{-preopen in } X\}$.

Theorem 3.20. *Let (X, w_X) be a w-space. Then for $A \subseteq X$, $wpI(A) = A \cap wI(wC(A))$.*

Proof. It is obtained from Theorem 3.18 and Lemma 3.19. □

Finally, we have the following theorem:

Theorem 3.21. *Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$. Then A is weakly wg-open if and only if $F \subseteq wpI(A)$ whenever $F \subseteq A$ and F is closed.*

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