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COUPLED QUANTALES AND A NON-COMMUTATIVE APPROACH TO BITOPOLOGICAL SPACES

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Abstract: The concept of a coupled quantales is introduced as a non-commutative extension of the concept of biframes. Also an approach to non-commutative bitopology is studied. Then an adjunction between the category of coupled quantales and the category of biquantum spaces is established.

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1. Introduction

In 1986 C.J. Mulvey [**7**] proposed the term quantale as a non-commutative extension of the concept of frame (a complete lattice satisfying the first infinite distributive law of finite meets over arbitrary sups). The purpose was to develop the concept of non-commutative topology, introduced by R. Giles and H. Kummer [**5**], while providing constructive foundations for the theory of quantum

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mechanics and non-commutative logic [**11**]. Nowadays, the notion of quantale can boast many areas of application, e. g., in the field of non-commutative topology [**8, 9, 3**]. Further details about quantales can be found in [**10**].

In 1989 F. Borceux and G. van den Bossche (c.f. [**2**])proposed a more general model of non-commutative topology strongly based on the notion of quantale; they define a quantum space as a family of open sets in which the intersection is substituted by a product $\otimes$, in such a manner that the lattice of open sets is a (right-sided and idempotent) quantale. Also a duality between spatial right-sided idempotent quantales and sober quantum spaces is proposed. Such duality is extended by Höhle [**6**] to an adjunction based on quantum spaces

In this paper we aim to introduce the notions of coupled quantale as a non-commutative extension of the concept of biframe [**1**]. Also, we aim to propose a model of non-commutative bitopology based on the notion of coupled quantale. Then, we will extend the dual adjunction between the category of right-sided idempotent quantales and the category of quantum spaces to one between the category of coupled quantales and the category of biquantum spaces.

2. Preliminaries

Definition 2.1. [**10**] A quantale $(Q, \leq, \otimes)$ is a complete lattice $(Q, \leq)$ equipped with an associative binary operation $\otimes : Q \times Q \longrightarrow Q$, called a tensor product such that:

$$a \otimes (\bigvee_j b_j) = \bigvee_j (a \otimes b_j) \text{ and } (\bigvee_j b_j) \otimes a = \bigvee_j (b_j \otimes a)$$

for all $a \in Q$ and $\{b_j\} \subseteq Q$.

Definition 2.2. [**10**] Let Q_1 and Q_2 be quantales. A function $h : Q_1 \longrightarrow Q_2$ is said to be:

(1) a quantale morphism if it preserves $\otimes$ and arbitrary sups.

(2) a strong quantale morphism if it preserves $\otimes$, arbitrary sups and $\top$.

By **Quant**(resp. **StQuant**), we mean the category of all quantales and quantale morphisms (resp. strong quantale morphisms). **StQuant** and **Quant** clearly share the same objects.

Definition 2.3. [**10**]A quantale $(Q, \leq, \otimes)$ is said to be:

(1) a unital quantale if whose multiplication $\otimes$ has an identity element $e \in L$ called the unit. **USQuant** denotes the category all unital quantale together with all quantale morphisms preserving the unit e.

(2) a commutative quantate if whose multiplication $\otimes$ satisfies that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in L$. **ComQuant** denotes the full subcategory of **Quant** of all commutative quantales.

(3) right-sided (resp., left-sided) iff $a \otimes \top \leq a$ (resp., $\top \otimes a \leq a$), for all $a \in L$.

(4) idempotent iff $a \otimes a = a$, for all $a \in L$.

By subquantale [**10**] of of a quantale $Q = (Q, \leq, \otimes)$ we mean subset $S \subseteq Q$ which is closed under the tensor product $\otimes$ and arbitrary sups.

Remark 2.4. [**10**] The intersection of two subquantales of quantale Q is also a subquantale of Q.

By a **quantic nucleus** [**10**] on a quantale Q, we mean a closure operator $c : Q \longrightarrow Q$ such that $c(a) \otimes c(b) \leq c(a \otimes b)$ for all $a, b \in Q$.

Lemma 2.5. [**6**] *In any right-sided and idempotent quantale Q the closure operator*

$$c : Q \to Q$$

defined by:

$$c(x) = \bigwedge \{p \in Q : p \ \ prime, \quad x \leq p\}, x \in Q \tag{1}$$

is a ***nucleus*** *on Q.*

For any isotone maps between $f : P \longrightarrow Q$, between posets, its right adjoint is the map, (which is necessarily unique), $f_* : Q \longrightarrow P$ such that $f(a) \leq b$ iff $a \leq f_*(b)$ for all $a \in P$ and $b \in Q$, explicitly given by $f_*(b) = \bigvee \{x \in P : f(x) \leq b\}$.

In 1989 F. Borceux and G. van den Bossche (c.f.[**2**]) proposed a more general model of non-commutative topology strongly based on the notion of quantale; they define a quantum space as a set X provided with a family $\mathcal{O}(X) \subseteq P(X)$ of open subsets and a multiplication:

$$\otimes : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(X)$$

in such a manner that the lattice of open sets, which called a non-commutative topology, is a right-sided and idempotent quantale as given in the following definition

Definition 2.6. [**2**] Let X be a non-empty set. A non-commutative topology on X is a subset $\mathcal{O}(X) \subseteq P(X)$ satisfying the following conditions:

(1) $\mathcal{O}(X)$ is closed under arbitrary union i.e., the set-inclusion $\mathcal{O}(X) \hookrightarrow P(X)$ is a join preserving map.

(2) $X \in \mathcal{O}(X)$.

(3) On $\mathcal{O}(X)$ there exists a binary operation:

$$\otimes : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(X)$$

such that

(4) $U, V \in \mathcal{O}(X) \Rightarrow U \cap V \subseteq U \otimes V$; and

(5) the pair $(\mathcal{O}(X), \otimes)$ is a right-sided and idempotent quantale.i.e.,

(S_1) $U \in \mathcal{O}(X) \Rightarrow U \otimes X = U$;

(S_2) $U, V, W \in \mathcal{O}(X) \Rightarrow U \otimes (V \otimes W) = (U \otimes V) \otimes W$;

(S_3) $U, V_j \in \mathcal{O}(X) \Rightarrow U \otimes (\bigcup_{j \in J} V_j) = \bigcup_{j \in J} (U \otimes V_j)$;

(S_4) $U, V_j \in \mathcal{O}(X) \Rightarrow (\bigcup_{j \in J} V_j) \otimes U = \bigcup_{j \in J} (V_j \otimes U)$.

If $\mathcal{O}(X)$ is a non-commutative topology on X, then the pair $(X, \mathcal{O}(X))$ is called a quantum space

An important corollary of the axioms of a non-commutative topology $\mathcal{O}(X)$ on X is the following property:

$$A \otimes B = A \cap (X \otimes B), A, B \in \mathcal{O}(X). \tag{2}$$

A *continuous mapping* **[2]** from a quantum space $(X, \mathcal{O}(X))$ to a quantum space $(Y, \mathcal{O}(Y))$ is a mapping $f : X \to Y$ such that

(C_1) $\forall V \in \mathcal{O}(Y) \Rightarrow f^{-1}(V) \in \mathcal{O}(X)$,

(C_2) $\forall V, W \in \mathcal{O}(Y) \Rightarrow f^{-1}(V \otimes W) \supseteq f^{-1}(V) \otimes f^{-1}(W)$.

A continuous mapping $f : X \to Y$ is said to by strict continuous if it satisfies the condition

(C_3) $\forall V, W \in \mathcal{O}(Y) \Rightarrow f^{-1}(V \otimes W) = f^{-1}(V) \otimes f^{-1}(W)$.

2.1. Biquantum Spaces

Definition 2.7. (The category of biquantum spaces)

(1) A biquantum space is a triple $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ consisting of a non-empty set X and two non-commutative topologies $\mathcal{O}_1(X)$ and $\mathcal{O}_2(X)$ of subsets of X.

(2) A bicontinuous map $f : X \to Y$ between biquantum spaces $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ and $(Y, \mathcal{O}_1(Y), \mathcal{O}_2(Y))$ is a function between their underlying sets for which

$$f : (X, \mathcal{O}_1(X)) \to (Y, \mathcal{O}_1(Y)) \text{ and } f : (X, \mathcal{O}_2(X)) \to (Y, \mathcal{O}_2(Y))$$

are continuous.

(3) The category of biquantum spaces and bicontinuous maps will be denoted by **BiQS**.

Example 2.8. Every bitopological space is clearly a biquantum space when defining

$$U \otimes V = U \cap V.$$

Definition 2.9. A strict bicontinuous map $f : X \to Y$ between biquantum spaces $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ and $(Y, \mathcal{O}_1(Y), \mathcal{O}_2(Y))$ is a function between their underlying sets for which

$$f : (X, \mathcal{O}_1(X)) \to (Y, \mathcal{O}_1(Y)) \text{ and } f : (X, \mathcal{O}_2(X)) \to (Y, \mathcal{O}_2(Y))$$

are strict continuous.

The biquantum spaces and their strict bicontinuous mappings constitute nevertheless an interesting category.

Definition 2.10. Let $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ be a biquantum space.

(1) We call $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ **pairwise** T_0 if for any two distinct points $a, b \in X$ there exists $G \in \mathcal{O}_1(X) \bigcup \mathcal{O}_2(X)$ containing exactly one of a, b.

(2) We call $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ **pairwise** T_1 if for any two distinct points $a, b \in X$ there exists $G \in \mathcal{O}_1(X) \bigcup \mathcal{O}_2(X)$ such that $a \in G$ and $b \notin G$.

(3) We call $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ **pairwise** T_2 or **pairwise Hausdorff** if for any two distinct points $a, b \in X$ there exist disjoint $G \in \mathcal{O}_1(X)$ and $H \in \mathcal{O}_2(X)$ such that $a \in G$ and $b \in H$ or there exist disjoint $G \in \mathcal{O}_2(X)$ and $H \in \mathcal{O}_1(X)$ with the same property.

Proposition 2.11. *$(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ is **pairwise** T_0 iff $\mathcal{O}_1(X) \vee \mathcal{O}_2(X)$ is T_0.*

Proof. it is clear that $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ **pairwise** T_0 implies $\mathcal{O}_1(X) \vee \mathcal{O}_2(X)$ is T_0. Conversely, assume $\mathcal{O}_1(X) \vee \mathcal{O}_2(X)$ is T_0 and $a \neq b$. Suppose $G \in \mathcal{O}_1(X)$ and $H \in \mathcal{O}_2(X)$ and $a \in G \cap H, b \notin G \cap H$. then $b \notin G$ or $b \notin H$. Thus $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ is **pairwise** T_0. □

Example 2.12. Let $X = \{0, 1\}$, $\mathcal{O}_1(X) = \{\phi, \{1\}, X\}$ and $\mathcal{O}_2(X) = \{\phi, \{0\}, X\}$, where

$$\otimes : X \times X \to X$$

defined by

$\otimes$	0	1
0	0	0
1	0	1

Then both $\mathcal{O}_1(X)$ and $\mathcal{O}_2(X)$ are a left-sided and idempotent quantales. Thus both are T_0, but not T_1.

Definition 2.13. We call a biquantum space $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ pairwise zero-dimensional if opens in $(X, \mathcal{O}_1(X))$ closed in $(X, \mathcal{O}_2(X))$ form a basis β_1 for $(X, \mathcal{O}_1(X))$ and opens in $(X, \mathcal{O}_2(X))$ closed in $(X, \mathcal{O}_1(X))$ form a basis β_2 for $(X, \mathcal{O}_2(X))$.

Example 2.14. $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$, in **Example 2.12**, is pairwise zero-dimensional and pairwise T_2.

Lemma 2.15. *Suppose that $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ is pairwise zero-dimensional. Then the following conditions are equivalent:*

(1) $(X, \mathcal{O}_1(X))$ is T_0.

(2) $(X, \mathcal{O}_2(X))$ is T_0.

(3) $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ is pairwise T_2.

(4) *For any two distinct points $x, y \in X$ there exist disjoint $U, V \in \mathcal{O}_1(X) \cup \mathcal{O}_2(X)$ such that $x \in U$ and $y \in V$.*

Proof. (1) $\Rightarrow$ (2) :Suppose that $(X, \mathcal{O}_1(X))$ is T_0 and x, y are two distinct points of X. Then there exists $U \in \mathcal{O}_1(X)$ containing exactly one of x, y. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ is pairwise zero-dimensional, there exists $V \in \beta_1$ such that $x \in V \subseteq U$. Therefore, $V^c \in \beta_2$, $y \in V^c$ and $x \notin V^c$. Thus, $(X, \mathcal{O}_2(X))$ is T_0.

(2) $\Rightarrow$ (3) : Suppose that $(X, \mathcal{O}_2(X))$ is T_0 and x, y are two distinct points of X. Then there exists $U \in \mathcal{O}_2(X)$ containing exactly one of x, y. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ is pairwise zero-dimensional, there exists $V \in \beta_2$ such that $x \in V \subseteq U$. Then $x \in V \in \beta_2$, $y \in V^c \in \beta_1$, and V, V^c are disjoint. Thus, $(X, \mathcal{O}_1(X), \mathcal{O}_2(X))$ is pairwise T_2.

(3) $\Rightarrow$ (4) :is obvious.

(4) $\Rightarrow$ (1) :is obvious.

□

2.2. Duality between BiQS and RSiCQuant

For a fixed $Q \in |\mathbf{Quant}|$, it follows, as a consequence of **Remark 2.4**, that the family of all subquantales of Q, ordered by inclusion, forms a complete lattice, with the meet $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ (the set-intersection), and the join $Q_1 \vee Q_2$ is the least subquantale of Q containing Q_1 and Q_2 (which is not their set-theoretical union). So, for a subset $K \subseteq Q$ of a quantale Q, the smallest subquantale of Q which contains K is defined to be the subquantale generated by K.

Definition 2.16. (The category of coupled quantales)

(1) A coupled quantale is a triple $Q = (Q_0, Q_1, Q_2)$ in which Q_0 is a quantale, Q_1 and Q_2 are subquantales of Q_0 such that $Q_1 \cup Q_2$ generates Q_0.

(2) A map $h : Q \to P$ between coupled quantales is a quantale morphism $Q_0 \to P_0$ for which the restrictions $h|_{Q_i} : Q_i \to P_i$ are quantale morphisms i.e., $h(Q_i) \subseteq P_i$ for $i = 1, 2$.

(3) The resulting category will be denoted by **CQuant**.

We refer to Q_0 as the total part of Q, and Q_1, Q_2 as its first and second parts, respectively.

The restrictions of a coupled quantale map $h : Q \to P$ to various parts will be written

$$h_0 : Q_0 \to P_0 \text{ and } h_i : Q_i \to P_i \text{ for } (i = 1, 2).$$

Definition 2.17. A coupled quantale $Q = (Q_0, Q_1, Q_2)$ is said to be:

(1) *unital* iff Q_0 is unital and e belongs to both Q_1 and Q_2. **UnCQuant** is the full subcategory of **CQuant** of all unital coupled quantales.

(2) right-sided (resp., left-sided iff the total part Q_0 is right-sided (resp., left-sided i.e., $a \otimes \top \leq a$ (resp., $\top \otimes a \leq a$) for all $a \in Q_0$.

(3) idempotent iff the total part Q_0 is idempotent i.e., $a \otimes a = a$ for all $a \in Q_0$.

(4) commutative if the operation $\otimes$ is commutative i.e.,$q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1 \in Q_i$ and $q_2 \in Q_k$. **ComCQuant** is the full subcategory of **CQuant** of all commutative coupled quantales

(5) A biframe [**1**] is a unital commutative coupled quantale whose multiplication and unit are $\wedge$ and $\top$ respectively and $\forall\, a \in Q_0, a = \bigvee\{b \otimes c : b \in Q_1$ and $c \in Q_2\}$.

Example 2.18. Let $Q = \{\perp, a, b, \top\}$ be the four Boolean lattice and let $\otimes : Q \times Q \to Q$ defined by

$\otimes$	$\perp$	a	b	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	a	$\perp$	a
b	$\perp$	$\perp$	b	b
$\top$	$\perp$	a	b	$\top$

It is clear that Q is a coupled quantales with the total part the $Q_0 = \{\perp, a, b, \top\}$, first part $Q_1 = \{\perp, a, \top\}$ second part $Q_2 = \{\perp, b, \top\}$.

Example 2.19. Any biframe [**1**] $A = (A_0, A_1, A_2)$ is a commutative coupled quantale provided that $\otimes = \wedge$ and any element of $a \in A_0$ can be expressed as $a = \bigvee\{a_1 \otimes a_2 : a_1 \in A_1, a_2 \in A_2\}$.

Through this paper we use **RSiCQuant** to denote the category of idempotent and right-sided coupled quantales with coupled quantales morphisms. In this context strictly bicontinuous maps can be viewed as strong homomorphisms w.r.t. the underlying idempotent and right-sided coupled quantales. This observation implies that there exists a functor

$$\Omega : \mathbf{BiQS} \rightarrow \mathbf{RSiCQuant}^{op}$$

sending a biquantum space to its underlying non-commutative bitopology.

i.e.,

$$\Omega(X, \mathcal{O}_1(X), \mathcal{O}_2(X)) = (\mathcal{O}_0(X), \mathcal{O}_1(X), \mathcal{O}_2(X))$$

where $\mathcal{O}_0(X) = \mathcal{O}_1(X) \vee \mathcal{O}_2(X)$, that is, the coarsest non-commutative topology containing $\mathcal{O}_1(X)$ and $\mathcal{O}_2(X)$.

For the strictly bicontinuous function

$$f : (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \longrightarrow (Y, \mathcal{O}_1(Y), \mathcal{O}_2(Y))$$

define the map $\Omega(f)$ by $\Omega(f)(A) = f^{-1}(A)$, which is clearly a coupled quantales map from $(\mathcal{O}_0(Y), \mathcal{O}_1(Y), \mathcal{O}_2(y))$ to $(\mathcal{O}_0(X), \mathcal{O}_1(X), \mathcal{O}_2(X))$.

We show that the functor Ω has a right adjoint. For this purpose we begin with the following observation.

As given in **[6]**, one can associate a non-commutative topology $\mathcal{O}_Q$ with every right-sided and idempotent quantale Q as follows:

Let $pt(Q)$ be the spectrum of Q i.e., the set of all prime elements of Q. Then for every $x \in Q$ the set

$$A_x = \{p \in pt(Q) : x \not\leq p\}$$

is in general not uniquely be determined by x, but there exists a largest element of Q with this property-namely $c(x)$ where c is the nucleus determined by **Eq.(1)**.

The complete sublattice

$$\mathcal{O}_Q = \{A_{c(x)} : x \in Q\}$$

of the power set $P(pt(Q))$ is a non-commutative topology **[6]** on $pt(Q)$.

For an $Q = (Q_0, Q_1, Q_2) \in |\mathbf{RSiCQuant}|$, we have a biquantum space $(pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_1})$ where

(1) $pt(Q_0)$ is the set of all prime elements of Q_0;

(2) $\mathcal{O}_{Q_i} = \{A_{c(x)} : x \in Q_i\}$, as usual, is a non-commutative topology on $pt(Q_0)$ for $(i = 1, 2)$.

i.e.,

$$(Q_0, Q_1, Q_2) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_1})$$

Also, the the strong homomorphism

$$f : (Q_0, Q_1, Q_2) \to (P_0, P_1, P_2)$$

takes the form

$$pt(f) : (pt(P_0), \mathcal{O}_{P_1}, \mathcal{O}_{P_2}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}).$$

Lemma 2.20. *For a strong* ***RSiCQuant****-homomorphism*

$$f : (Q_0, Q_1, Q_2) \to (P_0, P_1, P_2),$$

the mapping $pt(f) : (pt(P_0), \mathcal{O}_{P_1}, \mathcal{O}_{P_2}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})$ *is strictly bicontinuous.*

Proof. Let $f : (Q_0, Q_1, Q_2) \to (P_0, P_1, P_2)$ be an **RSiCQuant**-homomorphism. It is well known that the right adjoint map f_* of f preserves prime elements. Hence the restriction $pt(f)$ of f_* to $pt(P)$ gives the two maps

$$pt(f) : (pt(P_0), \mathcal{O}_{P_1}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1}) \text{ and}$$
$$pt(f) : (pt(P_0), \mathcal{O}_{P_2}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_2}).$$

First, we show that

$$pt(f) : (pt(P_0), \mathcal{O}_{P_1}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1})$$

is strictly continuous. For this purpose we choose $x \in Q_1$ and $p \in pt(P_0)$ with $pt(f)(p) \in A_x$. Because of $x \not\leq pt(f)(p) \Leftrightarrow f(x) \not\leq p$ the relation $(pt(f))^{-1}(A_x) = A_{f(x)}$ holds. Hence $pt(f)$ satisfies the condition (C_1) of the strict continuity which means that $pt(f)$ is strict continuous w.r.t. the non-commutative topologies $\mathcal{O}_{P_1}$ and $\mathcal{O}_{Q_1}$. Since $pt(Q_0) = A_\top$ and f is strong, i.e., $f(\top) = \top$, then the axiom (C_3) of the strict continuity follows from **Eq.(2)** and the following relation

$$\begin{aligned}(pt(f))^{-1}(pt(Q_0) \odot A_x) &= (pt(f))^{-1}(A_{\top\otimes x})\\ &= A_{f(\top\otimes x)}\\ &= A_{\top\otimes f(x)}\\ &= pt(P_0) \odot A_{f(x)}\\ &= pt(P_0) \odot (pt(f))^{-1}(A_x).\end{aligned}$$

Hence the first map:

$$pt(f) : (pt(P_0), \mathcal{O}_{P_1}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1})$$

is strictly continuous. Similarly, one can prove the strict continuity of the second map:

$$pt(f) : (pt(P_0), \mathcal{O}_{P_2}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_2}).$$

and this prove the strict bicontinuity of the map:

$$pt(f) : (pt(P_0), \mathcal{O}_{P_1}, \mathcal{O}_{P_2}) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})$$

and this completes the proof. □

Now we introduce a functor

$$PT : \mathbf{RSiCQuant}^{op} \to \mathbf{BiQS}$$

by

$$PT((Q_0, Q_1, Q_2)) = (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_1})$$

and

$$PT((Q_0, Q_1, Q_2) \stackrel{f}{\longrightarrow} (P_0, P_1, P_2)) = (pt(P_0), \mathcal{O}_{P_1}, \mathcal{O}_{P_2}) \stackrel{pt(f)}{\longrightarrow} (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}).$$

To study the adjunction between the functors:

$$PT : \mathbf{RSiCQuant}^{op} \longrightarrow \mathbf{BiQS}$$

and

$$\Omega : \mathbf{RSiCQuant}^{op} \longleftarrow \mathbf{BiQS}$$

we give the following definitions

For $(X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \in |\mathbf{BiQS}|$ and $Q = (Q_0, Q_1, Q_2) \in |\mathbf{RSiCQuant}|$ define the map:

$$\eta_X : (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \longrightarrow PT(\Omega(X, \mathcal{O}_1(X), \mathcal{O}_2(X))),$$

by setting,

$$\eta_X(x) = \cup\{V \in \mathcal{O}_1(X) \vee \mathcal{O}_2(X) : x \notin V\}; \forall\, x \in X,$$

and the map:

$$\varepsilon_Q^{op} : Q \longrightarrow \Omega(PT(Q))$$

by setting $\varepsilon_Q^{op}(x) = A_x$. It is clear that by definition ε_Q^{op} always surjective.

Lemma 2.21. *For $(X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \in |\boldsymbol{BiQS}|$.*

(1) *The map*

$$\eta_X : (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \longrightarrow PT(\Omega(X, \mathcal{O}_1(X), \mathcal{O}_2(X))),$$

is strictly bicontinuous, and

(2) *The system* $\eta = (\eta_X)_X$ *constitutes a natural transformation* $Id_{BiQS} \longrightarrow PT \circ \Omega$.

Proof. (1) Let$X = (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \in |\mathbf{BiQS}|$ and x be an element of X. To prove the strict bicontinuity of

$$\eta_X : (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \longrightarrow pt(\Omega(X, \mathcal{O}_1(X), \mathcal{O}_2(X))),$$

we need to prove the strict continuity of both the mappings

$$\eta_X : (X, \mathcal{O}_1(X)) \longrightarrow (pt(\mathcal{O}_0(X)), \mathcal{O}_{\mathcal{O}_1(X)})$$

and

$$\eta_X : (X, \mathcal{O}_2(X)) \longrightarrow (pt(\mathcal{O}_0(X)), \mathcal{O}_{\mathcal{O}_2(X)}),$$

Each open set in $(pt(\mathcal{O}_0(X)), \mathcal{O}_{\mathcal{O}_1(X)})$ is of the form A_V with $V \in \mathcal{O}_1(X)$. Therefore we have

(i) $\eta_X^{-1}(A_V) = \{x : G_x = A_V\} = V$, and

(ii) $\eta_X^{-1}(A_X \otimes A_V) = \eta_X^{-1}(A_{X \otimes V}) = X \otimes V$.

i,e., the first map $\eta_X : (X, \mathcal{O}_1(X)) \longrightarrow (pt(\mathcal{O}_0(X)), \mathcal{O}_{\mathcal{O}_1(X)})$ is strictly continuous. Similarly one can prove the strict continuity of the second map.

(2) Let $f : (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \longrightarrow (Y, \mathcal{O}_1(Y), \mathcal{O}_2(Y))$ be strictly bicontinuous. Since $\Omega(f)$ is the left adjoint to $pt(f)$ we have

$$\begin{aligned}(pt(\Omega(f)) \circ \eta_X)(x) &= pt(\Omega(f))(G_x) \text{ since } G_x = \bigcup\{U \in \tau_i, (i = 1, 2) : x \notin U\}.\\ &= (\Omega(f)^{-1}(G_x)\\ &= \{U : f^{-1}(U) \in G_x\}\\ &= \{U : x \notin f^{-1}(U)\}\\ &= \{U : f(x) \notin U\}\\ &= \eta_X(f(x)).\end{aligned}$$

which means that $\eta = (\eta_X)_X$ is a natural transformation from Id_{BiQS} to $PT \circ \Omega$. □

Theorem 2.22. *The functor:*

$$PT : \boldsymbol{RSiCQuant}^{op} \longrightarrow \boldsymbol{BiQS}$$

is right adjoint to the functor:

$$\Omega : \boldsymbol{RSiCQuant}^{op} \longleftarrow \boldsymbol{BiQS}$$

.

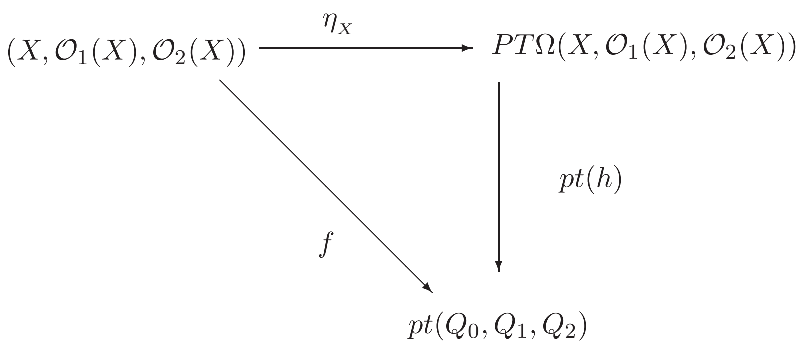
Proof. It is sufficient to show that for every $Q = (Q_0, Q_1, Q_2) \in$ **RSiCQuant** and a strictly bicontinuous map

$$f : (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \longrightarrow (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})$$

there exists a unique strong homomorphism

$$h : (Q_0, Q_1, Q_2) \longrightarrow (\mathcal{O}_0(X), \mathcal{O}_1(X), \mathcal{O}_2(X))$$

making the following diagram commutative:



To prove the existence, let $h(z) = f^{-1}(A_z)$. Obviously the mapping $h : Q_0 \longrightarrow \mathcal{O}_0(X)$ is join preserving and fulfills the following property:

$$h(z_1) \otimes h(z_2) = f^{-1}(A_{z_1} \otimes A_{z_2}) = f^{-1}(A_{z_1 \otimes z_2}) = h(z_1 \otimes z_2).$$

The same is true for the restrictions $h\,|_{Q_i}\colon Q_i \longrightarrow \mathcal{O}_i(X), i = 1, 2$. Hence h is a **RSiCQuant**-homomorphism. Further, for $z \in Q_0$ and $x \in X$ we observe:

$$h(z) \subseteq G_x \Leftrightarrow f^{-1}(A_z) \subseteq G_x \Leftrightarrow f(x) \notin A_z \Leftrightarrow z \leq f(x).$$

Thus for $G \in pt(O_0(X))$ we estimate from the right adjoint map $h_*(G) = \bigvee\{z \in Q : h(z) \leq G\}$ that the following relation holds:

$$pt(h) \circ \eta_X(x) = \bigvee\{z \in Q : z \leq f(x)\} = f(x).$$

Uniqueness of h follows from the observation that given another **RSiCQuant**-homomorphism

$$g : (Q_0, Q_1, Q_2) \longrightarrow (\mathcal{O}_0(X), \mathcal{O}_1(X), \mathcal{O}_2(X))$$

with the same property:

$\forall\, z \in Q_0, h(z) = (f^{-1}(A_z).$
$= (pt(g) \circ \eta_X)^{-1}(A_z)$
$= \eta_X^{-1} \circ pt(g)^{-1}(A_z)$
$= \eta_X^{-1}(A_{g(z)})$
$= g(z)$

Hence h is unique and it makes the diagram commutative. □

Definition 2.23. The coupled quantales $Q = (Q_0, Q_1, Q_2)$ is a spatial iff the total part Q_0 is spatial. Equivalently **[4]** the map

$$\varepsilon_Q^{op} : Q_0 \to \Omega_L(LPT(Q_0))$$

is a quantale isomorphism,

SpatCQuant will denote the full subcategory of the spatial coupled quantales in **CQuant**.

Lemma 2.24. *For all $Q = (Q_0, Q_1, Q_2) \in$ **CQuant**, $Q = (Q_0, Q_1, Q_2)$ is spatial if and only if the mapping*

$$\varepsilon_Q^{op} : (Q_0, Q_1, Q_2) \to \Omega_L(LPT(Q_0, Q_1, Q_2))$$

is a coupled quantale isomorphism.

Lemma 2.25. *For an $(X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \in |$**BiQS**$|$, the idempotent and right-sided coupled quantales $(\mathcal{O}_1(X) \vee \mathcal{O}_2(X), \mathcal{O}_1(X), \mathcal{O}_2(X))$ is spatial.*

Proof. Clearly the map

$$\varepsilon^{op}_{\mathcal{O}_1(X)\vee\mathcal{O}_2(X)} : (\mathcal{O}_1(X) \vee \mathcal{O}_2(X)) \to \Omega(PT(\mathcal{O}_1(X) \vee \mathcal{O}_2(X)) = A_x; x \in (\mathcal{O}_1(X) \vee \mathcal{O}_2(X))$$

is a quantale isomorphism, which implies that the quantale $\mathcal{O}_1(X) \vee \mathcal{O}_2(X)$ is a spatial and therefore, the coupled quantales

$$\Omega(X, \mathcal{O}_1(X), \mathcal{O}_2(X)) = (\mathcal{O}_1(X) \vee \mathcal{O}_2(X), \mathcal{O}_1(X), \mathcal{O}_2(X))$$

is spatial. □

Definition 2.26. An $(X, O_1(X), O_2(X)) \in |$**BiQS**$|$ is sober iff the map

$$\eta_X : (X, \mathcal{O}_1(X), \mathcal{O}_2(X)) \longrightarrow pt(\Omega(X, \mathcal{O}_1(X), \mathcal{O}_2(X))),$$

is a **BiQS**-isomorphism. **SoBiQS** will denote the full subcategory of the sober biquantum spaces in **BiQS**.

Lemma 2.27. *For an* $Q = (Q_0, Q_1, Q_2) \in |\boldsymbol{RSiCQuant}|$, *the biquantum space* $(pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})$ *is sober.*

Proof. Show bijectivity of the map

$$\eta_{PT(Q)} : PT(Q) \to PT(\Omega(PT(Q))) = PT(\mathcal{O}_{Q_1} \vee \mathcal{O}_{Q_2}, \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}) = (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})$$

For injectivity, let $p_1, p_2 \in pt(Q_0)$ with $p_1 \neq p_2$. Then there is $a \in Q_0$ with $p_1(a) \neq p_2(a)$ i.e., there is $\mathcal{O}_{Q_0}(a) \in pt(Q_0)$ such that

$$\eta_{PT(Q)}(p_1)(\mathcal{O}_{Q_0}(a)) = \mathcal{O}_{Q_0}(a)(p_1) = p_1(a) \neq p_2(a) = \mathcal{O}_{Q_0}(a)(p_2) = \eta_{PT(Q)}(p_2)(\mathcal{O}_{Q_0}(a))$$

which shows that $\eta_{PT(Q)}(p_1) \neq \eta_{PT(Q)}(p_2)$.Thus $\eta_{PT(Q)}$ is injective. To show the surjectivity of $\eta_{PT(Q)}$, let $q \in (pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})$ and put $p = q \circ \mathcal{O}_{Q_0}$. Clearly $p \in pt(Q_0)$. Furthermore, for all $a \in Q_0$, we have

$$\eta_{PT(Q)}(p)(a) = \mathcal{O}_{Q_0}(a)(p) = p(a) = q \circ \mathcal{O}_{Q_0}(a) = q(\mathcal{O}_{Q_0}(a)) \text{ So } \eta_{PT(Q)}(p) = q,$$

which means that $\eta_{PT(Q)}$ is surjective. From this it follows that $\eta_{PT(Q)}$ is bijective, so the biquantum space $(pt(Q_0), \mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})$ is sober, and the completes the proof. ☐

From the above results, we have the following theorem.

Theorem 2.28. *Sober biquantum spaces and spatial, right-sided and idempotent coupled quantales are equivalent concepts.*

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