SOME INTEGRAL TYPE WEAKLY COMPATIBLE CONTRACTION IN MODULAR METRIC SPACES

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Abstract: In this paper, we prove some fixed point theorems for integral type weakly compatible contraction in modular metric spaces which is more general than a metric. Our results generalize the results of Hossein Rahimpoor et.al., see [19].

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1. Introduction and Preliminaries

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Vijayaraju et. al. [10] extended this result for pair of mappings. Moreover, Rajaniand Moradi[11] prove the common fixed point theorem of integral type in Modular spaces. In 2010, V. V. Chistiyakov [12] introduced the more general concept of modular metric spaces. Later, many authors [13, 14, 15, 16, 17, 18] have extended and generalized the fixed point problems in modular metric spaces.

In this paper, we prove some coincidence and common fixed point theorems for a weakly contractive mappings of integral type in modular metric spaces, Our results generalize and extend the results of hossein Rahimpoor et. al. [19].

**Definition 1** (see [12]). A function $w_\lambda(x, y) : (0, \infty) \times X \times X \rightarrow [0, \infty]$ is said to be a metric modular on a non empty set $X$ if it satisfies the following three axioms:

(i) given $x, y \in X, w_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;

(iii) $w_{\lambda + \mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$. if instead of (i) we have only the condition;

(i)' $w_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then $w$ is said to be a metric pseudo-modular on $X$. And if $w$ satisfy (i)' and;

(ii)' given $x, y \in X$, if there exists $\lambda > 0$, possibly depending on $x$ and $y$ such that $w_\lambda(x, y) = 0$ then $x = y$, with the condition $w$ is called a convex modular on $X$. if instead of (iii) we replace the following condition:

(iii)' $w_{\lambda + \mu}(x, y) \leq \frac{\lambda}{\lambda + \mu} w_\lambda(x, z) + \frac{\mu}{\lambda + \mu} w_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$; then $w$ is called a convex modular on $X$.

**Example 2** (see [20]). Let $\lambda > 0$ and $x, y \in X$, then the following are simple examples of metric (pseudo) modulars on a set $X$.

(a) $w_\lambda(x, y) = \infty$ if $x \neq y, w_\lambda(x, y) = 0$ if $x = y$ and if $(X, d)$ is a (pseudo) metric space with (pseudo) metric $d$ then we also have:

(b) $w_\lambda(x, y) = \frac{d(x, y)}{\phi(\lambda)}$, where $\phi : (0, \infty) \rightarrow (0, \infty)$ is a non decreasing function;

(c) $w_\lambda(x, y) = \infty$ if $\lambda \leq d(x, y)$ and $w_\lambda(x, y) = 0$ if $\lambda > d(x, y)$;

(d) $w_\lambda(x, y) = \infty$ if $\lambda < d(x, y)$ and $w_\lambda(x, y) = 0$ if $\lambda \geq d(x, y)$;

**Remark 3** (see [19]). Let $w$ be a metric modular on a non empty set $X$ for given $x, y \in X$, the function $0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty]$ is non increasing on $(0, \infty)$. In fact, if $0 < \lambda < \mu$, then (iii), (i)' and (ii)' of above definition 1 implies that

\[ w_\mu(x, y) \leq w_{\mu - \lambda}(x, x) + w_\lambda(x, y) = w_\lambda(x, y) \text{ for all } x, y \in X. \] (1)
We conclude from above definition that if \( x_0 \in X \), the set \( X_w = \{ x \in X : \lim_{\lambda \to \infty} w_\lambda(x, x_0) = 0 \} \) is a metric space, called a modular space, whose metric is given by \( d_w(x, y) = \inf\{ \lambda > 0 : w_\lambda(x, y) \leq \lambda, x, y \in X_w \} \) and the modular set \( X_w \) is a metric space, see theorem 2.6 of [21]. We also put \( X_w^* \equiv X_w^*(x_0) = \{ x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } w_\lambda(x, x_0) < \infty \} \), and also we observe that from [21] that if \( X \) is a real linear space, \( \rho : X \to [0, \infty) \) and \( w_\lambda(x, y) = \rho(\frac{x-y}{\lambda}) \) for all \( \lambda > 0 \) and \( x, y \in X \), then \( \rho \) is a modular on \( X \) if and only if \( w \) is metric modular on \( X \).

**Definition 4.** [20] let \( X_w \) be a modular metric space.

(a) the sequence \( \{ x_n \}_{n=1}^{\infty} \) in \( X_w \) is said to be modular convergent (\( w \) – convergent) to \( x \in X \) if there exists a \( \lambda > 0 \), possibly depend on \( x_n \) and \( x \), such that \( w_\lambda(x_n, x) \to 0 \) as \( n \to \infty \) for all \( \lambda > 0 \).

(b) The sequence \( \{ x_n \}_{n=1}^{\infty} \) in \( X_w \) is said to be modular Cauchy (\( w \)–Cauchy) to \( x \in X \) if there exists a \( \lambda > 0 \) such that \( w_\lambda(x_n, x_m) \to 0 \) as \( m, n \to \infty \) for all \( \lambda > 0 \) i.e. for all \( \epsilon > 0 \) there exists \( n_0(\epsilon) \in \mathbb{N} \) such that for all \( n, m \geq n_0(\epsilon) \); \( w_\lambda(x_n, x_m) \leq \epsilon \).

(c) The modular metric space \( X_w \) is said to be complete (\( w \) – complete) if each modular Cauchy sequence from \( X_w \) is modular convergent in \( X_w \). i.e. if \( x_n \subset X_w \) and there exists \( \lambda > 0 \) such that \( w_\lambda(x_n, x_m) \to 0 \) as \( m, n \to \infty \) then there exists \( x \in X_w \) such that \( w_\lambda(x_n, x) \to 0 \).

(d)[14] A modular \( w \) on \( X \) is said to satisfy the \( \Delta_2 \)-condition if for a sequence \( \{ x_n \} \subset X_w \) and \( x \in X_w \), there exists a \( \lambda > 0 \), possibly depending on \( \{ x_n \} \) and \( x \), such that \( w_\lambda(x_n, x) \to 0 \) as \( n \to \infty \) for all \( \lambda > 0 \).

**Remark 5** (see [12]). Let \( X_w \) be a modular metric spaces, then the metric convergent (with respect \( d_w \)) implies the modular convergent i.e. \( d_w(x_n, x) \to 0 \) as \( n \to \infty \) if and only if \( w_\lambda(x_n, x) \to 0 \) as \( n \to \infty \) for all \( \lambda > 0 \).

**Definition 6** (see [19]). let \( w \) be a metric modular on a non empty set \( X \) and \( S \) and \( T \) are self maps on \( X_w \). A point \( x \in X_w \) is called a coincidence point of \( S \) and \( T \) if \( Tx = Sx \). The mappings \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence point i.e. \( TSx = STx \) Whenever \( Tx = Sx \).

Suppose \( T(X_w) \subset S(X_w) \). Let \( x_0 \) be an arbitrary point in \( X_w \). Since \( T(X_w) \subset S(X_w) \) there exist a point \( x_1 \in X_w \) such that \( Tx_0 = Sx_1 \). By repeating this process we construct the sequence \( \{ x_n \} \) in \( X_w \) such that \( Sx_n = Tx_{n-1} \) for all \( n \geq 1 \), we say that \( \{ Tx_n \} \) is a \( T - S \) sequence with initial point \( x_0 \).
2. Main Results

**Theorem 7.** Let $X_w$ be a modular metric space and $S, T : X_w \to X_w$ be two mappings such that $T(X_w) \subseteq S(X_w)$ be a $w$-complete subspaces of $X_w$. Suppose there exists numbers $\alpha, \beta, \gamma \in [0,1)$ such that the following assertion for all $x, y \in X_w$ and $\lambda > 0$ hold:

1. $(\alpha + 2\beta + 2\gamma) < 1$ for all $0 \leq \alpha, \beta, \gamma < 1$;

2. We have

$$\int_0^{w\lambda(Tx,Ty)} \phi(t) dt \leq \alpha \int_0^{w\lambda(Sx,Sy)} \phi(t) dt$$

$$+ \beta \left[ \int_0^{w\lambda(Sx,Tx)} \phi(t) dt + \int_0^{w\lambda(Sy,Ty)} \phi(t) dt \right]$$

$$+ \gamma \left[ \int_0^{2w\lambda(Sx,Ty)} \phi(t) dt + \int_0^{w\lambda(Tx,Tx)} \phi(t) dt \right],$$

where $\phi : R^+ \to R^+$ is a Lebesgue integral mapping which is summable, non-negative and for all

$$\epsilon > 0, \quad \int_0^\epsilon \phi(t) dt > 0.$$ (2)

3. If $\int_0^{w\lambda(Sx,Ty)} \phi(t) dt < \infty$, then $T$ and $S$ have a coincidence point.

**Proof.** Let $x_0$ be an arbitrary point in $X_w$. Since $T(X_w) \subseteq S(X_w)$ there exist a T-S sequence $\{Tx_n\}$ in $X_w$ such that $Sx_n = Tx_{n-1}$ for all $n \geq 1$. Now we take $x = x_n$ and $y = x_{n+1}$ in (2), we get

$$\int_0^{w\lambda(Tx_n,Tx_{n+1})} \phi(t) dt$$

$$\leq \alpha \int_0^{w\lambda(Sx_n,Sx_{n+1})} \phi(t) dt + \beta \left[ \int_0^{w\lambda(Sx_{n+1},Sx_{n+1})} \phi(t) dt + \int_0^{w\lambda(Tx_n,Tx_{n+1})} \phi(t) dt \right]$$

$$+ \gamma \left[ \int_0^{2w\lambda(Tx_{n-1},Tx_{n+1})} \phi(t) dt + \int_0^{w\lambda(Sx_{n+1},Sx_{n+1})} \phi(t) dt \right],$$

for all $\lambda > 0$.

On the other hand

$$\int_0^{w2\lambda(Tx_{n-1},Tx_{n-1})} \phi(t) dt \leq \int_0^{w\lambda(Tx_{n-1},Tx_n)} \phi(t) dt + \int_0^{w\lambda(Tx_n,Tx_{n+1})} \phi(t) dt$$
Some integral type weakly compatible...  

\[
\int_0^{\lambda} \phi(t) dt + \int_0^{\lambda} \phi(t) dt.
\]

So we obtain
\[
\int_0^{\lambda} \phi(t) dt \leq \int_0^{\lambda} \phi(t) dt + \beta \int_0^{\lambda} \phi(t) dt + \int_0^{\lambda} \phi(t) dt + \gamma \int_0^{\lambda} \phi(t) dt + \int_0^{\lambda} \phi(t) dt.
\]

this implies that
\[
\int_0^{\lambda} \phi(t) dt \leq k \int_0^{\lambda} \phi(t) dt \text{ for all } n \in N,
\]

where \( k = \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} < 1 \). So by induction we get
\[
\int_0^{\lambda} \phi(t) dt \leq k^n \int_0^{\lambda} \phi(t) dt \text{ for all } n \in N.
\]

By (2) in a straightforward way, we implies that \{ T_{x_n} \} is a \( w - Cauchy \) sequence. Since \( S(X_w) \) is a \( w - complete \), there exists \( u, v \in X_w \) such that \( u = S(v) \) and \( T_{x_n} \overset{w}{\rightarrow} u \) as \( n \to \infty \). Since \( w \) satisfy in the \( \Delta_2 \)-condition on \( X \) we get
\[
\lim_{n \to \infty} \int_0^{\lambda} \phi(t) dt = 0 \text{ for all } \lambda > 0.
\]

Therefore
\[
\lim_{n \to \infty} \int_0^{\lambda} \phi(t) dt = \int_0^{\lambda} \phi(t) dt = 0 \text{ for all } \lambda > 0. \quad (3)
\]

Now by taking \( x = x_n \) and \( y = v \) in (2), we obtain that
\[
\int_0^{\lambda} \phi(t) dt \leq \alpha \int_0^{\lambda} \phi(t) dt + \beta \int_0^{\lambda} \phi(t) dt + \int_0^{\lambda} \phi(t) dt + \gamma \int_0^{\lambda} \phi(t) dt + \int_0^{\lambda} \phi(t) dt.
\]
By Remark 3 the function \( \lambda \mapsto w_\lambda(x, y) \) is non-increasing, so we have

\[
\int_0^{w_\lambda(Tx_n, Tv)} \phi(t) dt \leq \int_0^{w_\lambda(Sx_n, Sv)} \phi(t) dt + \beta \int_0^{w_\lambda(Sx_n, Tx_n)} \phi(t) dt + \int_0^{w_\lambda(Tx_n, Tv)} \phi(t) dt + \int_0^{w_\lambda(Sx_n, Tn)} \phi(t) dt.
\]

Using (3) and letting \( n \to \infty \) in the above inequality, we get

\[
\int_0^{w_\lambda(Sv, Tv)} \phi(t) dt \leq \int_0^{w_\lambda(Sv, Sv)} \phi(t) dt + \beta \int_0^{w_\lambda(Sv, Tu)} \phi(t) dt + \int_0^{w_\lambda(Tv, Tv)} \phi(t) dt + \int_0^{w_\lambda(Sv, Sv)} \phi(t) dt.
\]

So, \( 1 - 2\beta - 2\gamma \int_0^{w_\lambda(Tv, Tv)} \phi(t) dt \leq 0 \) for all \( \lambda > 0 \), hence \( Sv = Tv = u \). Thus we have proved that \( S \) and \( T \) have a coincidence point.

**Remark 8.** If we take \( \phi(t) = 1 \) in above theorem then we get Theorem 10 of [19].

**Theorem 9.** In addition to the hypothesis of theorem 10, of [19] suppose that \( T \) and \( S \) are weakly compatible, then \( T \) and \( S \) have a unique common fixed point. Further, for any \( x_0 \in X_w \), the \( T - S \)–sequence \( \{ Tx_n \} \) with initial point \( x_0 \) modular converges to the common fixed point.

**Proof.** Assume that \( S, T \) are weakly compatible then \( S u = ST v = T S v = T u \). Now we want to show that \( T u = u = Tv \). Suppose \( \int_0^{w_\lambda(Tu, Tv)} \phi(t) dt > 0 \) for all \( \lambda > 0 \), by taking \( x = u \) and \( y = v \) in (2) we get

\[
\int_0^{w_\lambda(Tu, Tv)} \phi(t) dt \leq \int_0^{w_\lambda(Su, Sv)} \phi(t) dt + \beta \int_0^{w_\lambda(Su, Tu)} \phi(t) dt + \int_0^{w_\lambda(Tv, Tv)} \phi(t) dt + \int_0^{w_\lambda(Su, Tn)} \phi(t) dt.
\]
for all $u, v \in X_w$ and $\lambda > 0$, i.e.

\[
\int_0^{w_\lambda(Tu,Tv)} \phi(t) dt \leq \alpha \int_0^{w_\lambda(Tu,Tv)} \phi(t) dt + \beta \left[ \int_0^{w_\lambda(Tu,Tu)} \phi(t) dt + \int_0^{w_\lambda(Tv,Tv)} \phi(t) dt \right] + \gamma \left[ \int_0^{w_{2\lambda}(Tu,Tv)} \phi(t) dt + \int_0^{w_\lambda(Tv,Tu)} \phi(t) dt \right].
\]

By Remark 2 we have

\[
\int_0^{w_\lambda(Tu,Tv)} \phi(t) dt \leq \alpha \int_0^{w_\lambda(Tu,Tv)} \phi(t) dt + \gamma \left[ \int_0^{w_{2\lambda}(Tu,Tv)} \phi(t) dt + \int_0^{w_\lambda(Tv,Tu)} \phi(t) dt \right],
\]

for all $\lambda > 0$. This implies that $1 - \alpha - 2\gamma \int_0^{w_\lambda(Tu,Tv)} \phi(t) dt \leq 0$, which is a contradiction by assumption. Therefore $Su = Tu = Tv = u$, and hence $S, T$ have a common fixed point. For the uniqueness of common fixed point, Suppose that $u$ and $z$ be two common fixed points i.e. $Tu = Su = u$ and $Tv = Sz = z$.

Taking $x = u$ and $y = z$ in (2), we obtain that

\[
\int_0^{w_\lambda(Tu,Tz)} \phi(t) dt \leq \alpha \int_0^{w_\lambda(Su,Sz)} \phi(t) dt + \beta \left[ \int_0^{w_\lambda(Su,Tu)} \phi(t) dt + \int_0^{w_\lambda(Sz,Tz)} \phi(t) dt \right] + \gamma \left[ \int_0^{w_{2\lambda}(Su,Tz)} \phi(t) dt + \int_0^{w_\lambda(Sz,Tu)} \phi(t) dt \right],
\]

for all $\lambda > 0$.

So $(1 - \alpha - 2\gamma) \int_0^{w_\lambda(Tu,Tz)} \phi(t) dt \leq 0$ for all $\lambda > 0$, which is a contraction. Therefore $\int_0^{w_\lambda(u,z)} \phi(t) dt = 0$ for all $\lambda > 0$ and so $u = z$. Clearly, for any $x_0 \in X_w$, the T-S-sequence $\{ Tx_n \}$ with initial point $x_0$ converges to the unique common fixed point.

**Remark 10.** By putting $\phi(t) = 1$ in above theorem then it reduces the theorem 11 of [19]. By setting $S = IX_w$, we deduce the following result of fixed point for one self mapping from theorem 10 of [19].
Corollary 11. Let $X_w$ be a $w$–complete modular metric space and $T : X_w \rightarrow X_w$, such that for all $\lambda > 0$ and $x, y \in X_w$, $\int_0^{w_\lambda(x,Tx)} \phi(t)dt < \infty$ and $\int_0^{w_\lambda(Tx,Ty)} \phi(t)dt \leq \alpha \int_0^{w_\lambda(x,y)} \phi(t)dt + \beta [\int_0^{w_\lambda(x,Tx)} \phi(t)dt + \int_0^{w_\lambda(y,Ty)} \phi(t)dt] + \gamma [\int_0^{w_\lambda(x,Ty)} \phi(t)dt + \int_0^{w_\lambda(y,Tx)} \phi(t)dt]$, where $(\alpha + 2\beta + 2\gamma) < 1$, $0 \leq \alpha, \beta, \gamma < 1$. and $\phi : R^+ \rightarrow R^+$ is a Lebesgue integral mapping which is summable, non-negative and for all $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$. Then $T$ has a unique fixed point. Further, for any $x_0 \in X_w$, the Picard sequence \{ $T x_n$\} with initial point $x_0$ modular converges to the fixed point.

Corollary 12. Let $X_w$ be a $w$–complete modular metric space and $T : X_w \rightarrow X_w$, such that for all $\lambda > 0$ and $x, y \in X_w$, $\int_0^{w_\lambda(x,Tx)} \phi(t)dt < \infty$ and $\int_0^{w_\lambda(Tx,Ty)} \phi(t)dt \leq \alpha \int_0^{w_\lambda(x,y)} \phi(t)dt$ where $0 \leq \alpha < 1$ and $\phi : R^+ \rightarrow R^+$ is a Lebesgue integral mapping which is summable, non-negative and for all $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$. Then $T$ has a unique fixed point.

Corollary 13. Let $X_w$ be a $w$–complete modular metric space and $T : X_w \rightarrow X_w$, such that for all $\lambda > 0$ and $x, y \in X_w$, $\int_0^{w_\lambda(x,Tx)} \phi(t)dt < \infty$ and $\int_0^{w_\lambda(Tx,Ty)} \phi(t)dt \leq \beta [\int_0^{w_\lambda(x,Tx)} \phi(t)dt + \int_0^{w_\lambda(y,Ty)} \phi(t)dt]$ where $\beta \in [0, \frac{1}{2}]$ and $\phi : R^+ \rightarrow R^+$ is a Lebesgue integral mapping which is summable, non-negative and for $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$. then $T$ has a unique fixed point.

Theorem 14. Let $X_w$ be a modular metric space and $S, T : X_w \rightarrow X_w$ be two mappings such that $T(X_w) \subseteq S(X_w)$ be a $w$–complete subspace of $X_w$. Suppose there exists mappings $\alpha, \beta, \gamma, \mu : [0, 1)$ such that the following assertions for all $x, y \in X_w$ and $\lambda > 0$ holds:

1. $\alpha(Tx) \leq \alpha(Sx), \beta(Tx) \leq \beta(Sx), \gamma(Tx) \leq \gamma(Sx), \mu(Tx) \leq \mu(Sx)$;
2. $(\alpha + 2\beta + \gamma + \mu) < 1$ for all $0 \leq \alpha, \beta, \gamma, \mu < 1$;
3. $\int_0^{w_\lambda(Tx,Ty)} \phi(t)dt \leq \alpha \int_0^{w_\lambda(Sx, Sy)} \phi(t)dt + \beta \int_0^{w_\lambda(Sx,Tx)} \phi(t)dt + \gamma \int_0^{w_\lambda(Sy,Ty)} \phi(t)dt$ and
4. $\int_0^{w_\lambda(Sx,Ty)} \phi(t)dt < \infty$ where $\phi : R^+ \rightarrow R^+$ is a Lebesgue integrable mapping which is summable, non-negative and for all $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$. Then $T$ and $S$ have a unique common fixed point.

Proof. Let $x_0$ be an arbitrary point in $X_w$. Since $T(X_w) \subseteq S(X_w)$ there exist a $T - S$ sequence \{ $T x_n$\} in $X_w$ such that $Sx_n = Tx_{n-1}$ for all $n \geq 1$. . . . . . (4)
From (1), (3) and (4) we have

\[ \int_0^{\frac{w_\lambda(Sx_n, Sx_{n+1})}{\phi(t)}} \phi(t) dt = \int_0^{\frac{w_\lambda(Tx_{n-1}, Tx_n)}{\phi(t)}} \phi(t) dt \]

\[ \leq \alpha(Sx_{n-1}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt + \beta(Sx_{n-1}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Tx_{n-1})}{\phi(t)}} \phi(t) dt \]

\[ + \gamma(Sx_{n-1}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Tx_{n-1})}{\phi(t)}} \phi(t) dt + \mu(Sx_{n-1}) \int_0^{\frac{w_\lambda(Sx_n, Tx_n)}{\phi(t)}} \phi(t) dt \]

\[ = \alpha(Tx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt + \beta(Tx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_{n+1})}{\phi(t)}} \phi(t) dt \]

\[ + \gamma(Tx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_{n+1})}{\phi(t)}} \phi(t) dt + \mu(Tx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_{n+1})}{\phi(t)}} \phi(t) dt \]

\[ \leq \alpha(Sx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt + \beta(Sx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-2}, Sx_{n-1})}{\phi(t)}} \phi(t) dt \]

\[ + \gamma(Sx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-2}, Sx_{n-1})}{\phi(t)}} \phi(t) dt + \mu(Sx_{n-2}) \int_0^{\frac{w_\lambda(Sx_{n-2}, Sx_{n+1})}{\phi(t)}} \phi(t) dt \]

\[ \vdots \]

\[ \leq \alpha(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt + \beta(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_{n+1})}{\phi(t)}} \phi(t) dt \]

\[ + \gamma(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt + \mu(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt \]

\[ \leq \alpha(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt + \beta(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt \]

\[ + \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_{n+1})}{\phi(t)}} \phi(t) dt + \gamma(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_{n+1})}{\phi(t)}} \phi(t) dt \]

\[ + \mu(Sx_0) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_{n+1})}{\phi(t)}} \phi(t) dt. \]

This implies that

\[ \int_0^{\frac{w_\lambda(Sx_n, Sx_{n+1})}{\phi(t)}} \phi(t) dt \leq \left( \frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} \right) \int_0^{\frac{w_\lambda(Sx_{n-1}, Sx_n)}{\phi(t)}} \phi(t) dt, \]

for all \( n \geq 1. \)

Now we let

\[ k = \frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} < 1. \]
Repeating in (5), we get
\[
\int_0^{w_\lambda(Sx_n, Sx_{n+1})} \phi(t) dt + \leq k^n \int_0^{w_\lambda(Sx_1, Sx_0)} \phi(t) dt.
\] (6)

Now, for \(m > n \geq 1\), it follows from (6) that
\[
\int_0^{w_\lambda(Sx_n, Sx_m)} \phi(t) dt \leq \int_0^{w_\lambda(Sx_n, Sx_{n+1})} \phi(t) dt + \int_0^{w_\lambda(Sx_{n+1}, Sx_{n+2})} \phi(t) dt + \ldots + \int_0^{w_\lambda(Sx_{m-1}, Sx_m)} \phi(t) dt + \int_0^{w_\lambda(Sx_m, Sx_{m+1})} \phi(t) dt
\]
\[
\leq (k^n + k^{n+1} + \ldots + k^{m-1}) \int_0^{w_\lambda(Sx_0, Sx_1)} \phi(t) dt
\]
\[
= \frac{k^n - k^m}{1 - k} \int_0^{w_\lambda(Sx_0, Sx_1)} \phi(t) dt.
\]

Since \(0 \leq k < 1\), we conclude that \(\{Sx_n\}\) is a \(w-Cauchy\) sequence in \(S(X_w)\), on the other hand by hypothesis \(S(X_w)\) is a \(w-complete\) subspace of \(X_w\), therefore there exists a point \(u \in S(X_w)\) such that \(Sx_n \xrightarrow{w} u\) as \(n \to \infty\). By hypothesis \(w\) satisfy in \(\Delta_2-condition\) on \(X_w\), so \(\lim_{n \to \infty} \int_0^{w_\lambda(Sx_n, u)} \phi(t) dt = 0\) for all \(\lambda > 0\).

Now, we claim that \(u\) is a common fixed point. Suppose that \(Tu \neq u\) or \(Su \neq u\). Then we have
\[
0 < \inf \{ \int_0^{w_\lambda(Tx_n, u)} \phi(t) dt + \int_0^{w_\lambda(Sx_n, u)} \phi(t) dt + \int_0^{w_\lambda(Tx, Sx_n)} \phi(t) dt : x \in X_w \}
\]
\[
\leq \inf \{ \int_0^{w_\lambda(Tx_n, u)} \phi(t) dt + \int_0^{w_\lambda(Sx_n, u)} \phi(t) dt + \int_0^{w_\lambda(Tx_n, Sx_n)} \phi(t) dt : n \geq 1 \}
\]
\[
= \inf \{ \int_0^{w_\lambda(Sx_{n+1}, u)} \phi(t) dt + \int_0^{w_\lambda(Sx_n, u)} \phi(t) dt + \int_0^{w_\lambda(Sx_n, Sx_{n+1})} \phi(t) dt \}
\]
\[
\leq \inf \{ \int_0^{w_\lambda(Sx_{n+1}, u)} \phi(t) dt + \int_0^{w_\lambda(Sx_n, u)} \phi(t) dt + \int_0^{w_\lambda(Sx_n, u)} \phi(t) dt + \int_0^{w_\lambda(u, Sx_{n+1})} \phi(t) dt : n \geq 1 \} \to 0
\]
\[
+ \int_0^{w_\lambda(u, Sx_{n+1})} \phi(t) dt : n \geq 1 \} \to 0
\]
as \( n \to \infty \) for all \( \lambda > 0 \), which is a contraction. Therefore, this implies that \( u = Su = Tu \).

For the uniqueness of the common fixed point, Suppose \( Tu = Su = u \)
\( Tz = Sz = z \) be two common fixed points then by taking \( x = u \) and \( y = z \) in
(3), we obtain that
\[
\int_0^{w_\lambda(u,z)} \phi(t)dt \leq \alpha(u) \int_0^{w_\lambda(u,z)} \phi(t)dt + \beta(u) \int_0^{w_\lambda(u,z)} \phi(t)dt
+ \gamma(u) \int_0^{w_\lambda(u,u)} \phi(t)dt + \mu(u) \int_0^{w_\lambda(z,z)} \phi(t)dt
\leq \alpha(u) \int_0^{w_\lambda(u,z)} \phi(t)dt + \beta(u) \int_0^{w_\lambda(u,z)} \phi(t)dt.
\]

This implies that \((1 - \alpha - \beta)u \int_0^{w_\lambda(u,z)} \phi(t)dt \leq 0 \) which is a contraction
by assumption. By taking the mapping \( S \) in theorem 14 as \( IX_w \),
where \( IX_w \) is
an identity mapping on \( X_w \), we have the following corollary.

**Corollary 15.** Let \( X_w \) be a \( w \)-complete modular metric space and
\( T : X_w \to X_w \). Suppose that there exists mappings \( \alpha, \beta, \gamma, \mu : X_w \to [0,1) \) such
that the following assertion for all \( x, y \in X_w \) and \( \lambda > 0 \) hold:

1. \( \alpha(Tx) \leq \alpha(x), \beta(Tx) \leq \beta(x), \gamma(Tx) \leq \gamma(x), \mu(Tx) \leq \mu(x) \);

2. \( (\alpha + 2\beta + \gamma + \mu) < 1 \) for all \( 0 \leq \alpha, \beta, \gamma, \mu < 1 \);

3. \( \int_0^{w_\lambda(Tx,Ty)} \phi(t)dt \leq \alpha(x) \int_0^{w_\lambda(x,y)} \phi(t)dt + \beta(x) \int_0^{w_\lambda(x,y)} \phi(t)dt
+ \gamma(x) \int_0^{w_\lambda(Tx,Ty)} \phi(t)dt + \mu(x) \int_0^{w_\lambda(y,y)} \phi(t)dt; \)

4. \( \int_0^{w_\lambda(x,Ty)} \phi(t)dt < \infty \) where \( \phi : R^+ \to R^+ \) is a Lebesgue integrable
mapping which is summable, non negative and for all \( \epsilon > 0 \) \( \int_0^\epsilon \phi(t)dt > 0 \). Then
\( T \) has a unique common fixed point.

**References**


Berlin, 1983.


