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A COUPLED SYSTEM OF FUNCTIONAL DIFFERENTIAL EQUATIONS IN REFLEXIVE BANACH SPACES

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Abstract: We present an existence theorem for at least one weak solution for the coupled system of functional differential equations

$$\begin{aligned} x'(t) &= f_1(t, y'(t)),\ t \in (0, T], \\ y'(t) &= f_2(t, x'(t)),\ t \in (0, T] \end{aligned}$$

in reflexive Banach spaces.

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1. Introduction and Preliminaries

Let E be a reflexive Banach space with norm $\| \,.\, \|$ and dual E^*, and $L^1(I)$ be the space of Lebesgue integrable functions on the interval $I = [0, T]$. Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm $\| \,.\, \|_0$.

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Such systems appear in many problems of applied nature (see [1],[3], [4],[9]-[11] and [13]). Su [18] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [11] analyzed the solutions of coupled nonlinear fractional reaction-diffusion equations. The solvability of the coupled systems of integral equations in reflexive Banach space was proved in (see [5]-[7]).

In [15], O'Regan studied, in the reflexive Banach space, the existence of weak solutions of the initial value problem

$$\frac{d}{dt}x(t) \;=\; f(t, x(t)), \quad t \in (0, T].$$

Recently, the authors studied the existence of weak solutions of the initial value problem

$$\frac{d}{dt}x(t) \;=\; f(t, \frac{d}{dt}x(t)), \quad t \in (0, T].$$

in reflexive and nonreflexive Banach spaces (see [8]).

In this paper we study the existence of weak solutions for the coupled system of the functional differential equations

$$\frac{dx}{dt} = f_1(t, \frac{dy}{dt}),\ x(0) = x_0,\ t \;\in\; (0, T] \tag{1}$$

$$\frac{dy}{dt} = f_2(t, \frac{dx}{dt}),\ y(0) = y_0,\ t \;\in\; (0, T] \tag{2}$$

in the reflexive Banach space E. For this aim we study, firstly, the existence of weak solutions for the coupled system of the functional equations

$$u(t) = f_1(t, v(t)),\ t \in I \tag{3}$$

$$v(t) = f_2(t, u(t)),\ t \in I \tag{4}$$

in the reflexive Banach space E.

Now, we shall present some auxiliary results that will be needed in this work.

(1) A function $h : E \to E$ is said to be weakly Lipschitz if for every $\phi \in E^*$ there exists a positive constant K such that

$$\phi(h(x(.)) - h(y(.))) \leq K\ \phi(x(.) - y(.)).$$

(2) $h(.)$ is said to be weakly continuous if for every $\phi \in E^*$, $\phi(h(.))$ is continuous see ([12] and [14]).

(3) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

It is clear that (1) implies (2) and (2) implies (3). If h linear, then (2) and (3) are equivalent. The relation between weak and weak sequentially continuous of mapping is studied in details in [2].
Now, we have the following theorem due to Rubin (see [16]) and some propositions which will be used in the sequel (see [7] and [17]).

Theorem 1. *If E is metrizable (i.e., the topology is induced by a metric). Then the weakly sequentially continuous functions are weakly continuous.*

Proposition 1. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

The following result follows directly from the Hahn-Banach theorem.

Proposition 2. *Let E be a normed space with $x_0 \neq 0$. Then there exists a $\phi \in E^*$ with*
$\| \phi \| = 1$ and $\phi(x_0) = \| x_0 \|$.

Also, we have the following fixed point theorem, due to O'Regan, in reflexive Banach space (see [14]).

Theorem 2. *(O'Regan fixed point theorem) Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space E and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.*

2. Coupled System of Functional Equations

Here, we discuss the existence of weak solutions for the coupled system (3)-(4) in the reflexive Banach space E.
The coupled system (3)-(4) will be investigated under the following assumptions :
$f_i : I \times E \to E, \quad i = 1, 2$ such that

(i) For each $u \in C[I, E]$, $f_i(., u(.))$ are continuous on I;

(ii) For each $t \in I$, $f_i(t, .)$ are weakly Lipschitz with Lipschitz constants K_i,

where $a_i = \sup\{\phi(f_i(t,0)) : t \in I\}$.
Now, let X be the class of all ordered pairs $(u,v),\ u,v \in C[I,E]$ with the norm

$$||(u,v)|| = ||u|| + ||v||.$$

Definition 1. By a solution to the coupled system (3)-(4), we mean the pair of functions $(u,v) \in X,\ u,v \in C[I,E]$ which satisfies (3)-(4) weakly. This is equivalent to finding $(u,v) \in X,\ u,v \in C[I,E]$ with

$$\phi(u(t)) = \phi(f_1(t,v(t))),\ t \in I$$

$$\phi(v(t)) = \phi(f_2(t,u(t))),\ t \in I$$

for all $\phi \in E^*$.

Now, we can prove the following existence theorem.

Theorem 3. *Under the assumptions (i)-(ii), the coupled system (3)-(4) has at least one weak solution* $(u,v) \in X,\ \ u,v \in C[I,E]$.

Proof. Define the operator A by

$$A(u(t),v(t)) = (A_1v(t), A_2u(t))$$

where

$$A_1v(t) = f_1(t,v(t)),\ t \in I$$
$$A_2u(t) = f_2(t,u(t)),\ t \in I.$$

Now, define the set Ω by

$$\Omega = \{(u,v),\ \ u,v \in C[I,E] :\| v \|_0 \leq r_1,\ \ \| u \|_0 \ \leq\ r_2,\ r = r_1 + r_2\}.$$

The reminder of the proof will be given in four steps.

Step 1: The operator A maps X into itself. For this, let $u,v \in C[I,E]$. Let $t_1,\ t_2 \in I,\ t_2 > t_1$,(without loss of generality assume that $A(u,v)(t_2) - A(u,v)(t_1) \neq 0$), then

$$\begin{aligned} A_1v(t_2) - A_1v(t_1) &= f_1(t_2,v(t_2)) - f_1(t_1,v(t_1)) \\ &= f_1(t_2,v(t_2)) - f_1(t_1,v(t_1)) \\ &- f_1(t_2,v(t_1)) + f_1(t_2,v(t_1)) \end{aligned}$$

by proposition 2 we have

$$\| A_1v(t_2) - A_1v(t_1) \| = \phi(A_1v(t_2) - A_1v(t_1))$$

$$\begin{aligned}
&\leq \phi(f_1(t_2, v(t_2)) - f_1(t_2, v(t_1))) \\
&+ \phi(f_1(t_2, v(t_1)) - f_1(t_1, v(t_1))) \\
&\leq K_1 \phi(v(t_2) - v(t_1)) \\
&+ \phi(f_1(t_2, v(t_1)) - f_1(t_1, v(t_1))) \\
&\leq K_1 \parallel v(t_2) - v(t_1) \parallel \\
&+ \parallel f_1(t_2, v(t_1)) - f_1(t_1, v(t_1)) \parallel .
\end{aligned}$$

Similarity, we can show that

$$\begin{aligned}
\parallel A_2u(t_2) - A_2u(t_1) \parallel &\leq K_2 \parallel u(t_2) - u(t_1) \parallel \\
&+ \parallel f_2(t_2, u(t_1)) - f_2(t_1, u(t_1)) \parallel .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\parallel A(u,v)(t_2) - A(u,v)(t_1) \parallel &= \parallel (A_1v(t_2), A_2u(t_2)) \\
&- (A_1v(t_1), A_2u(t_1)) \parallel \\
&= \parallel (A_1v(t_2) - A_1v(t_1), A_2u(t_2) \\
&- A_2u(t_1)) \parallel \\
&= \parallel A_1v(t_2) - A_1v(t_1) \parallel \\
&+ \parallel A_2u(t_2) - A_2u(t_1)) \parallel \\
&\leq K_1 \parallel v(t_2) - v(t_1) \parallel \\
&+ \parallel f_1(t_2, v(t_1)) - f_1(t_1, v(t_1)) \parallel \\
&+ K_2 \parallel u(t_2) - u(t_1) \parallel \\
&+ \parallel f_2(t_2, u(t_1)) - f_2(t_1, u(t_1)) \parallel .
\end{aligned}$$

Which prove that $A : X \to X$.

Step 2: The operator A maps Ω into itself. Let $(u, v) \in \Omega$ then by proposition 2 we have

$$\begin{aligned}
\parallel A_1v(t) \parallel &= \phi(A_1v(t)) = \phi(f_1(t, v(t))) \\
&\leq \phi(f_1(t, 0)) + K_1\phi(v(t)) \\
&\leq a_1 + K_1 \parallel v \parallel_0 \\
&\leq a_1 + K_1r_1.
\end{aligned}$$

Then

$$\parallel A_1v \parallel_0 = \sup_{t\in I} \parallel A_1v(t) \parallel \leq a_1 + K_1r_1 = r_1, \ r_1 = \frac{a_1}{1 - K_1}.$$

Similarity, we can show that

$$\| A_2u \|_0 \leq a_2 + K_2r_2 = r_2, \ r_2 = \frac{a_2}{1-K_2}.$$

Now

$$\| A(u,v)(t) \| = \| (A_1v(t), A_2u(t)) \| \leq \| A_1v(t) \| + \| A_2u(t) \| \leq r.$$

Then

$$\| A(u,v) \|_0 \leq r.$$

Hence, for any $(u,v) \in \Omega$ we have $A(u,v) \in \Omega$, consequently $A\Omega \subset \Omega \Rightarrow A : \Omega \to \Omega$.

Step 3: $A\Omega(t)$ is relatively weakly compact in E. Note that Ω is nonempty, closed, convex and bounded subset of X. According to proposition 1, $A\Omega$ is bounded in X and closed in weak topology, hence $A\Omega$ is relatively weakly compact in X implies $A\Omega(t)$ is relatively weakly compact in E for each $t \in I$.

Step 4: The operator A is weakly sequentially continuous on Ω. Let $\{V_n\}$ be sequence in Ω converges weakly to V, then we have the two sequences $\{v_n(t)\}$, $\{u_n(t)\}$ converges weakly to $v(t), u(t)$, respectively for all $t \in I$. Since $f_1(t, v(t)), f_2(t, u(t))$ are weakly Lipschitz, then $f_1(t, v(t)), f_2(t, u(t))$ are weakly continuous, then we have that $f_1(t, v(t)), f_2(t, u(t))$ are weakly sequentially continuous in the second argument. Thus $f_1(t, v_n(t)), f_2(t, u_n(t))$ converges weakly to $f_1(t, v(t)), f_2(t, u(t))$ respectively, then $\phi(f_1(t, v_n(t)))$,$\phi(f_2(t, u_n(t)))$converges strongly to $\phi(f_1(t, v(t))), \phi(f_2(t, u(t)))$ respectively. Thus

$$\phi(A_1v_n(t)) \to \phi(A_1v(t)), \quad \phi(A_2u_n(t)) \to \phi(A_2u(t)),$$

i.e. $\| A_1v_n(t) \| \to \| A_1v(t) \|$ and $\| A_2u_n(t) \| \to \| A_2u(t) \|$.

Therefore,

$$\begin{aligned} \| AV_n(t) \| &= \| (A_1v_n(t), A_2u_n(t)) \| \\ &= \| A_1v_n(t) \| + \| A_2u_n(t) \| \\ &\to \| A_1v(t) \| + \| A_2u(t) \| \\ &\to \| (A_1v(t), A_2u(t)) \| \\ &\to \| AV(t) \| . \end{aligned}$$

This means that $\phi(AV_n(t)) \to \phi(AV(t)), \forall \phi \in E^*, t \in I$. Thus, $A : \Omega \to \Omega$ is weakly sequentially continuous.

Since all conditions of Theorem 2 are satisfied, then the operator A has at least one fixed point $(u,v) \in \Omega$, then the coupled system (3)-(4) has at least one weak solution. $\square$

3. Initial Value Problem

Consider now the coupled system of the initial value problems (1)-(2)

Theorem 4. *Under the assumptions of Theorem 3, the coupled system of the initial value problems (1)-(2) has at least one solution* $(x, y) \in X$.

Proof. Let $\frac{dx}{dt} = u \in C[I, E]$ and $\frac{dy}{dt} = v \in C[I, E]$, then

$$x(t) = x_0 + \int_0^t u(s)ds \in C[I, E],$$

and

$$y(t) = y_0 + \int_0^t v(s)ds \in C[I, E],$$

are differentiable and $(u(t), v(t))$ will be the solution of the coupled system of the functional equations

$$u(t) = f_1(t, v(t)), \quad t \in [0, T],$$

$$v(t) = f_2(t, u(t)), \quad t \in [0, T].$$

□

References

[1] Bashir Ahmad, Juan J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Computers and Mathematics with Applications.*, **58** (2009), 1838-1843.

[2] J.M. Ball, Weak continuity properties of mapping and semi-groups, *Proc. Royal Soc.Edinbourgh Sect.*, A**72** (1973), 275-280.

[3] C. Bai, J. Fang, The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations, *Appl. Math. Comput*, **150** (2004), 611-621.

[4] Y. Chen, H. An, Numerical solutions of coupled Burgers equations with time and space fractional derivatives, *Appl. Math. Comput*, **200** (2008), 87-95.

[5] A.M.A. EL-Sayed, H.H.G. Hashem, A coupled systems of fractional order integral equations in reflexive Banach spaces, *Commentationes Mathematicae*, No.**1** (2012) 21-28.

[6] A.M.A. El-Sayed, H.H.G. Hashem, Coupled systems of integral equations in reflexive Banach spaces, *Acta Math. Sci* **32**B(5) (2012), 1-8.

[7] A.M.A. EL-Sayed, H.H.G. Hashem, Coupled systems of Hammerstein and Urysohn integral equations in reflexive Banach spaces, *Differential Equation and Control Processes*, No.1(2012) 1-12.

[8] A.M.A. EL-Sayed, W. G. El-Sayed, A. A. H. Abd El-Mowla, On the existence of weakly continuous solutions of the nonlinear functional equations and functional differential equations in reflexive and nonreflexive Banach spaces, *Differential Equation and Control Processes*. No.1(2016) 32-39.

[9] V. Gafiychuk, B. Datsko, V. Meleshko, Mathematical modeling of time fractional reactiondiffusion systems, *J. Comput. Appl. Math.* **220** (2008), 215-225.

[10] V.D. Gejji, Positive solutions of a system of non-autonomous fractional differential equations, *J. Math. Anal. Appl*, **302** (2005), 56-64.

[11] V. Gafiychuk, B. Datsko, V. Meleshko, D. Blackmore, Analysis of the solutions of coupled nonlinear fractional reaction diffusion equations, *Chaos Solitons Fract*, **41** (2009), 1095-1104.

[12] W.J. Knight, Solutions of differential equations in B-spaces, *Duke Math. J*, **41** (1974), 437-442.

[13] M.P. Lazarevic, Finite time stability analysis of PD^{α} fractional control of robotic timedelay systems, *Mech. Res. Comm*, **33** (2006), 269-279.

[14] D. O'Regan, Fixed point theory for weakly sequentially continuous mapping, *Math. Comput. Modeling*. **27** (5)(1998), 1-14.

[15] D. O'Regan, Weak solutions of ordinary differential equations in Banach spaces, *Appl.Math. Lett* ., **12** (1999), 101-105.

[16] W. Rudin, *Functional Analysis*, McGraw-Hill, Inc. New york, Harlow(1991).

[17] H. A. H. Salem, A. M. A. El-Sayed, Weak solution for fractional order integral equations in reflexive Banach spaces, *Math. Slovaca*, **55** No.2(2005), 169-181.

[18] X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, *Appl. Math. Lett.* **22** (2009), 64-69.