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ASYMPTOTICALLY ω-PERIODIC SOLUTION FOR AN EVOLUTION DIFFERENTIAL EQUATION VIA ω-PERIODIC LIMIT FUNCTIONS

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Abstract: In this paper, we give sufficient conditions for the existence and uniqueness of asymptotically ω-periodic solutions for an evolution differential equation considering the class of ω-periodic limit functions. This is done using the Banach Fixed Point Theorem.

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1. Introduction

The existence of periodic solutions and its various extensions to abstract Cauchy problem has been studied in Several works([6]-[9]). Recently, the concept of ω-periodic limit functions has been introduced by Xie and Zhang in [10]. The class of ω-periodic limit functions generalize asymptotically ω-periodic func-

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tions ([11],[12]) in a different way from S-asymptotically ω-periodic functions ([1]-[5]) and have some relationship with asymptotically ω-periodic functions in the Stepanov sense ([6],[9]). The concept of S-asymptotically ω-periodic have many applications in functional differential equations, integro-differential equations, fractionnal differential equation. In [10], Xie and Zhang investigate some properties of ω-periodic limit functions in order to study the existence and uniqueness of asymptotically ω-periodic mild solutions of the following abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t))dt, \\ x(0) = c_0, \end{cases}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$. The aim of this work is to investigate properties of ω-periodic limit functions and to study the existence of asymptotically ω-periodic mild solution to the semilinear differential equation.

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t))dt, \\ x(0) = c_0, \end{cases} \tag{1}$$

where $c_0 \in \mathbb{X}$, and $A(t)$ generates an exponentially stable ω-perioic evolutionary family in $\mathbb{X}$. The paper is organised as follows. We recall in section 2, definitions and properties of ω-periodic limit functions. In section 2, we present also new properties of ω-periodic functions. In section 3, we study the existence and uniqueness of asymptotically ω-periodic solution of the equation (1). This result is obtained, using the Banach fixed point theorem.

2. Preliminaries

Let $\mathcal{C}_b(\mathbb{R}^+, \mathbb{X})$ the space consisting of bounded and continuous functions from $\mathbb{R}^+$ into $\mathbb{X}$, endowed with the uniform convergence norm $||\cdot||_\infty$. Set $C_0(\mathbb{R}^+, \mathbb{X}) = \{f \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{X}) = \lim_{t\to\infty} ||f(t)|| = 0\}$ and $P_\omega(\mathbb{R}^+, \mathbb{X}) = \{f \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{X}) : f \; is \; \omega - periodic\}$.

Definition 2.1. A function $f \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{X})$ is said to be asymptotically ω-periodic if it can be expressed as $f = g + h$, where $g \in P_\omega(\mathbb{R}^+, \mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$. The subspace of $\mathcal{C}_b(\mathbb{R}^+, \mathbb{X})$ consisting of the asymptotically ω-periodic functions will be denoted by $AP_\omega(\mathbb{R}^+, \mathbb{X})$.

Definition 2.2. Let $f \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{X})$ and $\omega > 0$. We call f ω-periodic limit if $g(t) = \lim_{n\to\infty} f(t + n\omega)$ is well defined for each $t \in \mathbb{R}^+$, where $n \in \mathbb{N}$. The collection of such functions will be denoted by $P_\omega L(\mathbb{R}^+, \mathbb{X})$.

Remark 2.1. The function g is measurable but not necessarily continuous.

Proposition 2.1. *(see [10]) If f, f_1 and f_2 are ω-periodic limit and $g(t) = \lim_{n\to\infty} f(t+n\omega)$ is well defined for each $t \in \mathbb{R}^+$, then the following statements are true:*

(1) $f_1 + f_2$ is ω-periodic limit;

(2) cf is ω-periodic limit for every scalac c;

(3) $g(t+\omega) = g(t)$ for each $t \in \mathbb{R}^+$;

(4) g is bounded on $\mathbb{R}^+$; moreover $||g||_\infty \leq ||f||_\infty$;

(5) $f_a(t) = f(t+a)$ is ω-periodic limit for each fixed $a \in \mathbb{R}^+$.

Theorem 2.2. *(see [10]) $P_\omega L(\mathbb{R}^+, \mathbb{X})$ is a Banach space.*

Proposition 2.3. *(see [10]) Let $f \in P_\omega L(\mathbb{R}^+, \mathbb{X})$ and $g(t) = \lim_{n\to\infty} f(t+n\omega)$ be well defined for each $t \in \mathbb{R}^+$. If $g(t) = \lim_{n\to\infty} f(t+n\omega)$ uniformly on $[0,\omega]$, then $f \in AP_\omega(\mathbb{R}^+, \mathbb{X})$.*

Corollary 2.4. *(see [10]) Let $f \in P_\omega L(\mathbb{R}^+, \mathbb{X})$ and $g(t) = \lim_{n\to\infty} f(t+n\omega)$ be well defined for each $t \in \mathbb{R}^+$. If $g(t) = \lim_{n\to\infty} f(t+n\omega)$ uniformly on $\mathbb{R}^+$, then $f \in AP_\omega(\mathbb{R}^+, \mathbb{X})$.*

Let $p \in [1, \infty[$. The space $BS^p(\mathbb{R}^+, \mathbb{X})$ of all Stepanov bounded functions, with the exponent p,consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{X}$ such that $f^b \in \mathbb{L}^\infty(\mathbb{R}, L^p([0,1]; \mathbb{X}))$, where f^b is the Bochner transform of f defined by $f^b(t,s) := f(t+s), t \in \mathbb{R}^+, s \in [0,1]$. $BS^p(\mathbb{R}^+, X)$ is a Banach space with the norm

$$||f||_{S^p} = ||f^b||_{\mathbb{L}^\infty(\mathbb{R}^+, L^p)} = \sup_{t\in\mathbb{R}^+} \Big(\int_t^{t+1} ||f(\tau)||^p d\tau \Big).$$

It is obvious that $L^p(\mathbb{R}, \mathbb{X}) \subset BS^p(\mathbb{R}, \mathbb{X}) \subset L^p_{loc}(\mathbb{R}, \mathbb{X})$ and $BS^p(\mathbb{R}, \mathbb{X}) \subset BS^q(\mathbb{R}, \mathbb{X})$ for $p \geq q \geq 1$. We denote by $BS^p_0(\mathbb{R}^+, \mathbb{X})$ the subspace of $BS^p(\mathbb{R}^+, \mathbb{X})$ consisting of functions f such that $\int_t^{t+1} ||f(s)||^p ds \to 0$ when $t \to \infty$.

Define the subspaces of $BS^p(\mathbb{R}^+\mathbb{X})$ by

$$S^pP_\omega(\mathbb{R}^+, X) = \left\{ f \in BS^p(\mathbb{R}^+, \mathbb{X}) : \int_t^{t+1} ||f(s+\omega) - f(s)||^p ds = 0, t \in \mathbb{R}^+ \right\}$$

and

$$BS_0^p(\mathbb{R}^+,\mathbb{X}) = \left\{ f \in BS^p(\mathbb{R}^+,\mathbb{X}) : \lim_{t\to\infty} \int_t^{t+1} ||f(s)||^p ds = 0 \right\}.$$

Definition 2.3. (see [9]) A function $f \in BS^p(\mathbb{R}^+,\mathbb{X})$ is called asymptotically ω-periodic in the Stepanov sense if it can be expressed as $f = g + h$, where $g \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ and $h \in BS_0^p(\mathbb{R}^+,\mathbb{X})$. The collection of such functions will be denoted by $S^pAP_\omega(\mathbb{R}^+,\mathbb{X})$.

Remark 2.2. $AP_\omega(\mathbb{R}^+,\mathbb{X}) \subset P_\omega L(\mathbb{R}^+,\mathbb{X}) \subset S^pP_\omega(\mathbb{R}^+,X)$

Theorem 2.5. *Let $\Phi : \mathbb{X} \to \mathbb{Y}$ be a function wich is uniformly continous on the bounded subsets of $\mathbb{X}$ and such that Φ maps bounded subsets of $\mathbb{X}$ into bounded subsets of $\mathbb{Y}$. Then for all $f \in P_\omega L(\mathbb{R}^+,\mathbb{X})$, the composition theorem $\Phi \circ f := [t \to \Phi(f(t))] \in P_\omega L(\mathbb{R}^+,\mathbb{Y})$.*

Proof. Since the range of f is bounded, we deduce that $\Phi(f(\cdot))$ is bounded. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $||\Phi(x) - \Phi(y)|| < \epsilon$ for all $x, y \in f(\mathbb{R}^+)$ with $||x - y|| \leq \delta$. Since $\delta > 0$, there exists $N = N(\delta) \in \mathbb{N}$ such that if $n \geq N$, $||f(t+n\omega) - g(t)|| < \delta$ for each $t \in \mathbb{R}^+$. Therefore, when $n \geq N$, $||\Phi(f(t+n\omega) - \Phi(g(t))|| < \epsilon$. □

Proposition 2.6. *Let $(\mathbb{X}, ||\cdot||)$ be a banach space over the field $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $a(t) \in PL_\omega(\mathbb{R}^+,\mathbb{R})$ and $f(t) \in PL_\omega(\mathbb{R}^+,\mathbb{X})$, then $a(t)f(t) \in PL_\omega(\mathbb{R}^+,\mathbb{X})$.*

Proof. We have $\lim_{n\to\infty} a(t+n\omega) = b(t)$ and $\lim_{n\to\infty} f(t+n\omega) = g(t)$ for all $t \geq 0$. Since $a(t)$ and $f(t)$ are bounded, there exists $M_1, M_2 \in \mathbb{R}^+$ such that $|a(t)| \leq M_1$ and $||f(t)|| \leq M_2$, for all $t \geq 0$. Since

$$||a(t+n\omega)f(t+n\omega) - b(t)g(t)|| \leq ||a(t+n\omega) - b(t)||\, M_1 + ||f(t+n\omega) - g(t)||\, M_2,$$

we deduce that $\lim_{n\to\infty} a(t+n\omega)f(t+n\omega) = b(t)g(t)$. □

Definition 2.4. A jointly continuous function $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is ω-periodic limit in $t \in \mathbb{R}^+$ uniformly for x in bounded subsets of $\mathbb{X}$ if for every bounded subset K of $\mathbb{X}$, $\{f(t,x) : t \in \mathbb{R}^+, x \in K\}$ is bounded and $\lim_{n\to\infty} f(t + n\omega, x) = g(t,x)$ exists for each $t \in \mathbb{R}^+$ and each $x \in K$. The collection of such functions will be denoted by $P_\omega L(\mathbb{R}^+ \times \mathbb{X},\mathbb{X})$.

Theorem 2.7. *If $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is ω-periodic limit in $t \in \mathbb{R}^+$ uniformly for x in bounded subsets of $\mathbb{X}$ and if f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}^+$, then g satisfies the same Lipschitz condition in x uniformly in t.*

Proof. f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}^+$:

$$||f(t,x) - f(t,y)|| \leq L||x-y||,$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant. let $\epsilon > 0$. Since $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is ω-periodic limit in $t \in \mathbb{R}^+$ uniformly for x in bounded subsets of $\mathbb{X}$ then there exists $N \in \mathbb{N}$ such that

$$||f(t+n\omega, x) - g(t,x)|| < \frac{\epsilon}{2}$$

and

$$||f(t+n\omega, y) - g(t,y)|| < \frac{\epsilon}{2}$$

when $n \geq N$ for each $t \in \mathbb{R}^+$. Observing that

$$\begin{aligned} ||g(t,x) - g(t,y)|| &\leq ||g(t,x) - f(t+n\omega, x)|| \\ &+ ||f(t+n\omega, x) - f(t+n\omega, y)|| \\ &+ ||f(t+n\omega, y) - g(t,y)||, \end{aligned}$$

we deduce so that

$$||g(t,x) - g(t,y)|| < \epsilon + L||x-y||$$

when $n \geq N$. Since ϵ is arbitrary, we can write

$$||g(t,x) - g(t,y)|| \leq L||x-y||.$$

□

Theorem 2.8. *(see [10]) Let $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ be ω-periodic limit on $t \in \mathbb{R}^+$ uniformly for x in bounded subsets of $\mathbb{X}$ and assume that f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}^+$:*

$$||f(t,x) - f(t,y)|| \leq L||x-y||,$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant. Let $\varphi : \mathbb{R}^+ \to \mathbb{X}$ be ω-periodic limit. Then the function $F : \mathbb{R}^+ \to \mathbb{X}$ defined by $F(t) = f(t, \varphi(t))$ is ω-periodic limit.

3. Main Results

Now we make the following hypthesis:

(H1) : The function $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is ω-periodic limit in $t \in \mathbb{R}^+$ uniformly for x in bounded subsets of $\mathbb{X}$ and f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}^+$:

$$||f(t,x) - f(t,y)|| \leq L||x-y||,$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant.

(H2): $A(t)$ generates an exponentially stable evolutionnary process $(U(t,s))_{t\geq s}$ in $\mathbb{X}$, that is, a two-parameter family of bounded linear operators that satisfies the following conditions:

1. $U(t,t) = I$ for all $t \in \mathbb{R}$ where I is the identity operator.
2. $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$.
3. The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbb{X}$.
4. $U(t+\omega, s+\omega) = U(t,s)$ for all $t \geq s$ (ω-periodicity).
5. There exist $K > 0$ and $a > 0$ such that $||U(t,s)|| \leq Ke^{-\delta(t-s)}$ for $t \geq s$.

Definition 3.1. We assume **(H1)** is satisfied and that $A(t)$ generates an evolutionary process $(U(t,s))_{t\geq s}$ in $\mathbb{X}$. The continuous function x given by

$$x(t) = U(t,0)c_0 + \int_0^t U(t,s)f(s,x(s))ds$$

is called the mild solution of equation (1).

Lemma 3.1. *We assume that* **(H2)** *is satisfied and that* $f \in P_\omega L(\mathbb{R}^+, \mathbb{X})$. *Then*

$$(\wedge f)(t) = \int_0^t U(t,s)f(s)ds \in AP_\omega(\mathbb{R}^+, \mathbb{X}), t \in \mathbb{R}^+.$$

Proof. We put

$$v(t) = \int_0^t U(t,s)f(s)ds.$$

Since $f \in P_\omega L(\mathbb{R}^+, \mathbb{X})$,

$$\lim_{n\to\infty} f(t+n\omega) = g(t)$$

is well defined for each $t \in \mathbb{R}^+$. There exists also a positive constant K such that $||g||_\infty \leq ||f||_\infty \leq K$ and $g(t) = g(t+\omega)$. We observe that

$$\begin{aligned} v(t+n\omega) &= \int_0^{t+n\omega} U(t+n\omega, s)f(s)ds \\ &= \int_{-n\omega}^{t} U(t+n\omega, s+n\omega)f(s+n\omega)ds \\ &= \int_{-n\omega}^{t} U(t,s)f(s+n\omega)ds \\ &= \int_{-n\omega}^{0} U(t,s)f(s+n\omega)ds + \int_0^t U(t,s)f(s+n\omega)ds \\ &= I_1(t,n) + I_2(t,n). \end{aligned}$$

Next we will prove that $I_1(t,n)$ is a Cauchy sequence in $\mathbb{X}$ for each $t \in \mathbb{R}^+$. Let $\epsilon > 0$. For any $p \in \mathbb{N}, n \in \mathbb{N}$, we observe that

$$\begin{aligned} I_1(t,n+p) - I_1(t,n) &= \int_{-(n+p)\omega}^{0} U(t,s)f(s+(n+p)\omega)ds \\ &- \int_{-n\omega}^{0} U(t,s)f(s+n\omega)ds \\ &= \int_{-(n+p)\omega}^{-n\omega} U(t,s)f(s+(n+p)\omega)ds \\ &+ \int_{-n\omega}^{0} U(t,s)\big(f(s+(n+p)\omega) - f(s+n\omega)\big)ds \\ &= I_3(t,n,p) + I_4(t,n,p). \end{aligned}$$

Consider the term $I_3(t,n,p)$.

$$\begin{aligned} ||I_3(t,n,p)|| &\leq \int_{-(n+p)\omega}^{-n\omega} ||U(t,s)||\ ||f(s+(n+p)\omega)||ds \\ &\leq \int_{-(n+p)\omega}^{-n\omega} Me^{-\delta(t-s)}Kds \\ &\leq \int_{n\omega}^{(n+p)\omega} MKe^{-\delta(t+s)}ds \\ &\leq KM\int_{n\omega}^{\infty} e^{-\delta(t+s)}ds \end{aligned}$$

$$\leq \frac{KM}{\delta} e^{-\delta n\omega}.$$

We can choose $N_1 \in \mathbb{N}$ such that $\frac{KM}{\delta} e^{-\delta n\omega} \leq \epsilon$ when $n \geq N_1$. Therefore $||I_3(t,n,p)|| \leq \epsilon$ whenever $n \geq N_1$ uniformly for $t \in \mathbb{R}^+$.
For $n \geq N_1$, consider $I_4(t,n,p)$.

$$\begin{aligned} I_4(t,n,p) &= \int_{-N_1\omega}^{0} U(t,s)\big(f(s+(n+p)\omega) - f(s+n\omega)\big)ds \\ &+ \int_{-n\omega}^{-N_1\omega} U(t,s)\big(f(s+(n+p)\omega) - f(s+n\omega)\big)ds \\ &= I_5(t,n,p) + I_6(t,n,p). \end{aligned}$$

We have

$$\begin{aligned} ||I_5(t,n,p)|| &\leq \int_{-N_1\omega}^{0} ||U(t,s)||\; ||(f(s+(n+p)\omega) - f(s+n\omega)||ds \\ &\leq \int_{-N_1\omega}^{0} Me^{-\delta(t-s)}\; ||(f(s+(n+p)\omega) - f(s+n\omega)||ds \\ &\leq \int_{0}^{N_1\omega} Me^{-\delta(t+s)}\; ||(f((n+p)\omega - s) - f(n\omega - s)||ds \\ &\leq \int_{0}^{N_1\omega} Me^{-\delta(t+N_1\omega-s)}\; ||(f((n-N_1+p)\omega + s) \\ &- f((n-N_1)\omega + s)||ds \\ &\leq \int_{0}^{N_1\omega} Me^{-\delta(N_1\omega-s)}\; ||(f((n-N_1+p)\omega + s) \\ &- f((n-N_1)\omega + s)||ds \\ &\leq \int_{0}^{N_1\omega} Me^{-\delta(N_1\omega-s)}\; ||(f((n-N_1+p)\omega + s) - g(s)||ds \\ &+ \int_{0}^{N_1\omega} Me^{-\delta(N_1\omega-s)}\; ||(f((n-N_1)\omega + s) - g(s)||ds. \end{aligned}$$

For each $s \in [0, N_1\omega]$, we have

$$Me^{-\delta(N_1\omega-s)}\; ||(f((n-N_1+p)\omega + s) - g(s)|| \leq 2MKe^{-\delta(N_1\omega-s)}$$

and

$$\int_{0}^{N_1\omega} 2MKe^{-\delta(N_1\omega-s)}ds = \frac{2MK}{\delta}(1 - e^{-\delta N_1\omega}).$$

Since $f \in P_\omega L(\mathbb{R}^+, \mathbb{X})$, for each $s \in [0, N_1\omega]$, we have

$$\lim_{n\to\infty} Me^{-\delta(N_1\omega - s)} \, ||(f((n-N_1+p)\omega + s) - g(s)|| = 0.$$

By the Lebesgue's Dominated Convergence Theorem, we deduce that

$$\lim_{n\to\infty} \int_0^{N_1\omega} Me^{-\delta(N_1\omega - s)} \, ||(f((n-N_1+p)\omega + s) - g(s)||ds = 0.$$

Similarly, we have

$$\lim_{n\to\infty} \int_0^{N_1\omega} Me^{-\delta(N_1\omega - s)} \, ||(f((n-N_1)\omega + s) - g(s)||ds = 0.$$

Therefore, we can select $N_2 \in \mathbb{N}\,(N_2 > N_1)$ such that $||I_5(t,n,p)|| \leq \epsilon$ whenever $n \geq N$ uniformly for $t \in \mathbb{R}^+$.

Now we consider the term $I_6(t,n,p)$.

$$\begin{aligned}
||I_6(t,n,p)|| &\leq \int_{-n\omega}^{-N_1\omega} ||U(t,s)|| \, ||f(s+(n+p)\omega) - f(s+n\omega)||ds \\
&\leq 2KM \int_{-n\omega}^{-N_1\omega} e^{-\delta(t-s)} ds \\
&\leq 2KM \int_{N_1\omega}^{n\omega} e^{-\delta(t+s)} ds \\
&\leq 2KM \int_{N_1\omega}^{\infty} e^{-\delta(t+s)} ds \\
&\leq \frac{2KM}{\delta} e^{-\delta N_1\omega} \\
&\leq 2\epsilon
\end{aligned}$$

uniformly for $t \in \mathbb{R}^+$. Since

$$||I_1(t,n+p) - I_1(t,n)|| \leq ||I_3(t,n,p)|| + ||I_5(t,n,p)|| + ||I_6(t,n,p)||$$

we deduce that $||I_1(t,n+p) - I_1(t,n)|| \leq 4\epsilon$ when $n \geq N_2$. Therefore $I_1(t,n)$ is a cauchy sequence. So we can denote $h(t) = \lim_{n\to\infty} I_1(t,n)$ for each $t \in \mathbb{R}^+$. Note also that $h(t) = \lim_{n\to\infty} I_1(t,n)$ uniformly for $t \in \mathbb{R}^+$.

Now consider the term $I_2(t,n)$. Since g is measurable, $\int_0^t U(t,s)g(s)ds$ is well defined for each $t \in \mathbb{R}^+$. For $m\omega \leq t < (m+1)\omega$, $m \in \mathbb{N}$, we have

$$||I_2(t,n) \quad - \quad \int_0^t U(t,s)g(s)ds|| \leq \int_0^t ||U(t,s)|| \, ||f(s+n\omega) - g(s)|| \, ds$$

$$\begin{aligned}
&\leq \int_0^t Me^{-\delta(t-s)} \,||f(s+n\omega)-g(s)||\, ds \\
&\leq \int_0^{m\omega} Me^{-\delta(t-s)} \,||f(s+n\omega)-g(s)||\, ds \\
&+ \int_{m\omega}^t Me^{-\delta(t-s)} \,||f(s+n\omega)-g(s)||\, ds \\
&\leq M\sum_{k=0}^{m-1}\int_{k\omega}^{(k+1)\omega} e^{-\delta(t-s)} \,||f(s+n\omega)-g(s)||\, ds \\
&+ M\int_{m\omega}^{(m+1)\omega} e^{-\delta(t-s)} \,||f(s+n\omega)-g(s)||\, ds \\
&\leq M\sum_{k=0}^{m-1} e^{-\delta(t-(k+1)\omega)}\int_{k\omega}^{(k+1)\omega} ||f(s+n\omega)-g(s)||\, ds \\
&+ M\int_{m\omega}^{(m+1)\omega} ||f(s+n\omega)-g(s)||\, ds
\end{aligned}$$

For each $s \in [0,\omega]$, we have $\lim_{n\to\infty} ||f(s+n\omega)-g(s)|| = 0$ and $||f(s+n\omega)-g(s)|| \leq 2K$. By Lebesgue's Dominated Convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^{\omega} ||f(s+n\omega)-g(s)|| = 0.$$

For $\epsilon > 0$, there existe $N_3 \in \mathbb{N}$ such that

$$\int_0^{\omega} ||f(s+n\omega)-g(s)|| \leq \epsilon$$

when $n \geq N_3$. For any $i \in \mathbb{N}$, we have

$$\begin{aligned}
\int_{i\omega}^{(i+1)\omega} ||f(s+n\omega)-g(s)||\, ds &= \int_0^{\omega} ||f(s+i\omega+n\omega)-g(s+i\omega)||\, ds \\
&= \int_0^{\omega} ||f(s+i\omega+n\omega)-g(s)||\, ds \\
&\leq \epsilon
\end{aligned}$$

when $n \geq N_3$. Therefore

$$||I_2(t,n) - \int_0^t U(t,s)g(s)ds|| \leq M\sum_{k=0}^{m-1} e^{-\delta(t-(k+1)\omega)}\epsilon + M\epsilon$$

$$\leq \Big(\frac{1}{1-e^{-\delta\omega}}+1\Big)M\epsilon$$

when $n \geq N_3$ uniformly for $t \in \mathbb{R}^+$. Therefore

$$\lim_{n\to\infty} I_2(t,n) = \int_0^t U(t,s)g(s)ds$$

uniformly for $t \in \mathbb{R}^+$. Now we have $\lim_{n\to\infty} v(t+n\omega) = \lim_{n\to\infty} I_1(t,n) + \lim_{n\to\infty} I_2(t,n) = h(t) + \int_0^t U(t,s)g(s)ds$ uniformly for $t \in \mathbb{R}^+$. We deduce that $v \in AP_\omega(\mathbb{R}^+, \mathbb{X})$. □

Theorem 3.2. *We assume that the hypothesis* **(H1)** *and* **(H2)** *are satisfied. Then (1) has a unique Asymptotically ω-periodic mild solution provided*

$$\Theta := \frac{LM}{a} < 1.$$

Proof. We define the nonlinear operator Γ by the expression

$$\begin{aligned}(\Gamma\phi)(t) &= U(t,0)c_0 + \int_0^t U(t,s)f(s,\phi(s))ds \\ &= U(t,0)c_0 + (\wedge_1\phi)(t)\end{aligned}$$

where

$$(\wedge_1\phi)(t) = \int_0^t U(t,s)f(s,\phi(s)).$$

According to the hypothesis **(H.2)**, we have

$$||U(t,0)|| \leq Me^{-at}$$

Therefore $\lim_{t\to\infty} ||U(t,0)|| = 0$.
According to the Theorem 2.8, the function $t \to f(t,u(t))$ belongs to $P_\omega L(\mathbb{R}_+, \mathbb{X})$.
According to the Lemma 3.1 the operator $\wedge_1$ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.
Therefore the operator Γ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.
We have

$$\begin{aligned}||(\Gamma\phi)(t) - \Gamma\psi)(t)|| &= \Big|\Big|\int_0^t U(t,s)\big(f(s,\phi(s)) - f(s,\psi(s))\big)ds\Big|\Big| \\ &\leq \int_0^t ||U(t,s)||\,||f(s,\phi(s)) - f(s,\psi(s))||ds \\ &\leq L\int_0^t ||U(t,s)||\,||\phi(s) - \psi(s)||ds\end{aligned}$$

$$
\begin{aligned}
&\leq LM\int_0^t e^{-a(t-s)}\,||\phi(s)-\psi(s)||ds\\
&\leq LM\int_0^t e^{-a(t-s)}\,||\phi-\psi||_\infty ds\\
&\leq LM\Big(\frac{1-e^{-at}}{a}\Big)||\phi-\psi||_\infty\\
&\leq \frac{LM}{a}||\phi-\psi||_\infty.
\end{aligned}
$$

Hence we have :

$$||\Gamma\phi-\Gamma\psi||_\infty \leq \frac{LM}{a}||\phi-\psi||_\infty$$

which proves that Γ is a contraction and we conclude that Γ has a unique fixed point in $AP_\omega(\mathbb{R}^+,\mathbb{X})$. The proof is complete. □

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