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FIXED POINT THEOREMS IN GP-METRIC SPACE USING α-CONTRACTION

Savita Rathee[1], Kusum Dhingra[2 §]
[1,2]Department of Mathematics
Maharshi Dayanand University
Rohtak, Haryana, 124001, INDIA

Abstract: In this paper, we consider α-contraction in the context of generalized partial metric space (GP-metric space) and give conditions to prove the existence and uniqueness of fixed point for such type of contraction mappings. We provide examples to support our results. In particular, we extend and generalize the results of Bilgili et.al., see [3].

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1. Introduction

Fixed point theory focuses on the strategies whether a function T, which is defined on some space X, has atleast one fixed point under some conditions imposed on T and on space. If the existence of fixed point is sure, then further uniqueness of fixed point is examined. The main tool to study contraction mappings is Banach contraction mapping[2]. He proved that every contraction has a fixed point in the setup of complete metric space. After then, many authors proved fixed point results for various contraction mappings and replacing complete metric space by generalized form of metric space.

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§Correspondence author

There are various generalizations of metric space such as 2-metric space, D-metric space,G-metric space and GP-metric space. The concept of 2-metric space was given by Gahler[6]. In 1984, Dhage[4] gave the concept of D-metric. Mustafa and Sims[7] introduced a new generalized version of metric space and is called it G-metric. Zand and Nezhad[11] recently introduced GP-metric spaces which are a combination of the notions of partial metric spaces and G-metric spaces. Then they proved a number of fixed point theorems on these new space for certain type of contractions. In this paper, we also prove fixed point theorem in GP-metric space using orbital admissible mapping.First, we review the necessary definitions and fundamental results produced on GP-metric space that we will need in this work.

2. Preliminaries

Now we recollect some definitions and results which will be use in the sequel.

Definition 1. *[11] Let X be a nonempty set. A function $G_p : X \times X \times X \to [0, \infty)$ is called a GP-metric if the following conditions are satisfied :*

$(GP1)$ $x = y = z$ *if* $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;

$(GP2)$ $0 \leq G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ *for all* $x, y, z \in X$;

$(GP3)$ $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) = ...$,*symmetry in all three variables;*

$(GP4)$ $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ *for any* $x, y, z, a \in X$.

Then the pair (X, G_p) is called a GP-metric space.

Proposition 2. *[11] Every GP-metric space (X, G_p) defines a metric space (X, D_{G_p}) where $D_{G_p}(x, y) = G_p(x, y, y) + G_p(y, x, x) - G_p(x, x, x) - G_p(y, y, y)$ for all $x, y \in X$.*

Proposition 3. *[11] Let (X, G_p) be a GP-metric space, then for any x, y, z and $a \in X$, it follows that*

(i) $G_p(x,y,z) \leq G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$;

(ii) $G_p(x,y,y) \leq 2G_p(x,x,y) - G_p(x,x,x)$;

(iii) $G_p(x,y,z) \leq G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$;

(iv) $G_p(x,y,z) \leq G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$.

Definition 4. *[10] Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be a function, then T is said to be α-orbital admissible if*

$$\alpha(x,Tx) \geq 1 \text{ implies } \alpha(Tx,T^2x) \geq 1.$$

Definition 5. *[10] Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be a function,then T is said to be triangular α-orbital admissible if T is α- orbital admissible and*

$$\alpha(x,y) \geq 1 \text{ and } \alpha(y,Ty) \geq 1 \text{ implies } \alpha(x,Ty) \geq 1.$$

Definition 6. *[3] A function $\phi : [0,\infty) \to [0,\infty)$ is called upper semi-continuous from the right if for each $t \geq 0$ and each sequence $\{t_n\}_{n\in N}$ such that $t_n \geq t$ and*

$$\lim_{n\to\infty} t_n = t, \text{ the equality holds } \limsup_{n\to\infty} \phi(t_n) \leq \phi(t).$$

Definition 7. *[3] Let (X,G_p) be a GP-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of sequence $\{x_n\}$ if*

$$\lim_{m,n\to\infty} G_p(x,x_m,x_n) = G_p(x,x,x).$$

Definition 8. *[3] Let (X,G_p) be a GP-metric space. A sequence $\{x_n\}$ is called a GP-Cauchy if and only if $\lim_{m,n,r\to\infty} G_p(x_n,x_m,x_r)$ exists and finite.*

Definition 9. *[3] A GP-metric space (X,G_p) is said to be GP-complete if and only if every GP-Cauchy sequence in X is GP-convergent to $x \in X$ such that*

$$\lim_{m,n,r\to\infty} G_p(x_n,x_m,x_r) = G_p(x,x,x).$$

Definition 10. *[1] Let (X, G_p) be a complete GP-metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is α-regular, if the following condition is satisfied:*
If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Lemma 11. *[10] Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in N$ with $n \prec m$.*

Proof. Since T is α- orbital admissible and $\alpha(x_1, Tx_1) \geq 1$ that is $\alpha(x_1, x_2) \geq 1$ we deduce that

$$\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1$$

By continuing this process, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$. Suppose that $\alpha(x_n, x_m) \geq 1$ and prove that $\alpha(x_n, x_{m+1}) \geq 1$ where $m \succ n$. Since T is triangular α -orbital admissible and $\alpha(x_m, x_{m+1}) \geq 1$ we get that $\alpha(x_n, x_{m+1}) \geq 1$. Hence we have proved that $\alpha(x_n, x_m) \geq 1$ for all $n, m \in N$ with $m \succ n$.

□

Definition 12. *[3] Let (X, d) be a GP-metric space and $T : (X, G_p) \to (X, G_p)$ be a map.Let*
$M(x, y, y) = \max\{G_p(x, y, y), G_p(x, Tx, Tx), G_p(y, Ty, Ty), \frac{1}{2}[G_p(x, Ty, Ty) + G_p(y, Tx, Tx)]\}$ for all $x, y \in X$.

Lemma 13. *[3] If (X, G_p) is GP-metric space and $T : X \to X$ is a map, Then for each $x \in X$, we have*
$M(x, Tx, Tx) = max\left\{G_p(x, Tx, Tx), G_p(Tx, T^2(x), T^2(x))\right\}$.

Proof. Let $x \in X$. Then $\max\{G_p(x, Tx, Tx), G_p(Tx, T^2(x), T^2(x))\}$

$$\begin{aligned}
&\leq M(x, Tx, Tx) \\
&\leq \max\{G_p(x, Tx, Tx), G_p(Tx, T^2(x), T^2(x)), \\
&\quad \frac{1}{2}[G_p(x, T^2(x), T^2(x)) + G_p(Tx, Tx, Tx)]\} \\
&\leq \max\{G_p(x, Tx, Tx), G_p(Tx, T^2(x), T^2(x)), \\
&\quad \frac{1}{2}[G_p(x, T(x), T(x)) + G_p(Tx, T^2x, T^2x)]\} \\
&= \max\{G_p(x, Tx, Tx), G_p(Tx, T^2(x), T^2(x))\}.
\end{aligned}$$

Hence $M(x,Tx,Tx) = max\{G_p(x,Tx,Tx), G_p(Tx,T^2(x),T^2(x)\}$. □

3. Main Results

Now we prove our first Result.

Theorem 14. *Let (X, G_p) be a complete GP-metric space and let $T : X \to X$ be a map such that $\alpha(x,y)G_p(Tx,Ty,Ty) \leq \phi(M(x,y,y))$, where,*

$$M(x,y,y) = \max\left\{G_p(x,y,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty), \frac{1}{2}[G_p(x,Ty,Ty) + G_p(y,Tx,Tx)]\right\},$$

for all $x, y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous function from the right such that $\phi(t) \prec t$ for all $t \succ 0$ and

1. *T is a triangular α-orbital admissible mapping;*
2. *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1, Tx_1) \geq 1$;*
3. *$\{x_n\}$ is α-regular.*

Then T has a fixed point $x^* \in X$. Moreover $G_p(x^*, x^*, x^*) = 0$.

Proof. Let $x_1 \in X$ such that $\alpha(x_1, Tx_1, Tx_1) \geq 1$. If for $x \in X$ there is $n \in N$ such that $T^n(x) = T^{n+1}(x)$, then $T^n(x)$ is a fixed point of T. Hence, we assume that $T^n(x) \neq T^{n+1}(x)$ for all $n \in N$. Now consider $x_0 = x$ and construct the sequence $(x_n)_{n\in N}$, where $x_n = T^n(x_0)$ for all $n \in N$. Thus $x_{n+1} = Tx_n$ and $G_p(x_n, x_{n+1}, x_{n+1}) \geq 0$ for all $n \in N$. Now by Lemma 11 we have $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \geq 1$. Now we shall prove that:

(a) $M(T^n(x), T^{n+1}(x), T^{n+1}(x)) = G_p(T^n(x), T^{n+1}(x), T^{n+1}(x))$ for all $n \in N$.

(b) For all $n \in N$:

$$\begin{aligned} G_p(T^n(x), T^{n+1}(x), T^{n+1}(x) &\leq \phi(G_p(T^{n-1}(x), T^n(x), T^n(x))) \\ &\leq G_p(T^{n-1}(x), T^n(x), T^n(x)). \end{aligned}$$

From Lemma 13 , we have

$$\begin{aligned} &M(T^n(x), T^{n+1}(x), T^{n+1}(x)) \\ &\qquad = \max G_p(T^n(x), T^{n+1}(x), T^{n+1}(x)), G_p(T^{n+1}(x), T^{n+2}(x), T^{n+2}(x)). \end{aligned}$$

Since $T^n(x) \neq T^{n+1}(x)$ for all $n \in N$, then by Lemma 11 , we get

$$G_p(T^n(x), T^{n+1}(x), T^{n+1}(x)) \succ 0,$$

for all $n \in N$.

Consequently $M(T^n(x), T^{n+1}(x), T^{n+1}(x)) \succ 0$. Now

$$\begin{aligned} G_p(T^{n+1}(x), T^{n+2}(x), T^{n+2}(x)) &= G_p(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\leq \alpha(x_n, x_{n+1}, x_{n+1}) G_p(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \phi(M(T^n(x), T^{n+1}(x), T^{n+1}(x))) \\ &\prec M(T^n(x), T^{n+1}(x), T^{n+1}(x)). \end{aligned}$$

This implies $G_p(T^{n+1}(x), T^{n+2}(x), T^{n+2}(x)) \prec G_p(T^n(x), T^{n+1}(x), T^{n+1}(x))$. Clearly, part (b) follows from all conditions and by part (a) above.

Then, we get, for all $n \geq 1$.

$$\begin{aligned} G_p(x_{n+1}, x_{n+2}, x_{n+2}) &= G_p(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \alpha(x_n, x_{n+1}) G_p(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \phi(M(x_n, x_{n+1}, x_{n+1})). \end{aligned} \tag{3.1}$$

By Lemma 13,

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+1}) &= M(x_n, Tx_n, Tx_n) \\ &= \max\{G_p(x_n, Tx_n, Tx_n), G_p(Tx_n, Tx_{n+1}, Tx_{n+1})\} \\ &= \max\{G_p(x_n, x_{n+1}, x_{n+1}), G_p(x_{n+1}, x_{n+2}, x_{n+2})\}. \end{aligned} \tag{3.2}$$

Now by equation (3.1)

$$\begin{aligned} G_p(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \phi(M(x_n, x_{n+1}, x_{n+1})) \\ &\prec M(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

This implies

$$G_p(x_{n+1}, x_{n+2}, x_{n+2}) \prec M(x_n, x_{n+1}, x_{n+1}).$$

Hence $M(x_n, x_{n+1}, x_{n+1}) = G_p(x_n, x_{n+1}, x_{n+1})$. Now we shall prove that

$$\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0.$$

Define the sequence $\{S_n\}$ by $S_n = G_p(x_n, x_{n+1}, x_{n+1})$ for all $n \in N$. By condition (b), we have $\{s_n\}$ is a non increasing sequence. Hence, there exists $c \in R^+$ such that $s_n \to c$ as $n \to \infty$. We shall show that $c = 0$. Let $c \succ 0$. By taking lim sup as $n \to \infty$ in condition (b) we get that

$$c = \limsup_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = \limsup_{n\to\infty} \phi(G_p(x_n, x_{n+1}, x_{n+1})),$$

and so, by upper semi-continuity from the right of the function ϕ, we have

$$c = \limsup_{n\to\infty} \phi(G_p(x_n, x_{n+1}, x_{n+1})) \leq \phi(c) \prec c,$$

which is contradiction. Hence $c = 0$. That is $\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0$. Now,we shall prove that x_n is a Cauchy sequence in complete GP-metric space (X, G_p),that is, we shall prove that $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0$.

Assume on the contrary, that there exists $\epsilon \succ 0$ and sequences $\{n_k\}_{k\in N}$, $\{m_k\}_{k\in N}$ in N with $m_k \geq n_k \geq k$ and such that $G_p(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon$ for all $k \in N$.

As we know $\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0$, we can suppose,without loss of generality, that $G_p(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \prec \epsilon$. For each $k \in N$,we have

$$\begin{aligned}\epsilon \leq & G_p(x_{n_k}, x_{m_k}, x_{m_k}) \\ \prec & \epsilon + G_p(x_{m_k-1}, x_{m_k}, x_{m_k}).\end{aligned}$$

Therefore $\lim\limits_{k\to\infty} G_p(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon$.

Now let $k_0 \in N$ be such that

$$G_p(x_{n_k}, x_{n_k+1}, x_{n_k+1}) \prec \epsilon$$

and

$$G_p(x_{m_k}, x_{m_k+1}, x_{m_k+1}) \prec \epsilon,$$

for all $k \geq k_0$. Then

$$\begin{aligned}G_p(x_{n_k}, x_{m_k}, x_{m_k}) \leq & M(x_{n_k}, x_{m_k}, x_{m_k}) \\ = & max\{G_p(x_{n_k}, x_{m_k}, x_{m_k}), G_p(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}), \\ & G_p(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}), \frac{1}{2}[G_p(x_{n_k}, x_{m_{k+1}}, x_{m_{k+1}}) \\ & + G_p(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}})]\},\end{aligned}$$

for all $k \geq k_0$. So $\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon$. Since $M(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon$ for all $k \in N$ and ϕ is upper semi-continuous from the right, we deduce that

$$\limsup_{k\to\infty} \phi(M(x_{n_k}, x_{m_k}, x_{m_k})) \leq \phi(\epsilon).$$

On the other hand, for each $k \in N$, we have

$$\begin{aligned}
\epsilon &\leq G_p(x_{n_k}, x_{m_k}, x_{m_k}) \\
&\leq G_p(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}) + G_p(x_{n_{k+1}}, x_{m_{k+1}}, x_{m_{k+1}}) + G_p(x_{m_{k+1}}, x_{m_k}, x_{m_k}) \\
&\leq G_p(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}) + \alpha(x_{n_k}, x_{m_k}) G_p(x_{n_{k+1}}, x_{m_{k+1}}, x_{m_{k+1}}) \\
&\quad + G_p(x_{m_{k+1}}, x_{m_k}, x_{m_k}) \\
&\leq G_p(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}) + \phi(M((x_{n_k}, x_{m_k}, x_{m_k}))) + G_p(x_{m_{k+1}}, x_{m_k}, x_{m_k}).
\end{aligned}$$

So $\epsilon \leq \limsup_{k\to\infty} \phi(M(x_{n_k}, x_{m_k}, x_{m_k})) \leq \phi(\epsilon)$ a contradiction because $\phi(\epsilon) \prec \epsilon$. Consequently, $\lim_{m,n\to\infty} G_p(x_n, x_m, x_m) = 0$ and that $\{x_n\}_{n\in N}$ is a Cauchy sequence in the GP-complete GP-metric space (X, G_p). Since (X, G_p) is a GP-complete metric space then every GP-Cauchy sequence in X is GP-convergent to $x^* \in X$ such that $\lim_{m,n\to\infty} G_p(x_n, x_m, x_m) = \lim_{n\to\infty} G_p(x^*, x_n, x_n) = G_p(x^*, x^*, x^*) = 0$. Now we claim that x^* is a fixed point of T. For this, we notice that $G_p(x^*, Tx^*, Tx^*) = \lim_{n\to\infty} M(x^*, x_n, x_n)$. So

$$\limsup_{n\to\infty} \phi(M(x^*, x_n, x_n)) \leq \phi(G_p(x^*, Tx^*, Tx^*)).$$

Also, for each $n \in N$,

$$G_p(x^*, Tx^*, Tx^*) \leq G_p(x^*, x_n, x_n) + G_p(x_n, Tx^*, Tx^*).$$

By using the fact that $\{x_n\}$ is α-regular. It follows that

$$\begin{aligned}
G_p(x^*, Tx^*, Tx^*) &\leq \limsup_{n\to\infty} (G_p(x^*, x_{n_k}, x_{n_k}) + G_p(x_{n_k}, Tx^*, Tx^*)) \\
&= \limsup_{n\to\infty} (G_p(x_{n_k}, Tx^*, Tx^*)) \\
&\leq \limsup_{n\to\infty} \alpha(x_{n_k - 1}, x^*) G_p(x_n, Tx^*, Tx^*) \\
&\leq \limsup_{n\to\infty} \phi(M(x_{n-1}, x^*, x^*)) \\
&\leq \phi(G_p(x^*, Tx^*, Tx^*)).
\end{aligned}$$

This implies $G_p(x^*, Tx^*, Tx^*) \leq \phi(G_p(x^*, Tx^*, Tx^*))$,

$$G_p(x^*, Tx^*, Tx^*) = 0, \qquad Tx^* = x^*.$$

This completes the proof. □

Example 15. *Let $X = [0, \infty)$ and $G_p : X \times X \times X \to [0, \infty)$ be defined by $G_p(x, y, z) = |x - y| + |y - z| + |z - x| + \max\{x, y, z\}$. Then (X, G_p) is a GP-complete metric space. Let $\phi : [0, \infty) \to [0, \infty)$ be defined by $\phi(t) = \frac{t}{3}$ and $T : X \to X$ defined by*

$$T(x) = \begin{cases} \frac{x}{4}, & \text{if } 0 \le x \le 1, \\ 3x, & \text{if } x \ge 1 \ , \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Firstly we check condition $\alpha(x, y)G_p(Tx, Ty, Ty) \le \phi(M(x, y, y))$ for all $x, y \in X$ where

$$M(x, y, y) = \max\{G_p(x, y, y), G_p(y, Ty, Ty), G_p(x, Tx, Tx), \frac{1}{2}[G_p(x, Ty, Ty) + G_p(y, Tx, Tx)]\}.$$

There are following three cases:

(i) When $x, y \in [0, 1]$ then $Tx, Ty \in [0, 1]$ For such values of x and y we have $\alpha(x, y) = 1$ and then we have to prove the inequality $G_p(Tx, Ty, Ty) \le \phi(M(x, y, y))$. $G_p(Tx, Ty, Ty) = G_p(\frac{x}{4}, \frac{y}{4}, \frac{y}{4})$ and $\phi(M(x, y, y)) = \frac{M(x,y,y)}{2}$ which shows that for all values of x and $y \in [0, 1]$ the inequality $G_p(Tx, Ty, Ty) \le \phi(M(x, y, y))$ satisfied.

(ii) When $x, y \in [1, \infty)$ then $\alpha(x, y) = 0$. then clearly inequality holds.

(iii) When one of x and y say $x \in [0, 1]$ and other say $y \in [1, \infty)$ also in this condition $\alpha(x, y) = 0$ and inequality holds trivially.This implies in all the three possibilities of values of x, y inequality $\alpha(x, y)G_p(Tx, Ty, Ty) \le \phi(M(x, y, y))$ is satisfied. Now to find fixed point we have to check whether T is triangular α orbital admissible mapping.

Consider $x, y \in [0, 1]$ then $\alpha(x, y) = 1$ and also $Tx, Ty \in [0, 1]$, then $\alpha(y, Ty) = 1$ which implies that $\alpha(x, Ty) = 1$.

Hence the conditions for T to be triangular α-orbital admissible mapping is satisfied. All the conditions of Theorem 14 are satisfied . So T has a fixed point and in this case that fixed point is 0. Hence $T(0) = 0$. □

The following corollary follows from Theorem 14.

Corollary 16. *Let (X, G_p)be a GP-complete metric space and let $T : X \to X$ be a map such that*

$$0 \le [\psi(z,z) - \psi(y,y)] \max\{G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)\} \\ + \psi(x,y)\psi(x,z)[\phi(Q(x,y,z)) - \alpha(x,y)G_p(Tx,Ty,Tz)],$$

where $\psi : X \times X \to [0,\infty)$ is a function and

$$Q(x,y,z) = \max\{G_p(x,y,z), G_p(x,Tx,Tx), G_p(y,Ty,Ty), G_p(z,Tz,Tz), \\ \frac{1}{2}[G_p(x,Ty,Ty) + G_p(y,Tx,Tx)], \frac{1}{2}[G_p(y,Ty,Ty) + G_p(z,Tz,Tz)], \\ \frac{1}{2}[G_p(x,Tz,Tz) + G_p(z,Tx,Tx)],$$

for all $x, y, z \in X$ and $\phi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous from the right such that $\phi(t) \prec t$ for all $t \succ 0$ and T is triangular α -orbital admissible and there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge 1$. Then T has a unique fixed point $x^* \in X$ and also $G_p(x^*, x^*, x^*) = 0$.

Proof. By considering $y = z$ in the hypothesis, we have

$$\alpha(x,y)G_p(Tx,Ty,Ty) \le \phi(Q(x,y,y)) = \phi(M(x,y,y)),$$

for all $x, y \in X$. Then all conditions of theorem(14) holds and hence T has a unique fixed point. □

Remark 17. *Theorem 2.1 of Bilgili et. al. [3] becomes a particular case of Theorem14 with $\alpha(x,y) = 1$ for all $x, y, z \in X$.*

Lemma 18. *[3] Let $\phi : [0,\infty) \to [0,\infty)$ be non decreasing and let $t \succ 0$. If $\lim_{n\to\infty} \phi^n(t) = 0$ then $\phi(t) \prec t$.*

Theorem 19. *Let (X, G_p) be a GP-complete metric space and $T : X \to X$ be a map such that $\alpha : X \times X \to R$ be a map such that*

$$\alpha(x,y)G_p(Tx,Ty,Ty) \le \phi(N(x,y,y)),$$

where $N(x,y,y) = max\,\{G_p(x,y,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty)\}$, for all $x, y \in X$ and $\phi : [0,\infty) \to [0,\infty)$ is a non decreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \succ 0$.

1. *T is triangular $\alpha-$ orbital admissible mapping.*

2. *there exist $x_1 \in X$ such that $\alpha(x_1, Tx_1, Tx_1) \ge 1$;*

3. $\{x_n\}$ *is* α*-regular.*

Then T *has fixed point.*

Proof. Let $x \in X$. If there is $n \in N$ such that $T^n(x) = T^{n+1}(x)$, then $T^n(x)$ is a fixed point of T. Hence , we will assume that $T^n(x) \neq T^{n+1}(x)$ for all $n \in N$. Put $x_0 = T^n(x_0)$ for all $n \in N$. Thus $x_{n+1} = T(x_n)$ and $G_p(x_n, x_{n+1}, x_{n+1}) \succ 0$ for all $n \in N$. We obtain

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \phi(G_p(x_{n-1}, x_n, x_n)) \tag{3.3}$$

for all $n \in N$. Then since ϕ is non decreasing we deduce that

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \phi^n(G_p(x_0, x_1, x_1)) \tag{3.4}$$

for all $n \in N$. Hence $\lim_{n\to\infty}(G_p(x_n, x_{n+1}, x_{n+1})) = 0$. Now choose an arbitrary $\epsilon \succ 0$. Since $\lim_{n\to\infty} \phi^n(\epsilon) = 0$. it follows from Lemma 18 that $\phi(\epsilon) \prec \epsilon$, so there is $n_\epsilon \in N^*$ such that:

$$G_p(x_n, x_{n+1}, x_{n+1}) \prec \epsilon - \phi(\epsilon). \tag{3.5}$$

for all $n \geq n_\epsilon$. Therefore,

$$\begin{aligned} G_p(x_n, x_{n+2}, x_{n+2}) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\prec \epsilon - \phi(\epsilon) + \alpha(x_n, x_{n+1}, x_{n+1})G_p(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\leq \epsilon - \phi(\epsilon) + \phi(G_p(x_n, x_{n+1}, x_{n+1})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(G_p(x_n, x_{n+1}, x_{n+1})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) \\ &= \epsilon. \end{aligned} \tag{3.6}$$

for all $n \geq n_\epsilon$. So,

$$\begin{aligned} G_p(x_n, x_{n+3}, x_{n+3}) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+3}, x_{n+3}) \\ &\prec \epsilon - \phi(\epsilon) + \alpha(x_n, x_{n+2}, x_{n+2})G_p(x_{n+1}, x_{n+3}, x_{n+3}) \\ &\prec \epsilon - \phi(\epsilon) + \phi(N(x_n, x_{n+2}, x_{n+2})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) \\ &= \epsilon \end{aligned} \tag{3.7}$$

and following this process,

$$G_p(x_n, x_{n+k}, x_{n+k}) \prec \epsilon \tag{3.8}$$

for all $n \geq n_\epsilon$ and $k \in N^*$. Consequently,

$$\lim_{n\to\infty} G_p(x_n, x_m, x_m) = 0. \tag{3.9}$$

and thus $\{x_n\}_{n\in N}$ is a Cauchy sequence in the G_p- metric space (X, G_P).

Hence, there is $z \in X$ such that

$$0 = \lim_{n,m\to\infty} G_p(x_n, x_m, X_m) = \lim_{n\to\infty} G_p(z, x_n, x_n) = G_p(z, z, z). \tag{3.10}$$

Now we claim that z is a fixed point of T . Assume on the contrary, z is not fixed point of T.

That is $G_p(z, Tz, Tz) \succ 0$ for each $n \in N^*$, From our assumption that $G_p(z, Tz, Tz) \succ 0.$, it follows that there is $n_0 \in N$ such that $N(z, z, x_{n-1}) = G_p(z, Tz, Tz)$ for all $n \geq n_0$.We have

$$\begin{aligned} G_p(z, Tz, Tz) &\leq G_p(z, x_{n_k}, x_{n_k}) + G_p(x_{n_k}, Tz, Tz); \\ &\leq \limsup_{n_k\to\infty} G_p(z, x_{n_k}, x_{n_k}) + G_p(x_{n_k}, Tz, Tz); \\ &\leq \limsup_{n_k\to\infty} G_p(z, x_{n_k}, x_{n_k}) + \alpha(x_{n_k-1}, z) G_p(x_{n_k}, Tz, Tz); \\ &\leq \limsup_{n_k\to\infty} G_p(z, x_{n_k}, x_{n_k}) + \phi(N(z, z, x_{n_k-1})) \end{aligned}$$

$$\begin{aligned} G_p(z, Tz, Tz) &\leq \phi(G_p(z, Tz, Tz)) \\ &\prec G_p(z, Tz, Tz) \\ G_p(z, Tz, Tz) &\prec G_p(z, Tz, Tz) \end{aligned}$$

a contradiction. Consequently, $G_p(z, Tz, Tz) = 0$. This implies $Tz = z$. □

Example 20. *Let* $X = [0, \infty), G_p : X \times X \times X \to [0, \infty)$ *defined by* $G_p(x, y, z) = |x - y| + |y - z| + |z - x| + \max\{x, y, z\}$. *Then* (X, G_p) *is a GP-complete GP-metric space . Let* $T : X \to X$ *be defined by*

$$Tx = \begin{cases} \frac{x^2}{4} & \text{if } 0 \leq x \leq 1 \\ \frac{1-x}{2} & \text{otherwise} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ *defined by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

and $\phi : [0, \infty) \to [0, \infty)$ *by* $\phi(t) = \frac{t}{2}$
All the conditions of Theorem 19 are satisfied. Hence by the result of theorem the mapping T *has a unique fixed point and under these definition of functions that fixed point is* 0.

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