International Journal of Pure and Applied Mathematics
Volume 113 No. 2 2017, 327-334
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i2.11
ijpam.eu

ON THE HOPFICITY OF COMPLETELY DECOMPOSABLE TORSION-FREE ABELIAN GROUPS

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Abstract: The property of Hopficity is studied in a well-known class of Abelian groups – completely decomposable torsion-free groups. Examples of non-Hopfian completely decomposable torsion-free group are constructed.

AMS Subject Classification: 16Txx, 20Kxx
Key Words: Hopficity, Abelian group, completely decomposable groups, homogeneous group, torsion-free group

1. Introduction

In 1932 Swiss mathemathician Heinz Hopf arised the following question: could the finitely generated group be isomorphic to its own factor group? Accordingly, G group is called *Hopfian* one if its any surjective endomorphism is the isomorphism. In a dual way, G group is called *co-Hopfian* one if its any injective endomorphism is the isomorphism.

Received: December 4, 2016
Revised: March 1, 2017
Published: March 19, 2017

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The first general result concerning Hopf's question was represented by Malcev theorem stating the Hopficity of arbitrary finitely generated F-approximating group [1]. The first example of the finitely generated non-Hopfian group was given by Neumann [2]; non-Hopfian group constructed by him has two forming elements but requires the infinite set of defining relationships. Higman has constructed the example of non-Hopfian group with three forming and two defining relationships [3]. Minimal in this respect examples of non-Hopfian groups with two forming and one defining relationships were specified in Baumslag and Solitar paper [4]. Later results concerning non-Hopfian groups are given in [5], [6], [7].

Evidently that the start of Hopfian groups systematic study was given by Hirshon's works [8], [9], [10], [11], [12], [13], [13], [15] in which there were found many sufficient signs of two groups direct product Hopficity and arisen the open problems. The examples of super-Hopfian[1] groups are given in [8].

Investigations concerning the Hopfian Abelian groups are very few and have the unfinished pattern. Nevertheless, the results of these investigations are expressive and valuable. Works of Baumslag [16], [17], [18], Corner [19], Takashi and Irwin [20], Goldmith and Gong [21], [22], [23], [24], Braun and Strungmann [25] are the most significant. Specifically, Corner has constructed the beautiful and at first sight quite surprising examples of A, B and C Hopfian Abelian torsion-free groups which are such that $A \oplus B$, $C \oplus C$ direct sums are not the Hopfian groups. Goldmith and Gong papers contain, along with Hopfian and *co-Hopfian*[2] Abelian groups, consideration of the super (co)-Hopfian and hereditarily (co)-Hopfian Abelian groups as well as some related problems.

In the present work we continue the Hopfian Abelian groups investigation which was begun in [26]. One condition of the completely decomposable torsion-free Abelian group Hopficity is presented in the second paragraph. The third paragraph contains the constructed examples of the non-Hopfian completely decomposable torsion-free Abelian groups.

Let's assume the following conditions. Hereinafter the term "group" defines the additively written Abelian group. All designations and terms are standard and comply with [27], [28], [29]. Necessary definitions and results are given as needed. Symbol □ means the proof end or its absence.

[1]Group is called the super-Hopfian (super-co-Hopfian) if its any epimorphic image is the Hopfian (co-Hopfian, respectively) group.

[2]G group is called the hereditarily Hopfian (hereditarily co-Hopfian) if any subgroup of G group is Hopfian (co-Hopfian, respectively) one.

2. Sufficient Condition of Completely Decomposable Torsion-Free Group Hopficity

There are exist quiet wide classes of the periodic groups being rather describable with the help of invariants, however the torsion-free group classes with the sufficiently developed structural theory are few and relatively small. Class of the completely decomposable groups is one of them. Torsion-free A group is called the *completely decomposable* if such group is the result of rank 1 groups direct sum. Free and divisible groups without torsion represent the simple examples of completely decomposable groups.

The theorem below partially answers the following question: Under what conditions the completely decomposable torsion-free group is the Hopfian one?

Theorem 1. *Let A is the completely decomposable torsion-free group and all its homogeneous components have the finite rank at that, and set of types of group rank 1 direct summands satisfies the minimality condition. Then A is the Hopfian group.*

Proving. Let's write direct decomposition $A = \bigoplus_{i \in I} A_i$, where A_i – A group homogenous components. Under hypothesis of theorem, set of types $\Omega(A) = \{t(A_i) \mid i \in I\}$ satisfies the minimality condition. For every $k \in I$ it can be written $A_k \otimes B_k$, where $B_k = \bigotimes_{i \neq k} B_i$.

Assume on the contrary, that A group is non-Hopfian. Let's fix some epimorphism ϕ of A group on itself which is not the automorphism. We'll prove that for any index $k \in I$ and any non-zero element $a_k \in A_k$ it is true that $\phi(a_k) \notin B_k$, i.e. $\phi(a_k)$ has non-zero coordinate in A_k (specifically $\phi(a_k \neq 0)$.

Let firstly index $k \in I$ is such that type $t(A_k)$ minimal in the set of types $\Omega(A)$. Assume that there was found the element $a_k \in A_k$ with the following properties: $\phi(a_k) \neq 0$ and $\phi(a_k) \in B_k$.

Recall that for every $i \in I$ the decomposition $A = A_i \oplus B_i$. As the type $t(A_k)$ is minimal among the types $t(A_i)$, then $Hom(B_k, A_k) = 0$. Let $\pi_k \colon A \to A_k$ is projection relative to direct decomposition $A = A_k \oplus B_k$. Then $A_k \subseteq \pi_k \phi A_k$ (or equivalently $A_k \subseteq \pi_k \phi|_{A_k} A_k$), taking into account that ϕ is the epimorphism. From $a_k \in Ker A_k \pi_k \phi$ we conclude that $r(\pi_k \phi A_k) = r(A_k / Ker \pi_k \phi) < r(A_k)$. But this inequality is contrary to injection $A_k \subseteq \pi_k \phi A_k$.

It is well known that the set of homogeneous component types of completely decomposable torsion-free group is the distributive lattice [28] (§85). We know also that in the partially ordered set the minimality and inductivity conditions are the equivalents [30] (chapter I, §85), therefore further study will be carried out for induction. Namely, let's suppose now that k is such index that type

$t(A_k)$ is not the minimal in the set of types $\Omega(A)$ and for all l, such that $t(A_t) < t(A_k)$, the statement is proven.

Let's first show the following. Let a_k is some non-zero element of homogeneous component A_k. If b is such element of A group that $\phi(b) = a_k$, it can be presented as $b = b_k + y$, where $b_k \in A_k, y_k \in B_k$ and $\phi(b) \in B_k$. Note that $b_k \neq 0$. Assume the contrary condition. Relative to direct decomposition $A = \bigoplus_{i \in I}$ we'll write: $b = b_1 + \ldots + b_l + b_k + c_1 + \ldots + c_t$. Here $b_1, \ldots, b_l$ are such elements that $\phi(b_1), \ldots, \phi(b_l)$ have the non-zero coordinate in A_k. According to assumption, such elements definitely exist. Elements $c_1, \ldots, c_l$, if any, are such that $\phi(c_1), \ldots, \phi(c_l)$ have zero coordinate in A_k, i.e. B_k contains the images of these elements ar epimorphism ϕ.

Let's write direct decomposition $A = A_1 \oplus \ldots \oplus A_l \oplus A_k \oplus A'_1 \oplus \ldots A'_t \oplus B$, where A_j and A'_n for all $j = 1, \ldots, l$ and $n = 1, \ldots, t$ are essentially suitable direct summands A_i from decomposition $A = \bigoplus_{i \in I} A_i$, B is additional summand. It is understandable that for all j the equalities $t(B_j) < t(a_k)$ are true. According to induction assumption, the elements $\phi(b_1), \ldots, \phi(b_l)$ have non-zero coordinates in $A_1, \ldots, A_l$, respectively. Let for the purpose of determinacy type $t(b_1)$ is minimal among the types $t(b_1), \ldots, t(b_l)$. Thus, $\phi(b_1)$ has non-zero coordinate in A_1. Clear that among the elements $c_1, \ldots, c_t$ there exist at the least one such that its image at epimorphism ϕ has non-zero coordinate in A_1. Let c_1 is this element. Then $t(c_1) < t(b_1) < t(a_k)$. According to inductive assumption, $\phi(c_1)$ has non-zero coordinate in A'_1. So, among the elements $b_1, \ldots, b_l, b_k, c_l, \ldots, c_t$ it will be found at the least one such that its image at epimorphism ϕ has non-zero coordinate in A'_1. Let's designate this element as x. Then $t(x) < t(c_1)$. Taking into account minimality of $t(b_1)$ and that $t(c_1) < t(a_k)$, we get that x is one of the elements $c_2, \ldots, c_l$. Let for the purpose of simplification $x = c_2$. Thus, $t(c_2) < t(c_1) < t(a_k)$. Hence conclude that $\phi(c_2)$ has non-zero coordinate in A_2. Then by analogy for the elements $c_2, \ldots, c_t$ and, due to finiteness of their number, we get conflict. Hence $b_1 = \ldots = b_l = 0$, and therefore $b = b_k + c_1 + \ldots + c_t = b_k + y$. Recalling that the elements $\phi(c_1), \ldots, \phi(c_t)$ have non-zero coordinate in A_k, we get $\phi(y) \in B_k$.

Let's return to proving of our statement. Assume that there was found the element $a_k \in A_k$ for which $\phi(a_k) \neq 0$ and $a_k \in B_k$. Let's select in A group the element b with property $\phi(b) = a_k$. According to proven, $b = b_k + y$, where $b_k \in A_k$, $y \in B_k$ and $\phi(y) \in B_k$ (as it was already noted, $b_k \neq 0$).

Applying lemma 86.8 [28], we'll get the direct decomposition $A_k =< a_k > * \oplus C$ for some C group. Let's write element b_k relative to this decomposition: $b_k = a' + c_k$. Here $c_k \neq 0$. Really, if $c_k = 0$, then $a_k = \phi(b) = \phi(b_k) + \phi(y) = \phi(a') + \phi(y)$. But $a' \in < a_k > *$ and $\phi(a_k) \in B_k$, therefore $\phi(a') \in B_k$, Hence

$a_k \in B_k$ that is impossible. Further, $\phi(b_k) = \phi(b) - \phi(y) \in < a_k > * \oplus B_k$ and $\phi(c_k) = \phi(b_k) - \phi(a') \in < a_k > * \oplus B_k$.

Now let's write the direct decomposition $A_k =< a_k > *\oplus < c_k > * \oplus E$ for some E group. Then we'll perform the similar step. Namely, there exists the element $d \in A$ with property $\phi(d) = c_k$. Then $d = d_k + z$, where $d_k \in A_k, d_k \neq 0, \phi(z) \in B_k$. Write $d_k = a'' + c'' + e_k$ where $a'' \in < a_k > *$, $c'' \in < c_k > *$, $e_k \in E$. Here again $e_k \neq 0$, as from $e_k = 0$ we get $c_k = \phi(d) = \phi(d_k) + \phi(z) = \phi(a'') + \phi(c'') + \phi(z)$, therefore $c_k \in < a_k > * \oplus B_k$. But this is impossible. Consequently, there takes place the direct decomposition $A_k =< a_k > *\oplus < c_k > *\oplus < e_k > * \oplus G$ for some G group, and $\phi(e_k) \in < a_k > *\oplus < c_k > * \oplus B_k$.

By analogy we'll get the direct decompositions of homogenous component A_k with any amount of the rank 1 direct summands. But it is impossible due to finiteness of A_k group rank. So, we get conflict. Thus, we have proven that for any $k \in I$ and any non-zero $a_k \in A_k$ there will be $\phi(a_k) \notin B_k$.

Let's select now in A group such non-zero element a, that $\phi(a) = 0$. Write $a = a_1 + \ldots + a_n$, where $a_n \in A_i$, and $\phi(a_i) \neq 0$ for every I at that. Let for the purpose of determinacy the type $t(a_1)$ is minimal among the types $t(a_1), \ldots, t(a_n)$. We state that $\phi(a_2) \in B_1, \ldots, \phi(a_n) \in B_1$. If, for example, $\phi(a_2) \notin B_1$, then $\phi(a_2)$ has non-zero coordinate w in A_1. Thus, $t(a_2) \leq t(w) = t(a_1)$. Strict inequality is impossible on base of $t(a_1)$ minimality. We have $t(a_2) = \ldots = t(a_n) = t(a_1)$, hence $\phi(a_2) \in B_1, \ldots, \phi(a_n) \in B_1$. It is clear that then $\phi(a_1) \in B_1$. We obtained that $a_1 \in A_1$, and $\phi(a_1) \in B_1$. But such situation is impossible as established above, therefore our assumption on A group non-Hopficity is incorrect. □

On base of proven theorem it is simply to understand that there exists the variety of non-Hopfian completely decomposable torsion-free groups. It is usefully to construct the examples of such groups.

3. Examples of Non-Hopfian Completely Decomposable Torsion-Free Group Hopficity

Prior to give the examples, let's formulate one auxiliary question and give the answer. Let A, B, C are the rank 1 torsion-free groups, and $t(B) < t(A)$ and $t(C) < t(A)$. Let's explore when the epimorphism $B \oplus C \to A$.

As $t(B) < t(A)$ and $t(C) < t(A)$, B and C can be considered as the subgroups in A. In this case the epimorphism $B \oplus C \to A$ existence is equivalence to $t(B + C) = t(A)$ equality satisfaction. Really, let $\phi\colon B \oplus C \to A$ is some epimorphism. Then, $\phi(B \oplus C) = \phi(B) + \phi(C) = A$. It is clear that the limitations $\phi|_B : B \to A$, $\phi|_C : C \to A$ are essentially the monomrphisms. Consequently, $\phi(B) \cong B$ and $\phi(C) \cong C$. If we consider that B and C are the subgroups in A, then $\phi(B)$ and B are quasiequal groups, and similarly $\phi(C)$ and C are quasiequal groups. Hence $B + C$ and A are quasiequal groups too. Consequently, $t(B + C) = t(A)$.

Let's assume inversely that $t(B + C) = t(A)$. Then $B + C$ and A are quasiequal groups. More precisely, $B+C = nA$ for some natural number n. Now it simply to construct the epimorphism $B \oplus C \to A$. Let $\phi\colon B \oplus C \to B + C$ is sum of some isomorphisms $B \cong B$ and $C \cong C$. Then let's take the isomorphism $\psi\colon nA \to A$, $\psi\colon na \to a$. Composition $\psi\phi$ will represent the epimorphism $B \oplus C \to A$.

Thus, the epimorphism $B \oplus C \to A$ existence is equavalently to equality $t(B + C) = t(A)$, if B and C are considered as the subgroups in A.

The next simple fact is initiated by theorem 1.4 [31] which contains description of rank 1 torsion-free factor group.

Lemma 3.1 [31, §1, exercise 1.6] *If B and C are the subgroups of rank 1 torsion-free group, then $t(B + C) = sup\{t(B), t(C)\}$.* □

This lemma along with the prevous reasoning leads to such result.

Consequence 3.2. Let A, B and C are the rank 1 torsion-free group, and $t(B) < t(A)$ and $t(C) < t(A)$. Then the epimorphism $B \oplus C \to A$ existence is equavalently to $sup\{t(B), t(C)\} = t(A)$ equality validity. □

Note that lemma 3.1 and consequence 3.2. can be apply to any finite set subgroups of rank 1 torsion-free group [31] (§1, exercise 1.6).

With the help of consequence 3.2 it is now easily to construct the non-Hopfian completely decomposable torsion-free groups where all homogeneous components have the finite rank. Namely, we can select the rank 1 torsion-free groups $A_1^{(1)}, A_2^{(1)}, A_2^{(2)}$ in such way that the epimorphism $A_2^{(1)} \oplus A_2^{(2)} \to A_1^{(1)}$ exists. Then let select the groups $A_3^{(1)}, A_3^{(2)}, A_3^{(3)}$ and $A_3^{(4)}$ for which the epimorphisms $A_3^{(1)} \oplus A_3^{(2)} \to A_2^{(1)}, A_3^{(3)} \oplus A_3^{(4)} \to A_2^{(2)}$ exist. Then we repeat the analogical constructions for every from groups $A_3^{(1)}, A_3^{(2)}, A_3^{(3)}, A_3^{(4)}$, etc. Let A means the direct sum

$$\oplus A_1^{(1)} \oplus (A_2^{(1)} \oplus A_2^{(2)}) \oplus (A_3^{(1)} \oplus A_3^{(2)} \oplus A_3^{(2)} \oplus A_3^{(4)}) \oplus \dots$$

of all constructed with specified method groups $A_i^{(j)}$, $i, j \in N$. Representation which transfers $A_1^{(1)}$ group to zero, coincides on $A_2^{(1)} \oplus A_2^{(2)}$ with any epimorphism $A_2^{(1)} \oplus A_2^{(2)} \to A_1^{(1)}$, on $A_3^{(1)} \oplus A_3^{(2)}$ with any epimorphism $A_3^{(1)} \oplus A_3^{(2)} \to A_2^{(1)}$, etc., will be the epimorphism but not the automorphism of A group. Therefore, A is non-Hopfian group.

Specifically, assuming that $t(A_1^{(1)}) = (1,1,1,1,\ldots)$, $t(A_2^{(1)}) = (1,0,1,0,\ldots)$, $t(A_2^{(2)}) = (0,1,0,1,\ldots)$, etc., we'll get the specific example of non-Hopfian completely decomposable torsion-free group.

Evidently that the more sophisticated non-Hopfian completely decomposable groups exist. In the knots of the correspondent trees showing such groups, the direct sums of rank 1 any groups may be present.

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