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ON LINEARLY TOPOLOGIZED MODULES OVER AN ARBITRARY RING

Patricia Couto G. Mauro[1], Dinamérico P. Pombo Jr.[2 §]

[1]Centro Interdisciplinar de Ciencias da Natureza
Universidade Federal da Integração Latino-Americana
Foz do Iguaçu, 85867-970, BRASIL

[2]Instituto de Matemática e Estatística
Universidade Federal Fluminense
Niterói, 24020-140, BRASIL

Abstract: Fundamental constructions in the category of linearly topologized modules over an arbitrary ring are studied, as well as certain linearly topologized modules of multilinear mappings over an arbitrary commutative ring.

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1. Introduction

The notion of a linear topology plays an important role in various branches of Mathematics, such as Algebraic Geometry [7], Algebraic Topology [11], Commutative Algebra [2], and Functional Analysis [9]. In the present paper certain aspects of the study of linear topologies on modules are investigated, as we shall briefly indicate below.

In Section 2, linearly topologized modules, that is, unitary left modules (over a given ring with an identity element) endowed with a linear topology are

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§Correspondence author

considered. Fundamental constructions in the category of linearly topologized modules, such as products, projective limits, direct sums and inductive limits, are discussed, as well as the pertinent universal properties. It is also shown that linear topologies may be characterized by means of discrete topologies. It should be mentioned that some of the results of this section have already been obtained in the special case where R is a discrete valuation ring [12].

In Section 3, linear topologies of uniform convergence on modules of multilinear mappings over a given commutative ring with an identity element are studied. The main result is a representation of the completion of certain linearly topologized modules of multilinear mappings which was inspired by Grothendieck's completeness theorem.

Throughout this work R will denote a (not necessarily commutative) ring with a non-zero identity element and "R-module" will mean "unitary left R-module".

2. Constructions of Linearly Topologized *R*-Modules and Universal Properties

In this paper we shall be dealing with the following notion [2, p. 236]:

Definition 2.1. A topology τ on an R-module E is said to be *linear*, and (E, τ) is said to be a *linearly topologized R-module*, if τ is translation-invariant and $0 \in E$ admits a fundamental system of τ-neighborhoods consisting of submodules of E.

Example 2.2. The chaotic topology and the discrete topology on an R-module are linear topologies.

Example 2.3. $[8, p.171]$ Let S be an ideal of R, E an R-module, and put

$$\mathcal{U} = \{S^n E \,|\, n = 1, 2, 3, \ldots\}$$

By Proposition 1, p.222 of [5], there is a unique group topology τ_S on the abelian group $(E, +)$ for which $\mathcal{U}$ is a fundamental system of τ_S- neighborhoods of 0. Thus, since the elements of $\mathcal{U}$ are submodules of E, it follows that τ_S is a linear topology on E, called the topology deduced from S.

Proposition 2.4. *Let $((E_i, \tau_i))_{i \in I}$ be a family of linearly topologized R-modules, E an R-module and $u_i : E \to E_i$ an R-linear mapping $(i \in I)$. If τ is the initial topology for the family $((E_i, \tau_i), u_i)_{i \in I}$, then τ is a linear topology on E. In particular, the following universal property holds: for each linearly topologized R-module (F, θ) and for each R-linear mapping $u : F \to E$, we have* *that $u : (F, \theta) \to (E, \tau)$ is continuous if and only if $u_i \circ u : (F, \theta) \to (E_i, \tau_i)$ is continuous for all $i \in I$.*

$$\begin{tikzcd}
(F,\theta) \arrow[rr, "u"] \arrow[dr, "u_i \circ u"'] & & (E,\tau) \arrow[dl, "u_i"] \\
& (E_i,\tau_i) &
\end{tikzcd}$$

Proof. Let x_0 be an arbitrary element of E and put $g(x) = x + x_0$ for $x \in E$. We claim that $g : (E, \tau) \to (E, \tau)$ is continuous. In fact, for each $i \in I$ let us consider the diagram

$$\begin{tikzcd}
(E,\tau) \arrow[rr, "g"] \arrow[dr, "u_i \circ g"'] & & (E,\tau) \arrow[dl, "u_i"] \\
& (E_i,\tau_i) &
\end{tikzcd}$$

Since $(u_i \circ g)(x) = u_i(x) + u_i(x_0)$ for all $i \in I$ and for all $x \in E$, it follows from Proposition 4, p. 30 of [5] that g is continuous. Consequently, τ is translation-invariant.

Now, for each $i \in I$ let $\mathcal{U}_i$ be a fundamental system of τ_i-neighborhoods of 0 in E_i consisting of submodules of E_i. Then $0 \in E$ admits a fundamental system of τ-neighborhoods consisting of submodules of E, since the set of all finite intersections of sets of the form $u_i^{-1}(U_i)$ $(i \in I, U_i \in \mathcal{U}_i)$ constitutes a fundamental system of τ-neighborhoods of 0 in E. Therefore τ is a linear topology.

Corollary 2.5. *(a) If $((E_i, \tau_i))_{i \in I}$ is a family of linearly topologized R-modules, then $(\prod_{i \in I} E_i, \prod_{i \in I} \tau_i)$ is a linearly topologized R-module, where $\prod_{i \in I} E_i$ is the product R-module and $\prod_{i \in I} \tau_i$ is the product topology on $\prod_{i \in I} E_i$.*
(b) If (E, τ) is a linearly topologized R-module and M is a submodule of E, then (M, τ_M) is a linearly topologized R-module, where τ_M is the topology induced by τ on M.
(c) If $(\tau_i)_{i \in I}$ is a family of linear topologies on an R-module E, then $\sup_{i \in I} \tau_i$

is a linear topology on E.

Proof. Follows immediately from Proposition 2.4.

We have already mentioned that the discrete topology on an R-module is linear. Now we shall prove that every linear topology comes from a family of discrete topologies. More precisely:

Theorem 2.6. *Let τ be a topology on an R-module E. In order that τ be a linear topology on E, it is necessary and sufficient that τ be an initial topology with respect to a family of discrete R-modules and R-linear mappings.*

Proof. The sufficiency follows from Example 2.2 and Proposition 2.4. In order to prove the necessity, we shall need the following auxiliary lemmas.

Lemma 2.7. *Let (E, τ) be a linearly topologized R-module and U a τ-neighborhood of 0 in E which is a submodule of E. Then U is τ-open and τ-closed.*

Proof. For each $x \in U$, we have that $x + U \subset U$, $x + U$ being a τ-neighborhood of x in E; thus U is τ-open. On the other hand, for each $x \in E \backslash U$, we have that $(x + U) \cap U = \emptyset$; thus U is τ-closed.

Lemma 2.8. *Let (E, τ) be a linearly topologized R-module.*
(a) If M is a submodule of E, the quotient R-module E/M endowed with the quotient topology is a linearly topologized R-module.
(b) If U is a τ-neighborhood of 0 in E which is a submodule of E, the quotient topology τ' on E/U is the discrete topology.

Proof.
(a): Straightforward.
(b): In fact, let $\pi : E \to E/U$ be the canonical surjection. Since $\pi : (E, \tau) \to (E/U, \tau')$ is an open mapping and U is τ-open (Lemma 2.7), then $\pi(U) = \{0\}$ is τ'-open. Consequently, τ' is the discrete topology.

Now, in order to prove the necessity, let us assume that τ is a linear topology on E and let $\mathcal{U}$ be a fundamental system of τ-neighborhoods of 0 in E consisting of submodules of E. For each $U \in \mathcal{U}$, E/U is a discrete R-module with respect to the quotient topology τ_U in view of Lemma 2.8(b). For each $U \in \mathcal{U}$ let $\pi_U : E \to E/U$ be the canonical surjection, and let $\tilde{\tau}$ be the initial topology for the family $((E/U, \tau_U), \pi_U)_{U \in \mathcal{U}}$ ($\tilde{\tau}$ is a linear topology on E in view of Example 2.2 and Proposition 2.4).

We claim that $\tau = \tilde{\tau}$. In fact, if $U \in \mathcal{U}$, the equality $U = \pi_U^{-1}(\{0\})$ implies that U is a $\tilde{\tau}$-neighborhood of 0 in E. Therefore τ is coarser than $\tilde{\tau}$. On the other hand, if U is a $\tilde{\tau}$-neighborhood of 0 in E, there are $U_1, \ldots, U_n \in \mathcal{U}$ so that $\bigcap_{i=1}^{n} \pi_{U_i}^{-1}(\{0\}) \subset U$. Therefore the equalities $\pi_{U_i}^{-1}(\{0\}) = U_i$ $(i = 1, \ldots, n)$ imply that U is a τ-neighborhood of 0 in E, and $\tilde{\tau}$ is coarser than τ.

This completes the proof of the theorem.

Definition 2.9. Let $(I, \leq)$ be a partially ordered set. $((E_i, \tau_i), u_{ij})_{i \in I}$ is said to be a *projective system of linearly topologized R-modules* if (E_i, τ_i) is a linearly topologized R-module for all $i \in I$, $u_{ij} : (E_j, \tau_j) \to (E_i, \tau_i)$ is a continuous R-linear mapping for $i, j \in I$ with $i \leq j$, $u_{ii} = 1_{E_i}$ for all $i \in I$, and $u_{ij} \circ u_{jk} = u_{ik}$ for $i, j, k \in I$ with $i \leq j \leq k$.

Consider the product R-module $\prod_{i \in I} E_i$ and, for each $j \in I$, the projection $pr_j : \prod_{i \in I} E_i \to E_j$ on the j-th factor. The submodule

$$\varprojlim E_i = \{x \in \prod_{i \in I} E_i \mid (u_{jk} \circ pr_k)(x) = pr_j(x) \text{ for all } j, k \in I \text{ with } j \leq k\}$$

of $\prod_{i \in I} E_i$ is said to be the projective limit of the projective system $(E_i, u_{ij})_{i \in I}$ of R-modules. For each $j \in I$ put $u_j = pr_{j|\varprojlim E_i}$, called the canonical R-linear mapping from $\varprojlim E_i$ into E_j. Then the diagram

$$\begin{tikzcd} E_k \arrow[rr, "u_{jk}"] & & E_j \\ & \varprojlim E_i \arrow[ul, "u_k"] \arrow[ur, "u_j"'] & \end{tikzcd}$$

commutes for $j \leq k$. If τ is the initial topology for the family $((E_j, \tau_j), u_j)_{j \in I}$, which is a linear topology by Proposition 2.4, $(\varprojlim E_i, \tau)$ is said to be the *topological projective limit* of the system $((E_i, \tau_i), u_{ij})_{i \in I}$.

Remark 2.10. It is easily seen that τ coincides with the topology induced by $\prod_{i \in I} \tau_i$ on $\varprojlim E_i$.

Remark 2.11. Let $((E_i, \tau_i))_{i \in I}$ be an arbitrary family of linearly topologized R-modules and consider the set I endowed with the equality relation (which is a partial order on I). If we define $u_{ii} = 1_{E_i}$ for all $i \in I$, it is clear that $((E_i, \tau_i), u_{ii})_{i\in I}$ is a projective system of linearly topologized R-modules whose topological projective limit $(\varprojlim E_i, \tau)$ coincides with $(\prod\limits_{i\in I} E_i, \prod\limits_{i\in I} \tau_i)$.

Example 2.12. (a) Let p be a prime natural number.

For each integer $m \geq 1$ let us consider the $\mathbb{Z}$-module $\mathbb{Z}/p^m\mathbb{Z}$ endowed with the discrete topology τ_m, which makes $\mathbb{Z}/p^m\mathbb{Z}$ into a compact linearly topologized $\mathbb{Z}$-module. Given two integers $1 \leq m \leq n$, the canonical $\mathbb{Z}$-linear mapping $u_{mn} : (\mathbb{Z}/p^n\mathbb{Z}, \tau_n) \to (\mathbb{Z}/p^m\mathbb{Z}, \tau_m)$ is continuous and $((\mathbb{Z}/p^m\mathbb{Z}, \tau_m), u_{mn})_{m\geq 1}$ is a projective system of linearly topologized $\mathbb{Z}$-modules. Therefore we may consider the topological projective limit of this system (see also [14, p. 11]).
(b) (based on Exercise 1, p.399 of [4])For each integer $m \geq 1$ let E_m be the $\mathbb{Z}$-module $\mathbb{Z}$ endowed with the discrete topology τ_m, which makes E_m into a linearly topologized $\mathbb{Z}$-module. For integers $1 \leq m \leq n$, consider the $\mathbb{Z}$-linear mapping $u_{mn} : E_n \to E_m$ given by $u_{mn}(x) = 3^{n-m}x$. Then $((E_m, \tau_m), u_{mn})_{m\geq 1}$ is a projective system of linearly topologized $\mathbb{Z}$-modules, and therefore we may consider the topological projective limit of this system.

Proposition 2.13. *Under the conditions of Definition 2.9, the following universal property holds: for each linearly topologized R-module (F, θ) and for each family $(\alpha_i : (F, \theta) \to (E_i, \tau_i))_{i\in I}$ of continuous R-linear mappings such that the diagram*

$$\begin{tikzcd} E_j \arrow[rr, "u_{ij}"] & & E_i \\ & F \arrow[ul, "\alpha_j"] \arrow[ur, "\alpha_i"'] & \end{tikzcd}$$

commutes for $i, j \in I$ with $i \leq j$, there exists a unique continuous R-linear mapping $u : (F, \theta) \to (\varprojlim E_i, \tau)$ such that the diagram

$$\begin{tikzcd} F \arrow[rr, "u"] \arrow[dr, "\alpha_i"'] & & \varprojlim E_i \arrow[dl, "u_i"] \\ & E_i & \end{tikzcd}$$

commutes for all $i \in I$.

Proof. Since the uniqueness is clear, let us prove the existence. For this purpose, let us consider the R-linear mapping $u : F \to \prod_{i \in I} E_i$ given by $u(y) = (\alpha_i(y))_{i \in I}$ for $y \in F$. Since

$$(u_{ij} \circ pr_j)(u(y)) = u_{ij}(\alpha_j(y)) = \alpha_i(y) = pr_i(u(y))$$

for all $y \in F$ and for all $i, j \in I$ with $i \leq j$, then $Im(u) \subset \varprojlim E_i$, and u may be regarded as an R-linear mapping from F into $\varprojlim E_i$. By definition, $u_i \circ u = \alpha_i$ for all $i \in I$. Finally, by Proposition 2.13, $u : (F, \theta) \to (\varprojlim E_i, \tau)$ is continuous because $\alpha_i : (F, \theta) \to (E_i, \tau_i)$ is continuous for all $i \in I$.

Corollary 2.14. *Let $((E_i, \tau_i), u_{ij})_{i \in I}$ and $((F_i, \theta_i), v_{ij})_{i \in I}$ be two projective systems of linearly topologized R-modules, and let $(\varprojlim E_i, \tau)$ and $(\varprojlim F_i, \theta)$ be the corresponding topological projective limits. For each $i \in I$ let $\beta_i : (E_i, \tau_i) \to (F_i, \theta_i)$ be a continuous R-linear mapping such that the diagram*

$$\begin{tikzcd} E_j \arrow[r, "\beta_j"] \arrow[d, "u_{ij}"'] & F_j \arrow[d, "v_{ij}"] \\ E_i \arrow[r, "\beta_i"'] & F_i \end{tikzcd}$$

commutes for $i, j \in I$ with $i \leq j$. Then there exists a unique continuous R-linear mapping $u : (\varprojlim E_i, \tau) \to (\varprojlim F_i, \theta)$ such that the diagram

$$\begin{tikzcd} \varprojlim E_i \arrow[r, "u"] \arrow[d, "u_i"'] & \varprojlim F_i \arrow[d, "v_i"] \\ E_i \arrow[r, "\beta_i"'] & F_i \end{tikzcd}$$

commutes for all $i \in I$, where u_i and v_i are the canonical R-linear mappings.

Proof. For each $i \in I$ let us consider the continuous R-linear mapping

$\alpha_i = \beta_i \circ u_i : (\varprojlim E_i, \tau) \to (F_i, \theta_i)$. Since

$$v_{ij} \circ \alpha_j = (v_{ij} \circ \beta_j) \circ u_j = (\beta_i \circ u_{ij}) \circ u_j = \beta_i \circ (u_{ij} \circ u_j) = \beta_i \circ u_i = \alpha_i$$

for $i \leq j$, Proposition 2.13 ensures the existence of a unique continuous R-linear mapping $u : (\varprojlim E_i, \tau) \to (\varprojlim F_i, \theta)$ such that $v_i \circ u = \alpha_i$ for all $i \in I$, as asserted.

Proposition 2.15. *Let $((E_i, \tau_i))_{i\in I}$ be a family of linearly topologized R-modules, E an R-module and $u_i : E_i \to E$ an R-linear mapping ($i \in I$). Then there exists a unique linear topology τ on E which is final for the family $((E_i, \tau_i), u_i)_{i\in I}$, in the following sense: for each linearly topologized R-module (F, θ) and for each R-linear mapping $u : E \to F$, we have that $u : (E, \tau) \to (F, \theta)$ is continuous if and only if $u \circ u_i : (E_i, \tau_i) \to (F, \theta)$ is continuous for all $i \in I$.*

$$\begin{tikzcd}
(E,\tau) \arrow[rr, "u"] & & (F,\theta) \\
& (E_i,\tau_i) \arrow[ul, "u_i"] \arrow[ur, "u \circ u_i"'] &
\end{tikzcd}$$

Proof. Put

$$\begin{aligned}
\mathcal{U} =& \{U \subset E \,|\, U \text{ is a submodule of } E \text{ and } u_i^{-1}(U) \text{ is a } \\
& \tau_i\text{-neighborhood of 0 in } E_i \text{ for all } i \in I\}, \\
\mathcal{U}(0) =& \{\tilde{U} \subset E \,|\, \text{ there is a } U \in \mathcal{U} \text{ with } U \subset \tilde{U}\},
\end{aligned}$$

and

$$\mathcal{U}(x) = \{x + \tilde{U} \,|\, \tilde{U} \in \mathcal{U}(0)\} \text{ for } x \in E.$$

Then, for each $x \in E$, the following properties hold:
(a) $x \in x + \tilde{U}$ for all $\tilde{U} \in \mathcal{U}(0)$ (obvious).
(b) If $\tilde{U} \in \mathcal{U}(0)$ and $x + \tilde{U} \subset V \subset E$, then $V \in \mathcal{U}(x)$ (obvious).
(c) If $\tilde{U}_1, \tilde{U}_2 \in \mathcal{U}(0)$, there are $U_1, U_2 \in \mathcal{U}$ with $U_1 \subset \tilde{U}_1$ and $U_2 \subset \tilde{U}_2$. Since

$$U_1 \cap U_2 \in \mathcal{U}, \; U_1 \cap U_2 \subset \tilde{U}_1 \cap \tilde{U}_2 \text{ and } (x + \tilde{U}_1) \cap (x + \tilde{U}_2) = x + (\tilde{U}_1 \cap \tilde{U}_2),$$

it follows that $(x+\tilde{U}_1)\cap(x+\tilde{U}_2)\in\mathcal{U}(x)$.
(d) If $\tilde{U}\in\mathcal{U}(0)$, there is a $U\in\mathcal{U}$ with $U\subset\tilde{U}$. Then $x+U\in\mathcal{U}(x)$ and $y+U\subset x+U+U=x+U\subset x+\tilde{U}$ for all $y\in x+U$, which guarantees that $x+\tilde{U}\in\mathcal{U}(y)$ for all $y\in x+U$.

Therefore, by Proposition 2, p. 19 of [5], there exists a unique topology τ on E for which $\mathcal{U}(x)$ is the set of all τ-neighborhoods of x for each $x\in E$. By construction, $\mathcal{U}$ is a fundamental system of τ-neighborhoods of 0 in E, and hence τ is linear. And, by construction, each $u_i:(E_i,\tau_i)\to(E,\tau)$ is continuous.

Now, let (F,θ) and u be as in the statement of the proposition. It is obvious that each $u\circ u_i$ is continuous if u is continuous. Conversely, assume that each $u\circ u_i$ is continuous and let V be a θ-neighborhood of 0 in F which is a submodule of F. By hypothesis, for each $i\in I$ there is a τ_i-neighborhood U_i of 0 in E_i such that $(u\circ u_i)(U_i)\subset V$. Thus the submodule $u^{-1}(V)$ of E is such that $U_i\subset(u\circ u_i)^{-1}(V)=u_i^{-1}(u^{-1}(V))$ for all $i\in I$, which implies that $u^{-1}(V)\in\mathcal{U}$. Consequently, u is continuous, and the existence is established.

In order to prove the uniqueness, let $\tilde{\tau}$ be a linear topology on E satisfying the property given in the statement of the proposition. By taking $(F,\theta)=(E,\tilde{\tau})$ and $u=1_E:(E,\tilde{\tau})\to(F,\theta)$, we conclude that each $u_i:(E_i,\tau_i)\to(E,\tilde{\tau})$ is continuous. Thus, by taking $(F,\theta)=(E,\tilde{\tau})$ and $u=1_E:(E,\tau)\to(F,\theta)$, we conclude that $1_E:(E,\tau)\to(E,\tilde{\tau})$ is continuous. Finally, by considering $(F,\theta)=(E,\tau)$ and $u=1_E:(E,\tilde{\tau})\to(F,\theta)$, we get the continuity of $1_E:(E,\tilde{\tau})\to(E,\tau)$. Therefore $\tau=\tilde{\tau}$.

Remark 2.16. Let (E,τ) be a linearly topologized R-module, M a submodule of E and $\pi:E\to E/M$ the canonical surjection. It is easily seen that the final linear topology for the pair $((E,\tau),\pi)$ coincides with the quotient topology on E/M.

Definition 2.17. Let $((E_i,\tau_i))_{i\in I}$ be a family of linearly topologized R-modules, E the R-module $\bigoplus_{i\in I}E_i$ and, for each $i\in I$, let $\lambda_i:E_i\to E$ be the canonical injection (which is an R-linear mapping). If τ is the final linear topology for the family $((E_i,\tau_i),\lambda_i)_{i\in I}$, (E,τ) is said to be the *topological direct sum* of the family $((E_i,\tau_i))_{i\in I}$.

Definition 2.18. Let $(I,\leq)$ be a partially ordered set. $((E_i,\tau_i),u_{ji})_{i\in I}$ is said to be an *inductive system of linearly topologized R-modules* if (E_i,τ_i) is a linearly topologized R-module for all $i\in I$, $u_{ji}:(E_i,\tau_i)\to(E_j,\tau_j)$ is a continuous R-linear mapping for $i,j\in I$ with $i\leq j$, $u_{ii}=1_{E_i}$ for all $i\in I$, and $u_{kj}\circ u_{ji}=u_{ki}$ for $i,j,k\in I$ with $i\leq j\leq k$.

Let E and λ_i be as in Definition 2.17 and consider the submodule N of E

generated by the set

$$\{\lambda_j(x_j) - (\lambda_k \circ u_{kj})(x_j) \,|\, x_j \in E_j,\, j,k \in I,\, j \leq k\}.$$

The quotient R-module E/N is said to be the inductive limit of the inductive system $(E_i, u_{ji})_{i\in I}$ of R-modules and is denoted by $\varinjlim E_i$. For each $j \in I$ put $u_j = \pi \circ \lambda_j$, where $\pi : E \to E/N$ is the canonical surjection; u_j is said to be the canonical R-linear mapping from E_j into $\varinjlim E_i$. Then the diagram

$$\begin{tikzcd} E_j \arrow[rr, "u_{kj}"] \arrow[dr, "u_j"'] & & E_k \arrow[dl, "u_k"] \\ & \varinjlim E_i & \end{tikzcd}$$

commutes for $j \leq k$. In fact, for all $x_j \in E_j$,

$$\begin{aligned} &(u_k \circ u_{kj})(x_j) = u_j(x_j) \Leftrightarrow (\pi \circ \lambda_k)(u_{kj}(x_j)) = (\pi \circ \lambda_j)(x_j) \\ &\Leftrightarrow \pi(\lambda_j x_j - (\lambda_k \circ u_{kj})(x_j)) = 0 \Leftrightarrow \lambda_j(x_j) - (\lambda_k \circ u_{kj})(x_j) \in N. \end{aligned}$$

But, since the last assertion is true, we have $u_k \circ u_{kj} = u_j$. If we endow $\varinjlim E_i$ with the final linear topology τ for the family $((E_j, \tau_j), u_j)_{j\in I}$, $(\varinjlim E_i, \tau)$ is said to be the *topological inductive limit* of the system $((E_i, \tau_i), u_{ji})_{i\in I}$.

Remark 2.19. Let $((E_i, \tau_i), u_{ji})_{i\in I}$, E, N and τ be as in Definition 2.18. Then it is easy to see that the quotient topology on $E/N = \varinjlim E_i$ coincides with τ.

Remark 2.20. Let $((E_i, \tau_i))_{i\in I}$ be an arbitrary family of linearly topologized R-modules and consider the set I endowed with the equality relation. If we define $u_{ii} = 1_{E_i}$ for all $i \in I$, it is clear that $((E_i, \tau_i), u_{ii})_{i\in I}$ is an inductive system of linearly topologized R-modules whose topological inductive limit coincides with the topological direct sum of the family $((E_i, \tau_i))_{i\in I}$.

Example 2.21. Let $(E_m)_{m\in\mathbb{N}}$ be a strictly increasing sequence of submodules of an R-module E and, for $m, n \in \mathbb{N}$ with $m \leq n$, let $u_{nm} : E_m \to E_n$ be the inclusion mapping. If each E_m is endowed with a linear topology τ_m such that the topology induced by τ_{m+1} on E_m is coarser than τ_m, then $((E_m, \tau_m), u_{nm})_{m\in\mathbb{N}}$ is an inductive system of linearly topologized R-modules. Therefore we may consider the topological inductive limit of this system.

Given two arbitrary R-modules G and H, we shall denote by $\mathcal{L}_a(G;H)$ the additive group of all R-linear mappings from G into H.

Proposition 2.22. *Under the conditions of Definition 2.18, the following universal property holds: for each linearly topologized R-module (F,θ) and for each family $(\alpha_i : (E_i,\tau_i) \to (F,\theta))_{i\in I}$ of continuous R-linear mappings such that the diagram*

$$\begin{tikzcd} E_i \arrow[rr, "u_{ji}"] \arrow[dr, "\alpha_i"'] & & E_j \arrow[dl, "\alpha_j"] \\ & F & \end{tikzcd}$$

commutes for all $i,j \in I$ with $i \leq j$, there exists a unique continuous R-linear mapping $u : (\varinjlim E_i, \tau) \to (F,\theta)$ such that the diagram

$$\begin{tikzcd} \varinjlim E_i \arrow[rr, "u"] & & F \\ & E_i \arrow[ul, "u_i"] \arrow[ur, "\alpha_i"'] & \end{tikzcd}$$

commutes for all $i \in I$.

Proof. Since the group homomorphism

$$w \in \mathcal{L}_a(\bigoplus_{i\in I} E_i; F) \longmapsto (w \circ \lambda_i)_{i\in I} \in \prod_{i\in I} \mathcal{L}_a(E_i;F)$$

is bijective [11, p. 128], there exists a unique $v \in \mathcal{L}_a(\bigoplus_{i\in I} E_i; F)$ such that $v \circ \lambda_i = \alpha_i$ for all $i \in I$. Moreover, $N \subset Ker(v)$ (N being as is Definition 2.18), because

$$v(\lambda_j(x_j) - (\lambda_k \circ u_{kj})(x_j)) = (v \circ \lambda_j)(x_j) - (v \circ \lambda_k)(u_{kj}(x_j))$$
$$= \alpha_j(x_j) - (\alpha_k \circ u_{kj})(x_j) = \alpha_j(x_j) - \alpha_j(x_j) = 0$$

for $x_j \in E_j$ and $j \leq k$. By the isomorphism theorem, there is a unique $u \in \mathcal{L}_a(\varinjlim E_i; F)$ making the diagram

$$\begin{tikzcd}
\bigoplus_{i\in I} E_i \arrow[rr, "v"] \arrow[dr, "\pi"'] & & F \\
& \varinjlim E_i \arrow[ur, "u"'] &
\end{tikzcd}$$

commutative (π being as in Definition 2.18). Finally,

$$u \circ u_i = u \circ (\pi \circ \lambda_i) = (u \circ \pi) \circ \lambda_i = v \circ \lambda_i = \alpha_i$$

for all $i \in I$, and $u : (\varinjlim E_i, \tau) \to (F, \theta)$ is continuous in view of Proposition 2.15, because each $\alpha_i : (E_i, \tau_i) \to (F, \theta)$ is continuous. Therefore the existence is proved.

To prove the uniqueness, let $\tilde{u} \in \mathcal{L}_a(\varinjlim E_i; F)$ be such that $\tilde{u} \circ u_i = \alpha_i$ for all $i \in I$ and put $\tilde{v} = \tilde{u} \circ \pi \in \mathcal{L}_a(\bigoplus_{i\in I} E_i; F)$. Then

$$\tilde{v} \circ \lambda_i = \tilde{u} \circ (\pi \circ \lambda_i) = \tilde{u} \circ u_i = \alpha_i$$

for all $i \in I$, which implies $\tilde{v} = v$. Consequently, $\tilde{u} = u$, and the uniqueness is proved.

Corollary 2.23. *Let $((E_i, \tau_i), u_{ji})_{i\in I}$ and $((F_i, \theta_i), v_{ji})_{i\in I}$ be two inductive systems of linearly topologized R-modules, and let $(\varinjlim E_i, \tau)$ and $(\varinjlim F_i, \theta)$ be the corresponding topological inductive limits. For each $i \in I$ let $\beta_i : (E_i, \tau_i) \to (F_i, \theta_i)$ be a continuous R-linear mapping such that the diagram*

$$\begin{tikzcd}
E_j \arrow[r, "\beta_j"] & F_j \\
E_i \arrow[u, "u_{ji}"] \arrow[r, "\beta_i"'] & F_i \arrow[u, "v_{ji}"']
\end{tikzcd}$$

commutes for $i, j \in I$ with $i \leq j$. Then there exists a unique continuous R-linear mapping $u : (\varinjlim E_i, \tau) \to (\varinjlim F_i, \theta)$ such that the diagram

$$\begin{tikzcd} \varinjlim E_i \arrow[r, "u"] & \varinjlim F_i \\ E_i \arrow[u, "u_i"] \arrow[r, "\beta_i"'] & F_i \arrow[u, "v_i"'] \end{tikzcd}$$

commutes for all $i \in I$, where u_i and v_i are the canonical R-linear mappings.

Proof. Analogous to that of Corollary 2.14, by applying Proposition 2.22 in place of Proposition 2.13.

Given two arbitrary linearly topologized R-modules (G, τ) and (H, θ), we shall denote by $\mathcal{L}(G; H)$ the subgroup of $\mathcal{L}_a(G; H)$ consisting of all continuous R-linear mappings from (G, τ) into (H, θ).

Let $((E_\alpha, \tau_\alpha), u_{\beta\alpha})_{\alpha \in I}$ be an inductive system of linearly topologized R-modules and (E, τ) its topological inductive limit, and let $((F_\lambda, \theta_\lambda), v_{\lambda\mu})_{\lambda \in J}$ be a projective system of linearly topologized R-modules and (F, θ) its topological projective limit. For (α, λ), $(\beta, \mu) \in I \times J$ with $(\alpha, \lambda) \leq (\beta, \mu)$ (that is, with $\alpha \leq \beta$ and $\lambda \leq \mu$), let us consider the group homomorphism

$$\Phi_{(\alpha,\lambda)(\beta,\mu)} : \mathcal{L}(E_\beta; F_\mu) \longrightarrow \mathcal{L}(E_\alpha; F_\lambda)$$

given by $\Phi_{(\alpha,\lambda)(\beta,\mu)}(f) = v_{\lambda\mu} \circ f \circ u_{\beta\alpha}$ for $f \in \mathcal{L}(E_\beta; F_\mu)$. It is easily seen that $(\mathcal{L}(E_\alpha; F_\lambda), \Phi_{(\alpha,\lambda)(\beta,\mu)})_{(\alpha,\lambda) \in I \times J}$ is a projective system of abelian groups. For each $(\alpha, \lambda) \in I \times J$ let us consider the group homomorphism

$$\Psi_{(\alpha,\lambda)} : \mathcal{L}(E; F) \longrightarrow \mathcal{L}(E_\alpha; F_\lambda)$$

given by $\Psi_{(\alpha,\lambda)}(w) = v_\lambda \circ w \circ u_\alpha$ for $w \in \mathcal{L}(E; F)$, where $u_\alpha : E_\alpha \to E$ and $v_\lambda : F \to F_\lambda$ are the canonical R-linear mappings. Then

$$\Psi(w) = (\Psi_{(\alpha,\lambda)}(w))_{(\alpha,\lambda) \in I \times J} \in \varprojlim \mathcal{L}(E_\alpha; F_\lambda)$$

for all $w \in \mathcal{L}(E; F)$. In fact, if $(\alpha, \lambda) \leq (\beta, \mu)$,

$$\begin{aligned}(\Phi_{(\alpha,\lambda)(\beta,\mu)} \circ pr_{(\beta,\mu)})(\Psi(w)) &= \Phi_{(\alpha,\lambda)(\beta,\mu)}(\Psi_{(\beta,\mu)}(w)) = \Phi_{(\alpha,\lambda)(\beta,\mu)}(v_\mu \circ w \circ u_\beta) \\ &= (v_{\lambda\mu} \circ v_\mu) \circ w \circ (u_\beta \circ u_{\beta\alpha}) = v_\lambda \circ w \circ u_\alpha = pr_{(\alpha,\lambda)}(\Psi(w)).\end{aligned}$$

Under the conditions above, we may state the following

Theorem 2.24. *The group homomorphism*

$$\begin{aligned}\Psi : \mathcal{L}(E;F) &\longrightarrow \varprojlim \mathcal{L}(E_\alpha;F_\lambda)\\ w &\longmapsto (\Psi_{(\alpha,\lambda)}(w))_{(\alpha,\lambda)\in I\times J}\end{aligned}$$

is an isomorphism. Moreover, $\mathcal{X} \subset \mathcal{L}_a(E;F)$ is equicontinuous if and only if $v_\lambda \circ \mathcal{X} \circ u_\alpha \subset \mathcal{L}_a(E_\alpha;F_\lambda)$ is equicontinuous for all $(\alpha,\lambda) \in I \times J$.

Proof. Firstly, let us prove that Ψ is an isomorphism. Indeed, let $g = (g_{(\alpha,\lambda)})_{(\alpha,\lambda)\in I\times J} \in \varprojlim \mathcal{L}(E_\alpha;F_\lambda)$ be arbitrary. For each $\lambda \in J$, let us consider the family $(g_{(\alpha,\lambda)})_{\alpha\in I}$. Since

$$g_{(\beta,\lambda)} \circ u_{\beta\alpha} = v_{\lambda\lambda} \circ g_{(\beta,\lambda)} \circ u_{\beta\lambda} = \Phi_{(\alpha,\lambda)(\beta,\lambda)}(g_{(\beta,\lambda)}) = g_{(\alpha,\lambda)}$$

for $\alpha, \beta \in I$ with $\alpha \leq \beta$, Proposition 2.22 guarantees the existence of a unique $w_\lambda \in \mathcal{L}(E;F_\lambda)$ such that $w_\lambda \circ u_\alpha = g_{(\alpha,\lambda)}$ for all $\alpha \in I$.

$$\begin{tikzcd} E_\alpha \arrow[rr, "u_{\beta\alpha}"] \arrow[dr, "g_{(\alpha,\lambda)}"'] & & E_\beta \arrow[dl, "g_{(\beta,\lambda)}"] \\ & F_\lambda & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "w_\lambda"] & & F_\lambda \\ & E_\alpha \arrow[ul, "u_\alpha"] \arrow[ur, "g_{(\alpha,\lambda)}"'] & \end{tikzcd}$$

Now, let $\lambda, \mu \in J$ with $\lambda \leq \mu$. We claim that $w_\lambda = v_{\lambda\mu} \circ w_\mu$. In fact, we have

$$w_\lambda \circ u_\alpha = g_{(\alpha,\lambda)} = \Phi_{(\alpha,\lambda)(\alpha,\mu)}(g_{(\alpha,\mu)}) = v_{\lambda\mu} \circ g_{(\alpha,\mu)} = v_{\lambda\mu} \circ w_\mu \circ u_\alpha$$

for all $\alpha \in I$. Since E coincides with the submodule generated by $\bigcup\limits_{\alpha\in I} Im(u_\alpha)$, it follows that $w_\lambda = v_{\lambda\mu} \circ w_\mu$. Thus, by Proposition 2.13, there exists a unique $w \in \mathcal{L}(E;F)$ such that $v_\lambda \circ w = w_\lambda$ for all $\lambda \in J$. Consequently,

$$\begin{tikzcd} F_\mu \arrow[rr, "v_{\lambda\mu}"] & & F_\lambda \\ & E \arrow[ul, "w_\mu"] \arrow[ur, "w_\lambda"'] & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "w"] \arrow[dr, "w_\lambda"'] & & F \arrow[dl, "v_\lambda"] \\ & F_\lambda & \end{tikzcd}$$

$$\Psi(w) = (\Psi_{(\alpha,\lambda)}(w))_{(\alpha,\lambda)\in I\times J} = ((v_\lambda \circ w) \circ u_\alpha)_{(\alpha,\lambda)\in I\times J}$$
$$= (w_\lambda \circ u_\alpha)_{(\alpha,\lambda)\in I\times J} = (g_{(\alpha,\lambda)})_{(\alpha,\lambda)\in I\times J} = g,$$

w being the unique element of $\mathcal{L}(E;F)$ whose image under Ψ is g. Therefore Ψ is an isomorphism.

In order to prove the second assertion, let $\mathcal{X} \subset \mathcal{L}_a(E;F)$ be fixed. Obviously, $v_\lambda \circ \mathcal{X} \circ u_\alpha$ is equicontinuous for all $(\alpha,\lambda) \in I \times J$ if $\mathcal{X}$ is equicontinuous. Conversely, let us assume that each set $v_\lambda \circ \mathcal{X} \circ u_\alpha$ is equicontinuous and show that $\mathcal{X}$ is equicontinuous. But, since θ is the initial topology for the family $((F_\lambda, \theta_\lambda), v_\lambda)_{\lambda\in J}$, it suffices to show that $v_\lambda \circ \mathcal{X}$ is equicontinuous for all $\lambda \in J$. So let $\lambda \in J$ be given and let V_λ be a θ_λ-neighborhood of 0 in F_λ which is a submodule of F_λ. By hypothesis, for each $\alpha \in I$ there is a τ_α-neighborhood U_α of 0 in E_α so that $(v_\lambda \circ \mathcal{X} \circ u_\alpha)(U_\alpha) \subset V_\lambda$. Hence, if U is the submodule of E generated by the set $\bigcup_{\alpha\in I} u_\alpha(U_\alpha)$, then U is a τ-neighborhood of 0 in E (because $U_\alpha \subset u_\alpha^{-1}(U)$ for all $\alpha \in I$) such that $(v_\lambda \circ \mathcal{X})(U) \subset V_\lambda$, showing the equicontinuity of $v_\lambda \circ \mathcal{X}$. Therefore $\mathcal{X}$ is equicontinuous.

This completes the proof of the theorem.

Corollary 2.25. *Under the conditions of Theorem 2.24, assume additionaly that θ_λ is the discrete topology on F_λ for all $\lambda \in J$ (special cases occur in Example 2.12). Then, in order that $\mathcal{X} \subset \mathcal{L}_a(E;F)$ be equicontinuous, it is necessary and sufficient that for each $(\alpha,\lambda) \in I \times J$ there exists a τ_α-neighborhood U_α of 0 in E_α so that $(v_\lambda \circ \mathcal{X} \circ u_\alpha)(U_\alpha) = \{0\}$.*

Proof. Follows immediately from Theorem 2.24.

Corollary 2.26. *Let $((E_\alpha, \tau_\alpha))_{\alpha\in I}$ and $((F_\lambda, \theta_\lambda))_{\lambda\in J}$ be two families of linearly topologized R-modules, (E,τ) the topological direct sum of the family $((E_\alpha, \tau_\alpha))_{\alpha\in I}$ and (F,θ) the topological product of the family $((F_\lambda, \theta_\lambda))_{\lambda\in J}$.*

Then the group homomorphism

$$\Psi : \mathcal{L}(E;F) \longrightarrow \prod_{(\alpha,\lambda)\in I\times J} \mathcal{L}(E_\alpha;F_\lambda)$$
$$w \longmapsto (pr_\lambda \circ w \circ i_\alpha)_{(\alpha,\lambda)\in I\times J}$$

is an isomorphism, where $i_\alpha : E_\alpha \to E$ *and* $pr_\lambda : F \to F_\lambda$ *are the canonical* R*-linear mappings. Moreover,* $\mathcal{X} \subset \mathcal{L}_a(E;F)$ *is equicontinuous if and only if* $pr_\lambda \circ \mathcal{X} \circ i_\alpha \subset \mathcal{L}_a(E_\alpha;F_\lambda)$ *is equicontinuous for all* $(\alpha,\lambda) \in I \times J$*.*

Proof. Follows immediately from Theorem 2.24, by remembering Remarks 2.11 and 2.20.

Remark 2.27. Assume that R is a linearly topologized ring [2, p. 212]. In the particular case where (E,τ) is the topological direct sum $R^{(I)}$ and (F,θ) is the topological product R^J, Corollary 2.26 guarantees that the R-modules $\mathcal{L}(E;F)$ and $(\mathcal{L}(R;R))^{I\times J}$ are isomorphic and furnishes a characterization of the equicontinuous subsets of $\mathcal{L}_a(E;F)$ in terms of equicontinuous subsets of $\mathcal{L}_a(R;R)$.

3. Linearly Topologized *R*-Modules of Multilinear Mappings

Throughout this section we shall assume that R *is commutative.*

Given an integer $n \geq 1$ and R-modules $E_1,\ldots,E_n,F$, we shall denote by $\mathcal{L}_a(E_1,\ldots,E_n;F)$ the R-module of all R-multilinear mappings from $E_1 \times \ldots \times E_n$ into F. If F is endowed with a linear topology τ and $\mathcal{M}_i$ is a set of subsets of E_i which is stable under finite unions ($i = 1,\ldots,n$), it is easily seen that the topology $\tau_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ of $(\mathcal{M}_1 \times \ldots \times \mathcal{M}_n)$ -convergence [6, p. 275] on $\mathcal{L}_a(E_1,\ldots,E_n;F)$ is linear. If $\mathcal{V}$ is a fundamental system of τ-neighborhoods of 0 in F consisting of submodules of F, all the submodules of $\mathcal{L}_a(E_1,\ldots,E_n;F)$ of the form

$$\mathcal{X}_{B_1,\ldots,B_n,V} = \{u \in \mathcal{L}_a(E_1,\ldots,E_n;F) \,|\, u(B_1 \times \ldots \times B_n) \subset V\}$$

$(B_1 \in \mathcal{M}_1,\ldots,B_n \in \mathcal{M}_n, V \in \mathcal{V})$ constitute a fundamental system of $\tau_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ -neighborhoods of 0 in $\mathcal{L}_a(E_1,\ldots,E_n;F)$. If $\mathcal{M}_i$ is a covering of E_i for $i = 1,\ldots,n$ and τ is a Hausdorff topology, then $\tau_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ is a Hausdorff topology.

If $\mathcal{M}_i$ is the set of all finite subsets of E_i for $i = 1,\ldots,n$, $\tau_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ is the topology of simple convergence on $\mathcal{L}_a(E_1,\ldots,E_n;F)$, denoted by τ_s.

Proposition 3.1. *Under the conditions above, assume additionally that (F, τ) is linearly compact [2, p. 237]. Then $(\mathcal{L}_a(E_1, \ldots, E_n; F), \tau_s)$ is a linearly compact R-module.*

Proof. By Exercise 15(d), p. 237 of [2], the R-module $\mathcal{F}(E_1, \ldots, E_n; F)$ of all mappings from $E_1 \times \ldots \times E_n$ into F, endowed with the Hausdorff linear topology of simple convergence (also denoted by τ_s), is linearly compact. Therefore $(\mathcal{L}_a(E_1, \ldots, E_n; F), \tau_s)$ is linearly compact, since it is easily seen that $\mathcal{L}_a(E_1, \ldots, E_n; F)$ is τ_s-closed in $\mathcal{F}(E_1, \ldots, E_n; F)$.

Corollary 3.2. *Let $E_1, \ldots, E_n$ be as above and let (F, τ) be as in Proposition 3.1. If $\mathcal{H}$ is a τ_s-closed submodule of $\mathcal{L}_a(E_1, \ldots, E_n; F)$, then $\mathcal{H}$ is a linearly compact R-module under the topology of simple convergence.*

Proof. Follows immediately from Proposition 3.1.

Corollary 3.3. *Let $E_1, \ldots, E_n$ be as above and let (F, τ) be as in Proposition 3.1. For each $i = 1, \ldots, n$ let B_i be a subset of E_i, and let V be a τ-neighborhood of* 0 *in F which is a submodule of F. Then the submodule*

$$\mathcal{H} = \{u \in \mathcal{L}_a(E_1, \ldots, E_n; F) \,|\, u(B_1 \times \ldots \times B_n) \subset V\}$$

of $\mathcal{L}_a(E_1, \ldots, E_n; F)$ is linearly compact under the topology of simple convergence. In particular, if (F, τ) is a linearly compact discrete R-module, then

$$\mathcal{H} = \{u \in \mathcal{L}_a(E_1, \ldots, E_n; F) \,|\, u(B_1 \times \ldots \times B_n) = \{0\}\}$$

is linearly compact under the topology of simple convergence.

Proof. Since the R-linear mapping

$$\delta_{(x_1, \ldots, x_n)} : u \in (\mathcal{L}_a(E_1, \ldots, E_n; F), \tau_s) \longmapsto u(x_1, \ldots, x_n) \in (F, \tau)$$

is continuous for each $(x_1, \ldots, x_n) \in E_1 \times \ldots \times E_n$ and since V is τ-closed in F (Lemma 2.7), it follows that

$$\mathcal{H} = \bigcap_{(x_1, \ldots, x_n) \in B_1 \times \ldots \times B_n} \delta^{-1}_{(x_1, \ldots, x_n)}(V)$$

is τ_s-closed in $\mathcal{L}_a(E_1, \ldots, E_n; F)$. Thus the result follows immediately from Corollary 3.2.

Remark 3.4. Since every discrete field is linearly compact, it follows from the second part of Corollary 3.3 that, if E_i is a vector space over a discrete field $\mathbb{K}$ and B_i is a subset of E_i for $i = 1, \ldots, n$, then

$$\mathcal{H} = \{u \in \mathcal{L}_a(E_1, \ldots, E_n; F) \,|\, u(B_1 \times \ldots \times B_n) = \{0\}\}$$

is linearly compact under the topology of simple convergence. In particular,

$$\mathcal{H} = \{\varphi \in E_1^* \,|\, \varphi(B_1) = \{0\}\}$$

is linearly compact under the topology of simple convergence, where E_1^* is the vector space over $\mathbb{K}$ of all $\mathbb{K}$-linear forms on E_1.

The next results were inspired by results which may be found in Chapter III of [3].

Proposition 3.5. *Let $E_1, \dots, E_n, F_1, \dots, F_n$ be R-modules and (F, τ) a linearly topologized R-module. For each $i = 1, \dots, n$ let $u_i \in \mathcal{L}_a(E_i; F_i)$, and let $\mathcal{M}_i$ (resp. $\mathcal{N}_i$) be a set of subsets of E_i (resp. F_i) which is stable under finite unions and such that $u_i(\mathcal{M}_i) \subset \mathcal{N}_i$. Then the R-linear mapping*

$$u \in \mathcal{L}_a(F_1, \dots, F_n; F) \longmapsto u \circ (u_1 \times \dots \times u_n) \in \mathcal{L}_a(E_1, \dots, E_n; F)$$

is $\tau_{\mathcal{N}_1 \times \dots \times \mathcal{N}_n} - \tau_{\mathcal{M}_1 \times \dots \times \mathcal{M}_n}$-continuous, where $u_1 \times \dots \times u_n \in \mathcal{L}_a(E_1 \times \dots \times E_n; F_1 \times \dots \times F_n)$ is given by $(u_1 \times \dots \times u_n)(x_1, \dots, x_n) = (u_1(x_1), \dots, u_n(x_n))$ for $(x_1, \dots, x_n) \in E_1 \times \dots \times E_n$.

Proof. In fact, if $B_i \in \mathcal{M}_i$ for $i = 1, \dots, n$ and V is a τ-neighborhood of 0 in F, then $C_i = u_i(B_i) \in \mathcal{N}_i$ for $i = 1, \dots, n$ and the relations $u \in \mathcal{L}_a(F_1, \dots, F_n; F)$, $u(C_1 \times \dots \times C_n) \subset V$ imply $(u \circ (u_1 \times \dots \times u_n))(B_1 \times \dots \times B_n) \subset V$.

Corollary 3.6. *Let $E_1, \dots, E_n$ be R-modules and (F, τ) a linearly topologized R-module. For each $i = 1, \dots n$ let M_i be a submodule of E_i, $\pi_i : E_i \to E_i/M_i$ the canonical surjection and $\mathcal{M}_i$ a set of subsets of E_i which is stable under finite unions. Then the R-linear mapping*

$$u \in \mathcal{L}_a(E_1/M_1, \dots, E_n/M_n; F) \longmapsto u \circ (\pi_1 \times \dots \times \pi_n) \in \mathcal{L}_a(E_1, \dots, E_n; F)$$

is injective and $\tau_{\pi_1(\mathcal{M}_1) \times \dots \times \pi_n(\mathcal{M}_n)} - \tau_{\mathcal{M}_1 \times \dots \times \mathcal{M}_n}$-continuous. Moreover, if $n = 1$, its image is the submodule $\{v \in \mathcal{L}_a(E_1; F) \,|\, v(M_1) = \{0\}\}$ of $\mathcal{L}_a(E_1; F)$.

Proof. It is obvious that $(u \circ (\pi_1 \times \dots \times \pi_n))(M_1 \times \dots \times M_n) = \{0\}$ for all $u \in \mathcal{L}_a(E_1/M_1, \dots, E_n/M_n; F)$. By Proposition 3.5, the mapping under consideration is continuous, and its injectivity is obvious. Moreover, if $n = 1$, $v \in \mathcal{L}_a(E_1; F)$ and $v(M_1) = \{0\}$, then $M_1 \subset Ker(v)$. Thus the mapping $u : x_1 + M_1 \in E_1/M_1 \mapsto v(x_1) \in F$ is well defined and it is clear that $u \in \mathcal{L}_a(E_1/M_1; F)$ and $u \circ \pi_1 = v$, which concludes the proof.

Proposition 3.7. *Let I be an index set and n an integer ≥ 1. For each $j = 1, \dots, n$ let $(E_i^j)_{i \in I}$ be a family of R-modules, E_j an R-module and*

$u_i^j \in \mathcal{L}_a(E_i^j; E_j)$ $(i \in I)$. *For each* $j = 1, \ldots, n$ *let* $\mathcal{M}_i^j$ *be a set of subsets of* E_i^j *which is stable under finite unions* $(i \in I)$, *and put*

$$\mathcal{M}^j = \{\bigcup_{k=1}^{l} u_{i_k}^j(B_{i_k}^j) \,|\, l = 1, 2, 3, \ldots,\ i_1, \ldots, i_l \in I,\ B_{i_k}^j \in \mathcal{M}_{i_k}^j,\ 1 \leq k \leq l\}.$$

Let (F, τ) *be a linearly topologized* R*-module and, for each* $(i_1, \ldots, i_n) \in I^n$, *consider* $u_{i_1}^1 \times \ldots \times u_{i_n}^n \in \mathcal{L}_a(E_{i_1}^1 \times \ldots \times E_{i_n}^n; E_1 \times \ldots \times E_n)$ *and the* R*-linear mapping*

$$u_{(i_1,\ldots,i_n)} : u \in \mathcal{L}_a(E_1, \ldots, E_n; F) \longmapsto u \circ (u_{i_1}^1 \times \ldots \times u_{i_n}^n) \in \mathcal{L}_a(E_{i_1}^1, \ldots, E_{i_n}^n; F).$$

If θ *is the initial topology for the family* $((\mathcal{L}_a(E_{i_1}^1, \ldots, E_{i_n}^n; F), \tau_{\mathcal{M}_{i_1}^1 \times \ldots \times \mathcal{M}_{i_n}^n}),$ $u_{(i_1,\ldots,i_n)}),$ $(i_1, \ldots, i_n) \in I^n$, *then* $\theta = \tau_{\mathcal{M}^1 \times \ldots \times \mathcal{M}^n}$.

Proof. Let V be a τ-neighborhood of 0 in F which is a submodule of F, $(i_1, \ldots, i_n) \in I^n$, $B_{i_1}^1 \in \mathcal{M}_{i_1}^1, \ldots, B_{i_n}^n \in \mathcal{M}_{i_n}^n$, and consider the subbasic θ-neighborhood

$$\mathcal{X} = \{u \in \mathcal{L}_a(E_1, \ldots, E_n; F) \,|\, u_{(i_1,\ldots,i_n)}(u)(B_{i_1}^1 \times \ldots \times B_{i_n}^n) \subset V\}$$

of 0 in $\mathcal{L}_a(E_1, \ldots, E_n; F)$. Then $C_1 = u_{i_1}^1(B_{i_1}^1) \in \mathcal{M}^1, \ldots, C_n = u_{i_n}^n(B_{i_n}^n) \in \mathcal{M}^n$ and

$$\mathcal{X}_{C_1,\ldots,C_n,V} \subset \mathcal{X},$$

showing that θ is coarser than $\tau_{\mathcal{M}^1 \times \ldots \times \mathcal{M}^n}$.

Conversely, let V be as above, $C_1 = \bigcup_{i_1=1}^{l_1} u_{i_1}^1(B_{i_1}^1) \in \mathcal{M}^1, \ldots, C_n = \bigcup_{i_n=1}^{l_n} u_{i_n}^n(B_{i_n}^n)$ $\in \mathcal{M}^n$ and consider the basic $\tau_{\mathcal{M}^1 \times \ldots \times \mathcal{M}^n}$-neighborhood $\mathcal{X}_{C_1,\ldots,C_n,V}$ of 0 in $\mathcal{L}_a(E_1, \ldots, E_n; F)$. Then

$$\mathcal{X} = \bigcap_{1 \leq i_1 \leq l_1, \ldots, 1 \leq i_n \leq l_n} u_{(i_1,\ldots,i_n)}^{-1}(\mathcal{X}_{B_{i_1}^1,\ldots,B_{i_n}^n,V})$$

is a θ-neighborhood of 0 in $\mathcal{L}_a(E_1, \ldots, E_n; F)$ such that $\mathcal{X} \subset \mathcal{X}_{C_1,\ldots,C_n,V}$, showing that $\tau_{\mathcal{M}^1 \times \ldots \times \mathcal{M}^n}$ is coarser than θ. Therefore $\theta = \tau_{\mathcal{M}^1 \times \ldots \times \mathcal{M}^n}$, as asserted.

Proposition 3.8. *Let* $E_1, \ldots, E_n$ *be* R*-modules and let* $\mathcal{M}_1, \ldots \mathcal{M}_n$ *be sets of subsets of* $E_1, \ldots, E_n$, *respectively, which are closed under finite unions. Let* (F_1, τ_1), (F_2, τ_2) *be linearly topologized* R*-modules and* $w : (F_1, \tau_1) \to (F_2, \tau_2)$ *a continuous* R*-linear mapping. Then the* R*-linear mapping*

$$u \in \mathcal{L}_a(E_1, \ldots, E_n; F_1) \longmapsto w \circ u \in \mathcal{L}_a(E_1, \ldots, E_n; F_2)$$

is $\tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n} - \tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}$*-continuous.*

Proof. In fact, let $B_1 \in \mathcal{M}_1, \ldots, B_n \in \mathcal{M}_n$ and V_2 a τ_2-neighborhood of 0 in F_2. Then $V_1 = w^{-1}(V_2)$ is a τ_1-neighborhood of 0 in F_1 and the relations $u \in \mathcal{L}_a(E_1, \ldots, E_n; F_1)$, $u(B_1 \times \ldots \times B_n) \subset V_1$ imply $(w \circ u)(B_1 \times \ldots \times B_n) \subset V_2$. Therefore the above-mentioned mapping is continuous.

Proposition 3.9. *Let $E_1, \ldots, E_n$ be R-modules and let $\mathcal{M}_1, \ldots \mathcal{M}_n$ be sets of subsets of $E_1, \ldots, E_n$, respectively, which are closed under finite unions. Let $((F_i, \tau_i))_{i \in I}$ be a family of linearly topologized R-modules, F an R-module and $u_i \in \mathcal{L}_a(F; F_i)$ $(i \in I)$, and consider F endowed with the initial topology τ for the family $((F_i, \tau_i), u_i)_{i \in I}$. For each $i \in I$ consider the R-linear mapping*

$$\Phi_i : u \in \mathcal{L}_a(E_1, \ldots, E_n; F) \longmapsto u_i \circ u \in \mathcal{L}_a(E_1, \ldots, E_n; F_i).$$

If θ is the initial topology for the family $((\mathcal{L}_a(E_1, \ldots, E_n; F_i), \tau^i_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}), \Phi_i)_{i \in I}$, then $\theta = \tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}$, where $\tau^i_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}$ denotes the topology of $(\mathcal{M}_1 \times \ldots \times \mathcal{M}_n)$-convergence on $\mathcal{L}_a(E_1, \ldots, E_n; F_i)$ $(i \in I)$.

Proof. Let $B_1 \in \mathcal{M}_1, \ldots, B_n \in \mathcal{M}_n, V$ a τ-neighborhood of 0 in F which is a submodule of F, and consider the basic $\tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}$-neighborhood $\mathcal{X}_{B_1, \ldots, B_n, V}$ of 0 in $\mathcal{L}_a(E_1, \ldots, E_n; F)$. Let I_0 be a finite subset of I and, for each $i \in I_0$, let V_i be a τ_i-neighborhood of 0 in F_i which is a submodule of F_i so that $\bigcap_{i \in I_0} u_i^{-1}(V_i) \subset V$. Then

$$\mathcal{X} = \bigcap_{i \in I_0} \Phi_i^{-1}(\mathcal{X}_{B_1, \ldots, B_n, V_i})$$

is a θ-neighborhood of 0 in $\mathcal{L}_a(E_1, \ldots, E_n; F)$ such that $\mathcal{X} \subset \mathcal{X}_{B_1, \ldots, B_n, V}$, proving that $\tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}$ is coarser than θ.

Conversely, let $i \in I$, $B_1 \in \mathcal{M}_1, \ldots, B_n \in \mathcal{M}_n$ and V_i a τ_i-neighborhood of 0 in F_i which is a submodule of F_i. Consider the subbasic θ-neighborhood $\Phi_i^{-1}(\mathcal{X}_{B_1, \ldots, B_n, V_i})$ of 0 in $\mathcal{L}_a(E_1, \ldots, E_n; F)$ and put $V = u_i^{-1}(V_i)$. Then

$$\mathcal{X}_{B_1, \ldots, B_n, V} \subset \Phi_i^{-1}(\mathcal{X}_{B_1, \ldots, B_n, V_i}),$$

proving that θ is coarser than $\tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}$. Therefore $\theta = \tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n}$, as asserted.

Before stating the main result of this section, let us recall the construction of the completion of an arbitrary Hausdorff linearly topologized R-module (E, τ) (see also Exercise 14(b), p. 236 of [2]). Indeed, let $(\hat{E}, \hat{\tau})$ be the additive topological group which is the completion of the Hausdorff additive topological group (E,τ) ([5], p. 249, Theorem 2). If R is endowed with the discrete topology, the $\mathbb{Z}$-bilinear mapping $(\lambda, x) \mapsto \lambda x$ from the product $R\times E$ of the additive groups R, E into the additive group E is continuous. In fact, if $(\lambda_0, x_0) \in R\times E$ is arbitrary and U is a τ-neighborhood of 0 in E which is a submodule of E, then $\{\lambda_0\}$ is a neighborhood of λ_0 in R and

$$\lambda_0 x - \lambda_0 x_0 = \lambda_0(x - x_0) \in U$$

for all $x \in x_0 + U$. By Theorem 1, p. 277 of [5], there exists a continuous $\mathbb{Z}$-bilinear mapping

$$(\lambda, \hat{x}) \in R \times \hat{E} \mapsto \lambda\hat{x} \in \hat{E}$$

extending the $\mathbb{Z}$-bilinear mapping just cited. Thus we have $(\lambda+\mu)\hat{x} = \lambda\hat{x} + \mu\hat{x}$ and $\lambda(\hat{x}+\hat{y}) = \lambda\hat{x} + \lambda\hat{y}$ and, by the principle of extension of identities, $1\hat{x} = \hat{x}$ and $\lambda(\mu\hat{x}) = (\lambda\mu)\hat{x}$ $(\lambda, \mu \in R, \hat{x}, \hat{y} \in \hat{E})$. Hence $\hat{E}$ is a unitary left R-module.

We claim that $(\hat{E},\hat{\tau})$ is a linearly topologized R-module. Since $\hat{\tau}$ is an additive group topology on $\hat{E}$, it remains to show that $\hat{\tau}$ is a linear topology. In fact, let $\mathcal{U}$ be a fundamental system of τ-neighborhoods of 0 in E consisting of submodules of E. It is known that $\overline{\mathcal{U}} = \{\overline{U} \,|\, U \in \mathcal{U}\}$ (closures in $(\hat{E},\hat{\tau})$) is a fundamental system of $\hat{\tau}$-neighborhoods of 0 in $\hat{E}$; and, by Proposition 1, p. 226 of [5], each $\overline{U}$ is an additive subgroup of E. Moreover, if $\lambda \in R$ and $x \in \overline{U}$, there is a net $(x_l)_{l\in L}$ in U converging to $\hat{x}$. Therefore, since $\lambda x_l \in U$ for all $l \in L$ and since the mapping $y \in (\hat{E},\hat{\tau}) \mapsto \lambda y \in (\hat{E},\hat{\tau})$ is continuous, we conclude that $\lambda\hat{x} = \lim\limits_{l\in L} \lambda x_l \in \overline{U}$. Consequently, $\overline{U}$ is a submodule of $\hat{E}$.

The Hausdorff linearly topologized R-module $(\hat{E},\hat{\tau})$ is the completion of (E,τ).

Given an integer $n \geq 1$, coverings $\mathcal{M}_1,\dots,\mathcal{M}_n$ of R-modules $E_1,\dots,E_n$, respectively, which are stable under finite unions, and a Hausdorff linearly topologized R-module (F,τ), let $\mathcal{H}$ be an arbitrary submodule of $\mathcal{L}_a(E_1,\dots,E_n ;F)$. By Corollary 2.5 (b), the topology $\tau^{\mathcal{H}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ induced by $\tau_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ on $\mathcal{H}$ is linear; moreover, $\tau^{\mathcal{H}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ is a Hausdorff topology.

Let $\mathcal{V}$ be a fundamental system of τ-neighborhoods of 0 in F consisting of submodules of F and put

$$\mathcal{Z} = \bigcap_{B_1\in\mathcal{M}_1,\dots,B_n\in\mathcal{M}_n, V\in\mathcal{V}} (\mathcal{H} + \mathcal{X}_{B_1,\dots,B_n,V}).$$

It is easily seen that $\mathcal{Z}$ is a submodule of $\mathcal{L}_a(E_1,\dots,E_n;F)$ containing $\mathcal{H}$ and that the topology induced by $\tau^{\mathcal{Z}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$ on $\mathcal{H}$ is $\tau^{\mathcal{H}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$. We shall prove the following result, inspired by [1], [7] and [13], which furnishes a representation of the completion of $(\mathcal{H}, \tau^{\mathcal{H}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n})$ when (F,τ) is complete.

Theorem 3.10. *If (F,τ) is complete, then $(\mathcal{Z}, \tau^{\mathcal{Z}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n})$ is the completion of $(\mathcal{H}, \tau^{\mathcal{H}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n})$.*

Proof.
Claim 1: $(\mathcal{Z}, \tau^{\mathcal{Z}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n})$ is complete.

Indeed, let $(u_l)_{l\in L}$ be a $\tau^{\mathcal{Z}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$-Cauchy net in $\mathcal{Z}$. Then, since $\mathcal{M}_i$ is a covering of E_i for $i=1,\ldots,n$, it follows that $(u_l(x_1,\ldots,x_n))_{l\in L}$ is a τ-Cauchy net in F for each $(x_1,\ldots,x_n)\in E_1\times\ldots\times E_n$. Thus, by the τ-completeness of F, $(u_l(x_1,\ldots,x_n))_{l\in L}$ converges to an element $u(x_1,\ldots,x_n)\in F$ for each $(x_1,\ldots,x_n)\in E_1\times\ldots\times E_n$. It is easily seen that the mapping $u: E_1\times\ldots\times E_n\to F$ so defined is R-multilinear.

We claim that $u\in\mathcal{Z}$. In fact, let $B_1\times\ldots\times B_n\in\mathcal{M}_1\times\ldots\times\mathcal{M}_n$ and $V\in\mathcal{V}$ be arbitrary, and let us show that $u\in\mathcal{H}+\mathcal{X}_{B_1,\ldots,B_n,V}$. Since $(u_l)_{l\in L}$ is a $\tau^{\mathcal{Z}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$-Cauchy net, there is an $l_0\in L$ so that $u_l-u_{l_0}\in\mathcal{X}_{B_1,\ldots,B_n,V}$ for all $l\geq l_0$, that is, so that

$$u_l(B_1\times\ldots\times B_n)\subset u_{l_0}(B_1\times\ldots\times B_n)+V$$

for all $l\geq l_0$. Thus, by Lemma 2.7 and the definition of u, we obtain

$$u(B_1\times\ldots\times B_n)\subset u_{l_0}(B_1\times\ldots\times B_n)+V,$$

which furnishes

$$u-u_{l_0}\in\mathcal{X}_{B_1,\ldots,B_n,V}.$$

On the other hand, $u_{l_0}\in\mathcal{H}+\mathcal{X}_{B_1,\ldots,B_n,V}$ and, consequently,

$$u=u_{l_0}+(u-u_{l_0})\in\mathcal{H}+\mathcal{X}_{B_1,\ldots,B_n,V}+\mathcal{X}_{B_1,\ldots,B_n,V}\subset\mathcal{H}+\mathcal{X}_{B_1,\ldots,B_n,V}.$$

Therefore $u\in\mathcal{Z}$, and it is clear that $(u_l)_{l\in L}$ converges to u for $\tau^{\mathcal{Z}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}$, thereby proving Claim 1.

Claim 2: $\mathcal{H}$ is dense in $(\mathcal{Z}, \tau^{\mathcal{Z}}_{\mathcal{M}_1\times\ldots\times\mathcal{M}_n})$.

Indeed, let $B_1,\ldots,B_n$ and V be as in the proof of Claim 1. If $v\in\mathcal{Z}$ is arbitrary, there are $v_1\in\mathcal{H}$ and $v_2\in\mathcal{X}_{B_1,\ldots,B_n,V}$ such that $v=v_1+v_2$, which implies

$$(v-v_1)(B_1\times\ldots\times B_n)\subset V$$

and guarantees the validity of Claim 2.

Finally, in view of Claims 1 and 2, the proof is concluded.

An obvious consequence of Theorem 3.10 reads:

Corollary 3.11. *If $E_1, \ldots, E_n$, $\mathcal{M}_1, \ldots, \mathcal{M}_n$ and (F, τ) are as above and (F, τ) is complete, then $(\mathcal{L}_a(E_1, \ldots, E_n; F), \tau_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n})$ is complete.*

Corollary 3.12. *Let $E_1, \ldots, E_n$, $\mathcal{M}_1, \ldots, \mathcal{M}_n$, (F, τ) and $\mathcal{V}$ be as above, with (F, τ) complete. In order that $(\mathcal{H}, \tau^{\mathcal{H}}_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n})$ be complete, it is necessary and sufficient that each $u \in \mathcal{L}_a(E_1, \ldots, E_n; F)$ belonging to every $\mathcal{H} + \mathcal{X}_{B_1, \ldots, B_n, V}$ $((B_1, \ldots, B_n) \in \mathcal{M}_1 \times \ldots \times \mathcal{M}_n$, $V \in \mathcal{V})$ be an element of $\mathcal{H}$.*

Proof. Follows immediately from Theorem 3.10.

Corollary 3.13. *Let $E_1, \ldots, E_n$, $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be as above, and let F be an R-module endowed with the discrete topology τ. For $B_1 \in \mathcal{M}_1, \ldots, B_n \in \mathcal{M}_n$, put*

$$\mathcal{X}_{B_1, \ldots, B_n} = \{u \in \mathcal{L}_a(E_1, \ldots, E_n; F) \,|\, u(B_1 \times \ldots \times B_n) = \{0\}\}.$$

If $\mathcal{H}$ is a submodule of $\mathcal{L}_a(E_1, \ldots, E_n; F)$ and

$$\mathcal{Z} = \bigcap_{B_1 \in \mathcal{M}_1, \ldots, B_n \in \mathcal{M}_n} (\mathcal{H} + \mathcal{X}_{B_1, \ldots, B_n}),$$

then $(\mathcal{Z}, \tau^{\mathcal{Z}}_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n})$ is the completion of $(\mathcal{H}, \tau^{\mathcal{H}}_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n})$.

Proof. Follows immediately from Theorem 3.10.

Corollary 3.14. *Let $(E_1, \tau_1), \ldots, (E_n, \tau_n)$ be linearly topologized R-modules, and let $\mathcal{M}_1, \ldots, \mathcal{M}_n$, (F, τ) and $\mathcal{V}$ be as Theorem 3.10. If $\mathcal{L}(E_1, \ldots, E_n; F)$ is the submodule of $\mathcal{L}_a(E_1, \ldots, E_n; F)$ consisting of all continuous R-multilinear mappings from $(E_1 \times \ldots \times E_n, \tau_1 \times \ldots \times \tau_n)$ into (F, τ) and*

$$\mathcal{Z} = \bigcap_{B_1 \in \mathcal{M}_1, \ldots, B_n \in \mathcal{M}_n, V \in \mathcal{V}} (\mathcal{L}(E_1, \ldots, E_n; F) + \mathcal{X}_{B_1, \ldots, B_n, V}),$$

then $(\mathcal{Z}, \tau^{\mathcal{Z}}_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n})$ is the completion of $(\mathcal{L}(E_1, \ldots, E_n; F), \tau^{\mathcal{L}}_{\mathcal{M}_1 \times \ldots \times \mathcal{M}_n})$, where $\mathcal{L} = \mathcal{L}(E_1, \ldots, E_n; F)$.

Proof. Follows immediately from Theorem 3.10.

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