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CENTROIDS AND QUASI-CENTROIDS OF FINITE DIMENSIONAL LEIBNIZ ALGEBRAS

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Abstract: We introduce the concept of centroid and quasicentroids for Leibniz algebras and study some of their properties. Then we determine the centroids of low-dimensional Leibniz algebras and give their classification.

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Key Words: Leibniz algebra, centroid, quasicentroids, derivation, central derivation

1. Introduction

Our main focus in the paper will be on centroids of Leibniz algebras. It is well-known that for a Lie algebra L, the centroid $\Gamma(L)$ is just the space of L-module homomorphisms φ on L such that $\varphi([x,y]) = [x,\varphi(y)]$ for all $x, y \in L$, (viewing L as an L-module under the adjoint action). Our interest in the centroid stems from the study of algebraic and geometric classification problems of the Leibniz algebras. In 1993, Loday introduced the notion of Leibniz algebra [4] which is generalization of Lie algebra, where the skew-symmetricity of the brackets is dropped and the Jacobi identity is replaced by the Leibniz identity. There, Loday also has show that the classical relationship between Lie algebras and

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associative algebras can be extended to the analogous relationship between Leibniz algebras and the diassociative algebras (see [5]). Since the introduction of Leibniz algebras, several results in the theory of Lie algebras have been extended to Leibniz algebras and this motivate us to study centroids of Leibniz algebras.

The centroid of algebras plays an important role in understanding the structure of algebras. All scalar extensions of a simple algebra remain simple if and only if its centroid just consists of the scalars in the base field. In particular, for finite-dimensional simple associative algebras, the centroid is critical in investigating Brauer groups and division algebras. Another area where the centroid occurs naturally is in the study of derivations of an algebra. If φ is an element of the centroid and d is a derivation of A, then $\varphi \circ d$ is also a derivation of A, so centroidal transformations can be used to construct derivations of algebras. The centroids of Lie algebras have been studied in [6]. It is well known that the centroid of a Lie algebra is a field, This plays an important role in the classification problem of finite dimensional extended affine Lie algebras over arbitrary field of characteristic zero (see [2]). Melville and Benkarta studied the centroids of nilpotent, extended affine and root graded Lie algebras, for more details see [2], [6], [7], and references therein. Our questions here is, can similar results be obtained for centroids of Leibniz algebras?

This paper deals with the problem of description of centroids of the Leibniz algebras. The concept of centroids in this case is easily imitated from that of finite-dimensional algebras. The algebra of centroids plays important role in the classification problems and in different applications of algebras. In the study we make use classification results of two, three, four-dimensional complex Leibniz algebras from [4], [3], [1].

2. Preliminaries

This section contains main definitions used and some results obtained for Leibniz algebras.

Definition 1. A Leibniz algebra L is a vector space over a field $\mathbb{K}$ equipped with a bilinear map

$$[\cdot,\cdot] : L \times L \to L$$

satisfying the Leibniz identity

$$\Big[x,[y,z]\Big] = \Big[[x,y],z\Big] - \Big[[x,z],y\Big] \quad \text{for all} \quad x,y,z \in L. \tag{1}$$

Example 2. Let L be a Lie algebra and let M be a L-module with an action $M \times L \to M$, $(m, x) \mapsto mx$. Let $\psi : M \to L$ be a L-equivariant linear map, this is $\psi(mx) = [\psi(m), x]$, for all $m \in M$ and $x \in L$, then one can define a Leibniz structure on M as follows:

$$[m, n]^{'} := m\psi(n), \quad \text{for all} \quad m, n \in M.$$

Definition 3. Let L and L_1 be two algebras over a field K. A linear mapping $\psi : L \longrightarrow L_1$ is a homomorphism if

$$\psi([x, y]_L) = [\psi(x), \psi(y)]_{L_1}, \quad \text{for all} \quad x, y \in L \tag{2}$$

For a Leibniz algebra L we define

$$L = L^1,\ L^{k+1} = [L^k, L],\ k \geqslant 1\,.$$

Clearly,

$$L^1 \supseteq L^2 \supseteq \cdots\,.$$

Definition 4. A Leibniz algebra L is called nilpotent if there is a positive integer $s \in \mathbb{N}$ such that $L = L^1 \supset L^2 \supset L^3 \supset \cdots \supset L^s = \{0\}$. The smallest integer s such that $L^s = \{0\}$ is called the nil-index of L.

For a subset B of L the subset

$$Z_L(B) = \{x \in L | [x, B] = [B, x] = 0\}$$

is said to be the centralizer of B in L. Obviously, $Z_L(L)$ is the center of L.

Definition 5. A linear transformation d of a Leibniz algebra L is called a derivation if for any $x, y \in L$

$$d([x, y]) = [d(x), y] + [x, d(y)]$$

holds.

The set of all derivations of Leibniz algebra L is a subspace of $End_K(L)$. This subspace equipped with the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$ is a Lie algebra denoted by $Der(L)$.

Definition 6. A derivation $d \in$ End(L) is said to be a qauasi-derivation, if there exist $d^{'}$ $\in$End(L) such that

$$[d(x), y] + [x, d(y)] = d^{'}[x, y].$$

Definition 7. Let L be a Leibniz algebra over a field K. The set

$$\Gamma_K(L) = \Big\{\varphi \in End_K(L) |\ \varphi[a,b] = [a,\varphi(b)] = [\varphi(a),b] \text{ for all } a,b \in L\Big\}$$

is called the centroids of L.

Also we can define the set

$$Q\Gamma_K(L) = \Big\{\varphi \in End_K(L) |\ [a,\varphi(b)] = [\varphi(a),b] \text{ for all } a,b \in L\Big\},$$

which is called the quasi-centroid of Leibniz algebra L.

Definition 8. Let L be Leibniz algebra and $\varphi \in End(L)$. Then φ is called a central derivation, if $\varphi(L) \subseteq Z(L)$ and $\varphi(L^1) = 0$.

The set of all central derivations of L is denoted by $C(L)$. It is a simple observation to see that $C(L) \subseteq \Gamma(L)$. In fact, $C(L)$ is an ideal of $\Gamma(L)$.
On the other hand, $\Gamma(L)$ is closed under composition and thus has Leibniz algebra structure. Hence, defining the bracket operation $[\varphi_1, \varphi_2] = \varphi_1 \circ \varphi_2 - \varphi_2 \circ \varphi_1$ for $\varphi_1, \varphi_2 \in \Gamma(L)$ as usual, we can consider $\Gamma(L)$ as a Lie subalgebra of $End(L)$.

Definition 9. Let L be Leibniz algebra. We say that L is indecomposable if it can not be written as a direct sum of its ideals. Otherwise the L is called decomposable.

Definition 10. Let L be Leibniz algebra. We say that L is indecomposable if it can not be written as a direct sum of its ideals. Otherwise the L is called decomposable.

Definition 11. Let L be an indecomposable Leibniz algebra. We say $\Gamma(L)$ is small if $\Gamma(L)$ is generated by central derivations and the scalars. The centroid of a decomposable Leibniz algebra is called small if the centroids of each indecomposable factors are small.

Example 12. Let L be a three-dimensional Leibniz algebra over complex field and $\Big\{e_1, e_2, e_3\Big\}$ be its basis of L given by $[e_2, e_2] = e_1, [e_3, e_3] = e_2$.
Let $\varphi(e_1) = ae_1$, $\varphi(e_2) = be_1 + ae_2$, and $\varphi(e_3) = ce_1 + ae_3$ for $a, b, c \in \mathbb{C}$.Then $\varphi \in \Gamma(L)$.

Next we develop some general result on centroids of Leibniz algebras.

3. Results

3.1. Properties of Centroids of Leibniz Algebras

In this section we declare the following results on properties of the centroids of Leibniz algebras.

Proposition 13. *Let L be Leibniz algebra. Then*

i) $\Gamma(L)(Der(L)) \subseteq Der(L)$.

ii) $[\Gamma(L), Der(L)] \subseteq \Gamma(L)$.

iii) $[\Gamma(L), \Gamma(L)](L) \subseteq C(L)$ *and* $[\Gamma(L), \Gamma(L)](L^1) = 0$.

iv) $\Gamma(L) \subset QDer(L)$.

v) *If* $L_1 \cong L_2$ *then* $\Gamma(L_1) \cong \Gamma(L_2)$

Proof. The proof of parts i) – iv) is straightforward by using definitions of derivation and centroid. As for part e) the mapping $\rho : \Gamma(L_1) \longrightarrow \Gamma(L_2)$ defined by $\rho(\varphi) = f \circ \varphi \circ f^{-1}$, for an isomorphism $f : L_1 \longrightarrow L_2$ and $\varphi \in \Gamma(L_1)$, gives the required isomorphism $\Gamma(L_1) \cong \Gamma(L_2)$. □

Theorem 14. *Let L be Leibniz algebra. Then for any $\varphi \in \Gamma(L)$ and $d \in Der(L)$ one has the following.*

a) $C(L) = \Gamma(L) \cap Der(L)$;

b) *$d \circ \varphi$ is contained in $\Gamma(L)$ if and only if $\varphi \circ d$ is a central derivation of L;*

c) *$d \circ \varphi$ is a derivation of L if and only if $[d, \varphi]$ is a central derivation of L.*

Proof. Let us prove a) If $\varphi \in \Gamma(L) \cap Der(L)$ then by definition of $\Gamma(L)$ and $Der(L)$, for all $x, y \in L$, We have $\varphi[x, y] = [\varphi(x), y] + [x, \varphi(y)]$ and $\varphi[x, y] = [\varphi(x), y] = [x, \varphi(y)]$, so $\varphi(L^1) = 0$ and $\varphi(L) \subseteq Z(L)$. It follows easily that $\Gamma(l) \cap Der(L) \subseteq C(L)$.
To show the inverse inclusion, let $\varphi \in C(L)$; then $0 = \varphi[x, y] = [\varphi(x), y] = [x, \varphi(y)]$. Thus $\varphi \in \Gamma(L) \cap Der(L)$. This implies $C(L) = \Gamma(L) \cap Der(L)$

Let us prove b). For any $\varphi \in \Gamma(L)$, $d \in Der(L)$, $\forall x, y \in L$ by saying $d \circ \varphi$ is contained in $\Gamma(L)$ is central derivation of L by a and b in Proposition (13) we have

$$(\varphi \circ d)[x, y] = [\varphi \circ d(x), y] + [x, \varphi \circ d(y)]$$

and

$$\begin{aligned}(\varphi, d)[x,y] &= [(\varphi,d)(x),y] = [x,(\varphi,d)(y)]\\ (\varphi\circ d)[x,y]-(d\circ\varphi)[x,y] &= [(\varphi\circ d)(x),y]-[(d\circ\varphi)(x),y]\\ &= [x,(\varphi\circ d)(y)]-[x,(d\circ\varphi)(y)].\end{aligned}$$

Then it holds $(\varphi\circ d)[x,y] = [(\varphi\circ d)(x),y]$ and $[x,(\varphi\circ d)(y)] = 0$. Similarly, $[(\varphi\circ d)(x),y] = 0$.
Finally, $(\varphi\circ d)[x,y] = 0$ and thus $\phi\circ d$ is an central derivation of A.
(2) If $d\circ\varphi \in Der(L)$: using $[d,\phi]\in\Gamma(L)$ we get

$$(d,\varphi)[x,y] = [(d,\varphi)(x),y] = [x,(d,\varphi)(y)]. \tag{3}$$

On the other hand $[d,\varphi] = d\circ\varphi-\varphi\circ d$ and $\varphi\circ d, d\circ\varphi\in Der(A)$. Therefore,

$$(d,\varphi)[x,y] = [(d\circ\varphi)(x),y]+[x,(d\circ\varphi)(y)]-[(\varphi\circ d)(x),y]-[x,(\varphi\circ d)(y)] \tag{4}$$

by equations (3) and (4), we get $x\Big([d,\varphi](y)\Big) = \Big([d,\varphi](x)\Big)y = 0$, and thus the necessity is gotten.
The sufficiency can be proved by $d\circ\varphi = [d,\varphi]-\varphi\circ d$ and Proposition 14 (i). □

If B is a $\Gamma(L)$-invariant ideal of L. Let $V(B) = \Big\{\varphi\in\Gamma(L)|\varphi(B) = 0\Big\}$ be its vanishing ideal.
Let $Hom(L/B, Z_L(B))$ be the a vector space of all linear map from L/B to $Z_L(B)$ over a field K. Define
$T(B) = \Big\{f\in Hom(L/B, Z_L(B))|f\overline{[x,y]} = f[\bar{x},\bar{y}] = [f(\bar{x}),\bar{y}] = [\bar{x},f(\bar{y})]\Big\}$
Where $\bar{x}$ and $\bar{y}\in L/B$. Then $T(B)$ is a subspace of $Hom(L/B, Z_L(B))$

Theorem 15. *Let L be a Leibniz algebra and B be $\Gamma(L)-$ invariant ideal of L. Then*

i. $V(B)$ is an isomorphic to $T(B)$ as vector space.

ii. If $\Gamma(B) = Kid_B$ then $\Gamma(L) = Kid_B\oplus V(B)$ as vector space.

Proof. i. Consider the map $\alpha : V(B)\to T(B)$, given by $\alpha\varphi(\bar{y}) = \varphi(y)$, where $\varphi\in V(B)$ and $\bar{y} = y+B\in L/B$. The map α is well defined. For if $\bar{y} = \bar{y_1}$, then $y-y_1\in B$ and so $\varphi(y-y_1) = 0$. i.e $\varphi(y) = \varphi(y_1)$. Hence $\alpha(\varphi)(\bar{y}) = \alpha(\varphi)(y_1)$ Its follows easily that α is an injective. if $\alpha\varphi(\bar{y}) = \alpha\varphi_1(\bar{y})$ for $\varphi,\varphi_1\in\Gamma(L)$ and for any $y\in L$, Viz. $\varphi(y) = \varphi_1(y)$, then $\varphi = \varphi_1$ We now show that α is onto. For $\varphi_f : L \to L$, $\varphi_f(x) = f(\bar{x})$ for $x \in L$ it follows from the definition of $\varphi_f[x, y] = f\overline{[x, y]} = f[\bar{x}, \bar{y}] = [f(\bar{x}), f(\bar{y})] = [\bar{x}, f(\bar{y})]$ for all $x, y \in L$ namely, $\varphi_f[x, y] = [\varphi_f(x), y] = [x, \varphi_f(y)]$. Thus $\varphi_f \in \Gamma(L)$ and $\varphi_f \in V(B)$ since $(\varphi_f)(B) = 0$. But $\alpha(\varphi_f) = f$ implies that α is onto. Its fairly easy to see that α preserve operation on vector space is linear. Thus α is an isomorphism of vector spaces. We now prove *ii*. If $\Gamma(L) = Kid_B$, then for all $\varphi \in \Gamma(L)$, $\varphi|_B = \lambda id_B$, for some $\lambda \in K$. If $\varphi \neq \lambda id_L$, let $\psi(x) = \lambda x$, for all $x \in L$, then $\psi \in \Gamma(L)$, $\psi - \varphi \in V(B)$. Clearly, $\varphi = \psi + (\varphi - \psi)$. Furthermore, $Kid_L \cap V(B) = 0$ so $\varphi(L) = Kid_L \oplus V(B)$. □

3.2. Centroids of Low Dimensional Leibniz Algebras

Let $\{e_1, e_2, e_3, \cdots, e_n\}$ be a basis of an n-dimensional Leibniz algebra L. Then

$$[e_i, e_j] = \sum_{k=1}^{n} \gamma_{ij}^k e_k \ , \quad i, j = 1, 2, \cdots, n.$$

L on the basis $\{e_1, e_2, e_3, \cdots, e_n\}$.

An element φ of the centroid $\Gamma(L)$ being a linear transformation of the vector space L is represented in a matrix form $[a_{ij}]_{i,j=1,2,\cdots,n}$, i.e. $\varphi(e_i) = \sum_{j=1}^{n} a_{ji} e_j$, $i = 1, 2, \cdots, n$. According to the definition of the centroid the entries a_{ij} $i, j = 1, 2, \cdots, n$ of the matrix $[a_{ij}]_{i,j=1,2,\cdots,n}$ must satisfy the following systems of equations:

$$\sum_{t=1}^{n} (\gamma_{ij}^t a_{kt} - a_{ti} \gamma_{tj}^k) = 0; \quad \text{and} \quad \sum_{t=1}^{n} (\gamma_{ij}^k a_{kt} - a_{tj} \gamma_{it}^k) = 0;$$

to solve the system of equations above with respect to a_{ij} $i, j = 1, 2, \cdots, n$ which can be done by using a computer software. algebras [3].

Theorem 16. *[4] Any two-dimensional Leibniz algebra L isomorphic to one of following non-isomorphic Leibniz algebras*

$L_1 :$ $[e_1, e_1] = e_2$
$L_2 :$ $[e_1, e_2] = -[e_2, e_1] = e_2$
$L_3 :$ $[e_1, e_2] = [e_2, e_2] = e_1$

Theorem 17. *The centroid of two dimensional complex Leibniz algebras are given as follows:*

Theorem 18. *[3] Up to isomorphism , there exist three one parametic families and six explicit representatives of non Lie complex Leibniz algebras of*

Table 1: Centroids of two-dimensional Leibniz algebras

Leibniz algebra **L**	**Centroid** $\Gamma(L)$	**Types of** $\Gamma(L)$
L_1	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{11} \end{pmatrix}$	not small
L_2	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	small
L_3,	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	small

dimension three:

$$
\begin{array}{lllll}
RR_1: & [e_1,e_3]=-2e_1, & [e_2,e_2]=e_1, & [e_3,e_2]=e_2, & [e_2,e_3]=-e_2 \\
RR_2: & [e_1,e_3]=\alpha e_1, & [e_3,e_3]=e_1, & [e_3,e_2]=e_2, & [e_2,e_3]=-e_2, \alpha\in\mathbb{C} \\
RR_3: & [e_2,e_2]=e_1, & [e_3,e_3]=\alpha e_1, & [e_2,e_3]=e_1 & \alpha\in\mathbb{C}/\{0\} \\
RR_4: & [e_2,e_2]=e_1, & [e_3,e_3]=e_2 & & \\
RR_5: & [e_1,e_3]=e_2, & [e_2,e_3]=e_1 & & \\
RR_6: & [e_1,e_3]=e_2, & [e_2,e_3]=\alpha e_1+e_2 & \alpha\in\mathbb{C} & \\
RR_7: & [e_1,e_3]=e_1, & [e_2,e_3]=e_2 & & \\
RR_8: & [e_3,e_3]=e_1, & [e_1,e_3]=e_2 & & \\
RR_9: & [e_3,e_3]=e_1, & [e_1,e_3]=e_1+e_2 & &
\end{array}
$$

$$
\begin{array}{llll}
RR_{10}: & [e_1,e_2]=e_1 & & \\
RR_{11}: & [e_1,e_3]=e_1+e_2, & [e_2,e_3]=e_2 & \\
RR_{12}: & [e_1,e_3]=2e_1, & [e_2,e_3]=-e_2, & [e_1,e_2]=e_3 \\
RR_{13}: & [e_1,e_3]=e_1, & [e_2,e_3]=\alpha e_3 & \alpha\in\mathbb{C}/\{0\}
\end{array}
$$

Theorem 19. *The centroid of three dimensional complex Leibniz algebras are given as follows:*

Table 2: Centroids of three-dimensional Leibniz algebras

Leibniz algebra L	**Centroid** $\Gamma(L)$		**Types of** $\Gamma(L)$
RR_1	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$		small
RR_2	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$		small
RR_3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$		small
RR_4	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$		not small
RR_5	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$		small
RR_6	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	$(\alpha = 0)$	not small
	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	$(\alpha \neq 0)$	small
RR_7	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$		small
RR_8	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$		small

Theorem 20. *[1] The isomorphism class of four-dimensional complex nilpotent leibniz algebras are given by the following representatives*

Table 3: Centroids of three-dimensional Leibniz algebras

Leibniz algebra L	**Centroid** $\Gamma(L)$	**Types of** $\Gamma(L)$
RR_9	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	small
RR_{10}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$	not small
RR_{11}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	small
RR_{12}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	small
RR_{13}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	small

$$\begin{array}{lllll}
R_1: & [e_1,e_1]=e_2, & [e_2,e_1]=e_3, & [e_3,e_1]=e_4 & \\
R_2: & [e_1,e_1]=e_3, & [e_1,e_2]=e_4, & [e_2,e_1]=e_3, & [e_3,e_1]=e_4 \\
R_3: & [e_1,e_1]=e_3, & [e_2,e_1]=e_3, & [e_3,e_1]=e_4 & \\
R_4(\alpha): & [e_1,e_1]=e_3, & [e_1,e_2]=\alpha e_4, & [e_2,e_1]=e_3, & [e_2,e_2]=e_4, \\
 & [e_3,e_1]=e_4 & \alpha\in\{0,1\}; & & \\
R_5: & [e_1,e_1]=e_3, & [e_1,e_2]=e_4, & [e_3,e_1]=e_4 & \\
R_6: & [e_1,e_1]=e_3, & [e_2,e_2]=e_4, & [e_3,e_1]=e_4 & \\
R_7: & [e_1,e_1]=e_4, & [e_2,e_1]=e_3, & [e_3,e_1]=e_4, & [e_1,e_2]=-e_3, \\
 & [e_1,e_3]=-e_4; & & &
\end{array}$$

$R_8:$	$[e_1,e_1]=e_4,$	$[e_2,e_1]=e_3,$	$[e_3,e_1]=e_4,$	$[e_1,e_2]=-e_4+e_4;$
	$[e_1,e_3]=-e_4;$			
$R_9:$	$[e_1,e_1]=e_4,$	$[e_2,e_1]=e_3,$	$[e_2,e_2]=e_4,$	$[e_3,e_1]=e_4;$
	$[e_1,e_2]=-e_3+2e_4,$	$[e_1,e_3]=-e_4;$		
$R_{10}:$	$[e_1,e_1]=e_4,$	$[e_2,e_1]=e_3,$	$[e_2,e_2]=e_4,$	$[e_3,e_1]=e_4,$
	$[e_1,e_2]=-e_3,$	$[e_1,e_3]=-e_4;$		
$R_{11}:$	$[e_1,e_1]=e_4,$	$[e_1,e_2]=e_3,$	$[e_2,e_1]=-e_3,$	$[e_2,e_2]=-2e_3+e_4$
$R_{12}:$	$[e_1,e_2]=e_3,$	$[e_2,e_1]=e_4,$	$[e_2,e_2]=-e_3$	
$R_{13}(\alpha):$	$[e_1,e_1]=e_3,$	$[e_1,e_2]=e_4,$	$[e_2,e_1]=-\alpha e_3,$	$[e_2,e_2]=-e_4$
	$\alpha\in\mathbb{C}$			
$R_{14}(\alpha):$	$[e_1,e_1]=e_4,$	$[e_1,e_2]=\alpha e_4,$	$[e_2,e_1]=-\alpha e_4,$	$[e_2,e_2]=e_4;$
	$[e_3,e_3]=e_4$	$\alpha\in\mathbb{C};$		
$R_{15}:$	$[e_1,e_2]=e_4,$	$[e_1,e_3]=e_4,$	$[e_2,e_1]=-e_4,$	$[e_2,e_2]=e_4$
$R_{16}:$	$[e_1,e_1]=e_4,$	$[e_1,e_2]=e_4,$	$[e_2,e_1]=-e_4,$	$[e_3,e_3]=e_4$
$R_{17}:$	$[e_1,e_2]=e_3,$	$[e_2,e_1]=e_4$		
$R_{18}:$	$[e_1,e_2]=e_3,$	$[e_2,e_2]=e_4,$	$[e_2,e_1]=-e_3$	
$R_{19}:$	$[e_2,e_1]=e_4,$	$[e_2,e_2]=e_3$		
$R_{20}(\alpha):$	$[e_1,e_2]=e_4,$	$[e_2,e_2]=e_3,$	$[e_2,e_1]=\frac{1+\alpha}{1-\alpha}e_4$	$\alpha\in\mathbb{C}/\{1\}$
$R_{21}:$	$[e_1,e_2]=e_4,$	$[e_2,e_1]=-e_4,$	$[e_3,e_3]=e_4$	

Theorem 21. *The centroid of four-dimensional complex Leibniz algebras are given as follows:*

Table 4: Centroids of four-dimensional Leibniz algebras

Leibniz algebra L	**Centroid** $\Gamma(L)$	**Types of** $\Gamma(L)$
R_1	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	not small
R_2	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small

Corollary 22. i) *The centroid of a two-dimensional Leibniz algebra are small.*

ii) *The dimensions of the centroids of two-dimensional Leibniz algebras vary between one and two.*

Corollary 23. i) *The centroids of three-dimensional Leibniz algebras except for the classes RR_4, RR_6, RR_{10} are small.*

Leibniz algebra L	**Centroid** $\Gamma(L)$	**Types of** $\Gamma(L)$
R_3	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_4	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_5	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{43} & a_{11} & 0 & 0 \\ a_{43} & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	not small
R_6	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_7	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_8	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_9	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_{10}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small

ii) *The dimensions of the centroids of three-dimensional complex Leibniz algebras vary between one and three.*

Leibniz algebra L	**Centroid** $\Gamma(L)$	**Types of** $\Gamma(L)$
R_{11}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_{12}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_{13}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_{14}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	small
R_{15}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	small
R_{16}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	small
R_{17}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small
R_{18}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	small

4. Quasi-Centroids of Leibniz Algebras

This section is devoted to the description of quasi-centroids of two and three-dimensional complex Leibniz algebras.

Leibniz algebra L	Centroid $\Gamma(L)$		Types of $\Gamma(L)$
R_{19}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$		small
R_{20}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	$\alpha \neq 0$	not small
	$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{21} & a_{11} \end{pmatrix}$	$\alpha = 0$	not small
R_{21}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$		small

Theorem 24. *The Quasi-centroids of two dimensional complex Leibniz algebras are given as follows:*

Table 5: Quasi-centroids of two-dimensional Leibniz algebras

Leibniz algebra L	Quasi-centroids $Q\Gamma(L)$	Types of $Q\Gamma(L)$
L_1	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$	small
L_2	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	small
L_3,	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	small

Theorem 25. *The centroid of three dimensional complex Leibniz algebras are given as follows:*

Table 6: Description of Quasicentroids of three-dimensional Leibniz algebras

Leibniz algebra L	Quasicentroids $Q\Gamma(L)$	Dim	Leibniz algebra L	Quasicentroids $Q\Gamma(L)$	Dim
RR_1	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	1	RR_2	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	2
RR_3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{22} \end{pmatrix}$	4	RR_4	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{32} \\ 0 & a_{32} & a_{33} \end{pmatrix}$	1

Leibniz algebra L	**Quasicentroids** $Q\Gamma(L)$	**Dim**	**Leibniz algebra** L	**Quasicentroids** $Q\Gamma(L)$	**Dim**
RR_5	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	1	RR_6	$\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	4
RR_7	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$	4	RR_8	$\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	5
RR_9	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	4	RR_{10}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	5
RR_{11}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	1	RR_{12}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	1
RR_{13}	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$	1			

References

[1] Albeverio, S., Omirov, B. and Rakhimov, I., Classification of 4-dimensional nilpotent complex Leibniz algebras, *Extracta Math .*, **3**(1), (2006), 197-210.

[2] G. Benkart, E. Neher, The centroid of extended affine and root graded Lie algebras, *Journal of Pure and Applied Algebra*, **205**(1), (2006), 117–145.

[3] Casas, J., Insua, M., M. and Ladra, S., An algorithm for the classification of 3-dimensional complex Leibniz algebras, *Linear Algebra and Its Applications.*, **40**(8), (2012), 2962-2982.

[4] Loday J.-L.,Une version non commutative dés de Lie: les de Leibniz. *L'Ens. Math.*, *Publ. Math. I.H.E.S.*, 39, (1993), 269–293.

[5] Loday J.-L., Frabetti A., Chapoton F., Goichot F. *Dialgebras and Related Operads*, Lecture Notes in Math., 1763, Springer, Berlin, (2001).

[6] D.J. Melville, Centroids of nilpotent Lie algebras, *Comm. Algebra*, **20**(12), (1992), 3649–3682.

[7] Melville D.J., Centroids and Cartan algebras, *Comm. Algebra*, **21**(8), (1993),2791–2818.