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# SELF FP-INJECTIVE AMALGAMATED DUPLICATION OF A RING ALONG AN IDEAL

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**Abstract:** Let $R$ be commutative ring and let $I$ be an ideal of $R$. The amalgamated duplication of $R$ along $I$ is subring of $R \times R$ given by $R \bowtie I = \{(r, r+i)/r \in R, i \in I\}$. In this paper, we characterize the amalgamated duplication of a ring along an ideal to be self FP-injective provided $I$ is finitely generated. Hence, we deduce a characterization of this construction to be IF-ring, and to be quasi-Frobenius ring.

**AMS Subject Classification:** 13D05, 13D07
**Key Words:** amalgamated duplication of a ring along an ideal, FP-injective dimension.

## 1. Introduction

All rings in this paper are unitary and commutative, and all modules are unital. An exact sequence of $R$-modules $0 \to F \to M \to G \to 0$ is pure if it remains exact when tensoring it with any $R$-module. Then, we say that $F$ is a *pure submodule* of $M$. When $M$ is flat, it is well known that $G$ is flat if and only if $F$ is a pure submodule of $M$. An R-module $M$ is FP-*injective* if $\mathrm{Ext}^1_R(F, M) = 0$ for any finitely presented $R$-module $F$. We recall that a module $M$ is FP-injective if and only if it is a pure submodule of every overmodule.

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When $R$ is FP-injective $R$-module, $R$ is said to be *self* FP-*injective ring*. An $R$-module $G$ is called *almost* FP-*injective* if there exist an FP-injective module $M$ and a pure submodule $D$ such that $G \cong M/D$ ([3]). It is proved that a ring $R$ is self FP-injective if and only if every flat $R$-module is almost FP-injective ([3, Proposition 3]). A ring $R$ is said to be IF [7] if each injective $R$-module is flat (or equivalently, each FP-injective $R$-module is flat). Clearly, an IF-ring is self FP-injective. The converse holds when $R$ is coherent ([7, Theorem 3.8]). One can see IF-rings by using the Gorenstein flat modules ([8]). Recall that an $R$-module $M$ is called *Gorenstein flat*, if there exists an exact sequence of flat $R$-modules:

$$\mathbf{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that the functor $I \otimes_R -$ leaves $\mathbf{F}$ exact whenever $I$ is an injective $R$-module. The complex $\mathbf{F}$ is called a *complete flat resolution*. It is proved that a ring $R$ is IF if and only if every $R$-module is Gorenstein flat ([11, Proposition 2.3]).

Let $R$ be a ring and $I$ and ideal of $R$, and $\pi : R \to R/I$ the canonical surjection. The amalgamated duplication of $R$ along $I$, denoted by $R \bowtie I$, is the special pullback (or fiber product) of $\pi$ and $\pi$; i.e., the subring of $R \times R$ given by

$$R \bowtie I := \pi \times_{R/I} \pi = \{(r, r+i) \mid r \in R,\, i \in I\}$$

This construction was introduced and its basic properties were studied by D'Anna and Fontana in [6, 5] and then it was investigated by DAnna in [4] with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated duplication of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve [4, Theorem 14 and Corollary 17].

This short paper investigates necessary and sufficient conditions for an amalgamated duplication of a ring along a finitely generated ideal to be self FP-injective ring. Hence, we deduce a characterization of this construction to be IF-ring, and to be quasi-Frobenius ring.

## 2. Results

The main result of this paper is formulaes as follows.

**Theorem 1.** *Let $R$ be a ring and $I$ a finitely generated ideal of $R$. Then, $R \bowtie I$ is self* FP-*injective if and only if $R$ is self* FP-*injective and $I$ is generated by an idempotent.*

The proof of our main result will be divided into several lemmas.

**Lemma 2.** *Let $R$ be a ring and $I$ a finitely generated ideal of $R$. If $R \bowtie I$ is self* FP*-injective, then $I$ is generated by an idempotent.*

*Proof.* Since $I$ is finitely generated ideal of $R$, $I \times I$ is a finitely generated ideal of $R \bowtie I$. Indeed, if $I := \sum_{j=1}^{n} a_j R$, then $I \times I = \sum_{j=1}^{n}(a_j, 0)R \bowtie I + \sum_{j=1}^{n}(0, a_j)R \bowtie I$. Hence, $\frac{R \bowtie I}{I \times I}$ is a finitely presented $R \bowtie I$-module. Note also that $R \times R$ is an $R \bowtie I$-module via the naturel injection $R \bowtie I \hookrightarrow R \times R$. Consider the following morphism of $R \bowtie I$-modules:

$$\begin{array}{rccc} \psi: & R \times R & \longrightarrow & \frac{R \bowtie I}{I \times I} \\ & (a, b) & \longmapsto & \overline{(b-a, b-a)} \end{array}$$

For each $a \in R$, we have $\overline{(a, a)} = \psi(0, a)$, and so $\psi$ is surjective, and its kernel is

$$\ker(\psi) = \{(a, b)/b - a \in I\} = R \bowtie I.$$

Therefore, we have the following short exact sequence of $R \bowtie I$-modules.

$$0 \longrightarrow R \bowtie I \longrightarrow R \times R \xrightarrow{\psi} \frac{R \bowtie I}{I \times I} \longrightarrow 0 \ (*)$$

Since $R \bowtie I$ is self FP-injective and $\frac{R \bowtie I}{I \times I}$ is finitely presented, $(*)$ splits, and thus there exist an $R \bowtie I$-morphism $\mu : \frac{R \bowtie I}{I \times I} \to R \times R$ such that $\psi \circ \mu = \mathrm{id}_{\frac{R \bowtie I}{I \times I}}$, and so there exist $a, b \in R$ and $i, j \in I$, such that

$$(1, 0) = (a, a+i) + \mu \circ \psi((1, 0)) \quad \text{and} \quad (0, 1) = (b, b+j) + \mu \circ \psi((0, 1))$$

and so, we have

$$\begin{array}{rcl} (1, 1) & = & (a, a+i) + \mu \circ \psi((1, 0)) + (b, b+j) + \mu \circ \psi((0, 1)) \\ & = & (a, a+i) + (b, b+j) + \mu \circ \psi((1, 1)) \\ & = & (a, a+i) + (b, b+j) \end{array}$$

For any $x \in I$, since$(x, 0)$ and $(0, x)$ belong to $R \bowtie I$, we have

$$(x, 0) = (x, 0)(1, 0) = (ax, 0) + \mu \circ \psi((x, 0)) = (ax, 0).$$

Then, $1 - a = b \in \mathrm{ann}_R(I)$, and so

$$(0, x) = (0, x)(0, 1) = (0, (b+j)x) + \mu \circ \psi((0, x)) = (0, xj)$$

Thus, $1 - j \in \mathrm{ann}_R(I)$. Consequently, for each $a \in I$, $a(1-j) = 0$, and so $a = aj$ and in particular $j^2 = j$. Hence, $j$ is a idempotent element of $I$ and $I = Rj$. □

**Lemma 3.** *Let $R$ be a ring and $I$ an ideal of $R$ generated by an idempotent element. We have the following isomorphism of rings:*

$$R \bowtie I \cong R \times \frac{R}{\mathrm{ann}_R(I)}$$

*Proof.* Let $e$ be an idempotent element of $R$ such that $I = Re$, and consider the following map

$$\begin{array}{rccc} \varphi : & R \bowtie I & \longrightarrow & R \times \frac{R}{\mathrm{ann}_R(I)} \\ & (a, a+i) & \longmapsto & (a, \overline{a+i}) \end{array}$$

It's easy to see that $\varphi$ is a morphism of rings. Moreover, for each $(a, \overline{b}) \in R \times \frac{R}{\mathrm{ann}_R(I)}$, we have $(a, \overline{b}) = \varphi(a, a + e(b-a))$, and so $\varphi$ is surjective. Now, let $(a, a+i) \in \ker(\varphi)$. Then, $a = 0$ and $i \in I \cap \mathrm{ann}_R(I) = (0)$. Consequently, $\varphi$ is bijective. □

**Lemma 4.** *Let $R$ and $R'$ be two rings. Then, $R \times R'$ is FP-injective if and only if $R$ and $R'$ are* FP*-injective.*

*Proof.* Suppose that $R \times R'$ is self FP-injective, and let $M$ be a finitely presented $R$-module. It is clear that $M$ is also an $R \times R'$ module (via the canonical projection $R \times R' \to R$). With this modulation, and by using [9, Theorem 2.1.8], $M$ is a finitely presented $R\times R'$-module. Thus, $\mathrm{Ext}^1_{R\times R'}(M, R\times R') = 0$. From [13, Theorem 10.75],

$$\mathrm{Ext}^1_{R\times R}(M, R\times R') = \mathrm{Ext}^1_R(M, \mathrm{Hom}_{R\times R'}(R, R\times R')) = \mathrm{Ext}^1_R(M, R)$$

Hence, $\mathrm{Ext}^1_R(M, R) = 0$. Consequently, $R$ is self FP-injective. Similarly, $R'$ is self FP-injective.
Conversely, assume that $R$ and $R'$ are self FP-injective rings, and let $M$ a finitely presented $R \times R'$-module. Then,

$$\begin{array}{rcl} \mathrm{Ext}^1_{R\times R'}(M, R\times R') & = & \mathrm{Ext}^1_{R\times R'}(M, (R\times 0)\oplus(0\times R')) \\ & = & \mathrm{Ext}^1_{R\times R'}(M, R\times 0)\oplus \mathrm{Ext}^1_{R\times R'}(M, 0\times R') \\ & = & \mathrm{Ext}^1_{R\times R'}(M, R)\oplus \mathrm{Ext}^1_{R\times R'}(M, R') \\ \text{(by [13, Theorem 10.75])} & = & \mathrm{Ext}^1_R(M\otimes_{R\times R'} R, R)\oplus \mathrm{Ext}^1_{R'}(M\otimes_{R\times R'} R', R') \end{array}$$

On the other hand, by [9, Theorem 2.1.8], $M \otimes_{R\times R'} R \cong M/(0\times R')M$ (resp. $M\otimes_{R\times R'} R' \cong M/(R\times 0)M$) is a finitely presented $R$-module (resp. $R'$-module). Thus, $\mathrm{Ext}^1_R(M\otimes_{R\times R'} R, R) = 0$ and $\mathrm{Ext}^1_{R'}(M\otimes_{R\times R'} R', R') = 0$. Consequently, $\mathrm{Ext}^1_{R\times R'}(M, R\times R') = 0$, and so $R\times R'$ is self FP-injective ring. □

*Proof of Theorem 1.* Note that, if $I$ is generated by an idempotent element then we have the following isomorphism of rings:

$$R \cong \frac{R}{I} \times \frac{R}{\mathrm{ann}_R(I)}$$

Thus, our proof is a deduced by combining Lemmas 2, 3, and 4. □

Using our main result, we can deduce the characterization of $R \bowtie I$ to be IF-ring.

**Corollary 5.** *Let $R$ be a ring and $I$ a finitely generated ideal of $R$. Then, $R \bowtie I$ is an* IF*-ring if and only if $R$ is an* IF*-ring and $I$ is generated by an idempotent.*

*Proof.* Recall that a ring $R$ is IF if and only if $R$ is coherent and self FP-injective ([11, Proposition 2.3]). On the other hand, since $I$ is finitely generated, $R \bowtie I$ is coherent if and only if $R$ is coherent ([10, Corollary 2.8]). Thus, our result is a direct consequence of Theorem 1. □

Recall that a ring $R$ is called *quasi-Frobenius* [12] if it Noetherian and self injective. The quotient $R/I$ where $R$ is a principal ideal domain and $I$ is any nonzero ideal of $R$ is a classical example of quasi-Frobenius ring. Several characterizations of quasi-Frobenius rings were given in [12]. The characterization of $R \bowtie I$ to be quasi-Frobenius was done in [2]. However, we will find it again by using Theorem 1.

**Corollary 6** ([2, Proposition 2.6])**.** *Let $R$ be a ring and $I$ an ideal of $R$. Then, $R \bowtie I$ is quasi-Frobenius if and only if $R$ is quasi-Frobenius and $I$ is generated by an idempotent.*

*Proof.* Using [1, Theorem 2.2] and [11, Proposition 2.3 & 2.10], we can deduce that a ring is quasi-Frobenius if and only it is a Noetherian and IF-ring. Moreover, it is proved that $R \bowtie I$ is Noetherian if and only if $R$ is Noetherian ([6, Corollary3.3]). Thus, if $R \bowtie I$ or $R$ is quasi-Frobenius, then $I$ is generated by an idempotent. Hence, our result is deduced directly from Corollary 5. □

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