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APPROXIMATION BY GENERALIZED FABER SERIES IN BERS SPACES ON QUASIDISKS

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Abstract: In this paper, by the conformal natural reflection introduced by Earle and Nag ,we establish an integral representation for Bers space $B(D)$ where D is a quasidisk, and then we define generalized Faber series of functions in $B(D)$. We find a subspace $B_1(D)$ of $B(D)$ such that each $\varphi \in B_1(D)$ converges uniformly on compact subsets of D. Also, if a series $\sum_{m=3}^{\infty} a_m F_m^{'''}(z)$ is convergent to $\varphi \in B_1(D)$ in the norm $\|\|_D$, we show that the a_m are the generalized Faber coefficients $a_m(\varphi)$ of φ.

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1. Introduction

Let D be a quasidisk in the extended complex plane $\overline{C}$ i.e., D is the image of the open unit disk Δ under the quasiconformal self mapping of $\overline{C}$. We denote by $B(D)$ the Bers space of holomorphic functions $\varphi(z)$ in D such that

$$\|\varphi\|_D = \sup\{|\varphi(z)|\lambda_D^{-2}(z) : z \in D\} < \infty,$$

where λ_D is the hyperbolic density on D(If $\infty \in D$ and $\varphi \in B(D)$, then $\varphi(z)$ has a zero of order ≥ 4 at ∞). It is well known that the space $B(D)$ is regarded as the tangent space of the Teichmüller space $T(D)$ (see [2], [7], [8], [9], [11], [12], [14], [15]). Due to the importance of the Bers space $B(D)$ in extremal quasiconformal mappings theory and Teichmüller spaces theory, it is meaningful to consider the polynomial approximation problem in $B(D)$.

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Suppose $\Psi(w) \in \Sigma_0$ maps $\Delta^*(= \overline{C}\backslash\overline{\Delta}\ )$ onto $D^*(= \overline{C}\backslash\overline{D}\ )$,where Σ_0 is the class of univalent functions of the form $g(w) = w + b_1 w^{-1} + b_2 w^{-2} + \cdots$ for $w \in \Delta^*$. It is known that $\overline{D}$ is a compact subset of the closed ball $\{z : |z| \leq 2\}$([16]).

Faber polynomials play an important role in the study of univalent functions. Recall that the mth Faber polynomial F_m for $\Psi(w) \in \Sigma_0$ is a polynomial of degree m determined by the following expression:

$$\log \frac{\Psi(w) - z}{w} = -\sum_{m=1}^{\infty} \frac{1}{m} F_m(z) w^{-m}, z \in C, w \to \infty.$$

If $F_m(z)$ is the mth Faber polynomial for $\overline{D}$, then it is known that for every $(z, w) \in \overline{D} \times \Delta^*$,

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{m=0}^{\infty} \frac{F_m(z)}{w^{m+1}} \tag{1}$$

and the series converges uniformly and absolutely on compact subsets of $\overline{D}\times\Delta^*$. From (1), we get

$$\frac{6\Psi'(w)}{(\Psi(w) - z)^4} = \sum_{m=3}^{\infty} \frac{F_m'''(z)}{w^{m+1}}, (z, w) \in \overline{D} \times \Delta^* \tag{2}$$

where the series converges uniformly and absolutely on compact subsets of $\overline{D} \times \Delta^*$. More information for Faber polynomials and Faber expansions can be found in [10] and [17].

This paper is arranged as follows. In Section 2, by the conformal natural reflection introduced by Earle and Nag, we establish an integral representation for Bers space $B(D)$ where D is a quasidisk, and then we define a generalized Faber series of a function $\varphi \in B(D)$ to be the form $\sum_{m=3}^{\infty} a_m(\varphi)F_m'''(z)$. In Section 3, we define a subspace $B_1(D)$ of $B(D)$ such that each $\varphi \in B_1(D)$ converges uniformly on compact subsets of D. Also, if a series $\sum_{m=3}^{\infty} a_m F_m'''(z)$ is convergent to $\varphi \in B_1(D)$ in the norm $\|\|_D$, we show that the a_m are the generalized Faber coefficients $a_m(\varphi)$ of φ. Note that polynomial approximation problem has been studied by Cavus in Bergman spaces[4] and Israfilov in $A(\overline{D})$ where $A(\overline{D})$ denotes the class of functions which are analytic in D and continuous in $\overline{D}$[13] respectively.

2. Integral Representation of $\varphi \in B(D)$ by Conformal Natural Reflection

Theorem 1. *[1] If D is a Jordan domain, then D is a quasidisk if and only if there exists a quasiconformal reflection with respect to the boundary ∂D. When $\infty \in \partial D$, there exists a quasiconformal reflection satisfying a uniform Lipschitz condition.*

Theorem 2. *[3] Suppose D is a quasidisk with $\infty \in \partial D$. If h is a quasiconformal reflection with respect to ∂D and satisfies a uniform Lipschitz condition, then every $\varphi \in B(D)$ has the following reproducing formula*

$$\varphi(z) = -\frac{3}{\pi}\iint_{D^*}\frac{(\zeta - h(\zeta))^2}{(\zeta - z)^4} h_{\bar{\zeta}}(\zeta)\varphi(h(\zeta))d\xi d\eta, z \in D. \tag{3}$$

In Theorem 1 Ahlfors gave a characterization of quasidisks by quasiconformal reflections and in Theorem 2 Bers established an integral representation of $\varphi \in B(D)$ where D is a quasidisk with $\infty \in \partial D$. We want to establishes an integral representation of $\varphi \in B(D)$ where D is a quasidisk without the restriction $\infty \in \partial D$.

If $\infty \notin \partial D$, there exists a Möbius transformation $\gamma : \overline{C} \to \overline{C}$ such that $\infty \in \partial\gamma(D)$. By Theorem 1, there exists a quasiconformal reflection $\widehat{h}$ with respect to $\partial\gamma(D)$ and $\widehat{h}$ satisfies a uniform Lipschitz condition. Then $h = \gamma^{-1} \circ \widehat{h} \circ \gamma$ is a quasiconformal reflection with respect to ∂D. By simple computations, we have the following

Theorem 3. *Suppose D is a quasidisk. There exists a quasiconformal reflection h with respect to ∂D such that every $\varphi \in B(D)$ has the following reproducing formula*

$$\varphi(z) = -\frac{3}{\pi}\iint_{D^*}\frac{(\zeta - h(\zeta))^2}{(\zeta - z)^4} h_{\bar{\zeta}}(\zeta)\varphi(h(\zeta))d\xi d\eta, z \in D. \tag{4}$$

Earle and Nag introduced the concept of conformally natural quasiconformal reflection. Suppose S is a Jordan curve, D and D^* are two complementary components of S in $\overline{C}$. Two conformal mappings $f : \Delta \to D$ and $f^* : \Delta \to D^*$ determine the homeomorphism $f^{-1} \circ f^*$ and $f^{-1} \circ f^*$ has a conformally natural extension $ex(f^{-1} \circ f^*) : \overline{\Delta} \to \overline{\Delta}$[5]. The mapping $j : \overline{C} \to \overline{C}$

$$j(z) = f^* \circ ex(f^{-1} \circ f^*)^{-1} \circ f^{-1}, z \in \overline{D}$$

$$j(z) = f \circ ex(f^{-1} \circ f^*) \circ (f^*)^{-1}, z \in \overline{D}^*$$

is called the conformally natural reflection of D. Earle and Nag[6] proved that j depends only on the domain D, independent of the choose of f and f^*; j is a real analytic homeomorphism and is a quasiconformal reflection with respect to the boundary ∂D whenever D is a quasidisk; if j is the conformally natural reflection of D and $\gamma : \overline{C} \to \overline{C}$ is a Möbius transformation, then $\gamma \circ j \circ \gamma^{-1}$ is the conformally natural reflection of $\gamma(D)$. Earle, Gardiner and Lakic proved that for the quasidisk D with $\infty \in D$, the conformally natural quasiconformal reflection j satisfies a uniform Lipschitz condition. Hence by the proof of Theorem 3, we have the following result

Theorem 4. *Let D be a quasidisk and j be the conformally natural quasiconformal reflection with respect to the boundary ∂D. Then for every $\varphi \in B(D)$, we have*

$$\varphi(z) = -\frac{3}{\pi}\iint_{D^*} \frac{(\zeta - j(\zeta))^2}{(\zeta - z)^4} j_{\bar{\zeta}}(\zeta)\varphi(j(\zeta))d\xi d\eta, z \in D. \tag{5}$$

Substituting $\zeta = \Psi(w)$ in (5), we get

$$\begin{aligned}\varphi(z) &= -\tfrac{3}{\pi}\iint_{\Delta^*} \tfrac{(\Psi(w)-j(\Psi(w)))^2}{(\Psi(w)-z)^4} j_{\bar{\zeta}}(\Psi(w))\varphi(j(\Psi(w)))|\Psi'(w)|^2 dudv \\ &= -\tfrac{3}{\pi}\iint_{\Delta^*} \varphi(j(\Psi(w)))(\Psi(w) - j(\Psi(w)))^2\overline{\Psi'(w)} j_{\bar{\zeta}}(\Psi(w)) \\ &\cdot \tfrac{\Psi'(w)}{(\Psi(w)-z)^4} dudv.\end{aligned}$$

If we consider (2) and define coefficients $a_m(\varphi)$ by

$$\begin{aligned}a_m(\varphi) &= -\tfrac{1}{2\pi}\iint_{\Delta^*} \tfrac{\varphi(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2\overline{\Psi'(w)}}{w^{m+1}} \\ &\cdot j_{\bar{\zeta}}(\Psi(w))dudv, m = 3, 4, ...\end{aligned} \tag{6}$$

then we can associate a formal series $\sum\limits_{m=3}^{\infty} a_m(\varphi)F_m'''(z)$ with $\varphi \in B(D)$, i.e.,

$$\varphi(z) \sim \sum_{m=3}^{\infty} a_m(\varphi)F_m'''(z). \tag{7}$$

We call this formal series a generalized Faber series of $\varphi \in B(D)$, and the coefficients $a_m(\varphi)$ are called generalized Faber coefficients of φ.

3. The Main Results and Proofs

Lemma 5. *The series $\sum_{m=3}^{\infty} \frac{|F_m'''(z)|^2}{m+1}$ is convergent uniformly on compact subsets of D.*

Proof. Let z be a fixed point in D. Then the power series $\sum_{m=3}^{\infty}\frac{F_m'''(z)}{m+1}w^{m+1}$ defines an analytic function $A(z,w)$ in Δ, i.e.,

$$A(z,w) := \sum_{m=3}^{\infty}\frac{F_m'''(z)}{m+1}w^{m+1}, w\in\Delta. \tag{8}$$

Thus, by taking derivative of (8) with respect to w and considering (2) we get

$$A'(z,w) = \sum_{m=3}^{\infty}F_m'''(z)w^m = \frac{6\Psi'(1/w)}{(\Psi(1/w)-z)^4 w}, w\in\Delta. \tag{9}$$

Let $0<r<1$. Since $\sum_{m=3}^{\infty}F_m'''(z)w^m$ is convergent uniformly and absolutely on the closed disc $\bar{B}(0,r)$, it follows that

$$\iint_{\bar{B}(0,r)}|A'(z,w)|^2 dudv = \pi\sum_{m=3}^{\infty}\frac{|F_m'''(z)|^2}{m+1}r^{2m+2}. \tag{10}$$

(9) and (10) show that

$$\pi\sum_{m=3}^{\infty}\frac{|F_m'''(z)|^2}{m+1}r^{2m+2} = 36\iint_{\bar{B}(0,r)}\left|\frac{\Psi'(1/w)}{(\Psi(1/w)-z)^4 w}\right|^2 dudv. \tag{11}$$

Now we claim that

$$\iint_{\Delta^*}\left|\frac{\Psi'(w)}{(\Psi(w)-z)^4}\right|^2 dudv < +\infty.$$

Indeed by (2), we have

$$\lim_{w\to\infty} w^4\frac{\Psi'(w)}{(\Psi(w)-z)^4} = 1.$$

Hence there exists some $R>1$ such that $\left|\frac{\Psi'(w)}{(\Psi(w)-z)^4}\right| \le \frac{2}{|w|^4}$ when $|w|>R$. Thus

$$\iint_{|w|>R}\left|\frac{\Psi'(w)}{(\Psi(w)-z)^4}\right|^2 dudv \le \frac{4\pi}{3R^6}.$$

We choose some R' such that $|\Psi(w)-z|\le R'$ whenever $z\in D$ and $1<|w|<R$. Now for any $z\in D$, set $d(z,\partial D)=\inf\{|\zeta-z| : \zeta\in\partial D\}$ so that $|\Psi(w)-z|\ge d(z,\partial D)$ whenever $w\in\Delta^*$. Thus

$$\begin{aligned}\iint_{1<|w|<R}\left|\frac{\Psi'(w)}{(\Psi(w)-z)^4}\right|^2 dudv &= \iint_{\Psi(1<|w|<R)}\frac{1}{|\zeta-z|^8}d\xi d\eta\\ &\le \iint_{d(z,\partial D)<|\zeta-z|<R'}\frac{1}{|\zeta-z|^8}d\xi d\eta\\ &\le \frac{\pi}{3d^6(z,\partial D)}.\end{aligned}$$

Since

$$\lim_{r\to 1^-}\iint_{\bar{B}(0,r)}\left|\frac{\Psi'(1/w)}{(\Psi(1/w)-z)^4 w}\right|^2 dudv=\iint_{\Delta}\left|\frac{\Psi'(1/w)}{(\Psi(1/w)-z)^4 w}\right|^2 dudv$$

and

$$\iint_{\Delta}\left|\frac{\Psi'(1/w)}{(\Psi(1/w)-z)^4 w}\right|^2 dudv=\iint_{\Delta^*}\left|\frac{\Psi'(w)}{(\Psi(w)-z)^4}\right|^2 dudv<+\infty,$$

we get

$$\pi\sum_{m=3}^{\infty}\frac{|F_m'''(z)|^2}{m+1}=36\iint_{\Delta}\left|\frac{\Psi'(1/w)}{(\Psi(1/w)-z)^4 w}\right|^2 dudv.$$

On the other hand, it can be easily proved that

$$\iint_{\Delta}\left|\frac{\Psi'(1/w)}{(\Psi(1/w)-z)^4 w}\right|^2 dudv$$

is continuous in D, by Dini's theorem the series $\sum_{m=3}^{\infty}\frac{|F_m'''(z)|^2}{m+1}$ is convergent uniformly on compact subsets of D. The proof Lemma 5 is completed. □

Definition 6. Denote by $B_1(D)$ the space of functions in $B(D)$ such that for every $\varphi\in B(D)$ there exists a constant M satisfying

$$\iint_{D^*}|\varphi(j(\zeta))|^2|(\zeta-j(\zeta)|^4|j_{\bar{\zeta}}(\zeta)|^2 d\xi d\eta\le M\|\varphi\|_D^2,$$

where j is the conformal natural quasiconformal reflection with respect to ∂D.

It is easy to see that $B_1(D)$ is a proper subspace of $B(D)$. Then we have the following

Theorem 7. *Let $\varphi\in B_1(D)$. If $\Sigma_{m=3}^{\infty}a_m(\varphi)F_m'''(z)$ is a generalized Faber series of φ, then the series $\Sigma_{m=3}^{\infty}a_m(\varphi)F_m'''(z)$ converges uniformly to φ on compact subsets of D.*

Proof. Let E be a compact subset of D. For every $z \in E$ and $\varphi \in B_1(D)$,

$$\begin{aligned}
&\left|\varphi(z) - \sum_{m=3}^{n} a_m(\varphi)F_m'''(z)\right| \\
&= \tfrac{3}{\pi}\left|\iint_{\Delta^*} \varphi(j(\Psi(w)))(\Psi(w) - j(\Psi(w)))^2\overline{\Psi'(w)}j_{\bar{\zeta}}(\Psi(w))\left(\sum_{m=n+1}^{\infty} \tfrac{F_m'''(z)}{6w^{m+1}}\right) dudv\right| \\
&\le \left(\iint_{\Delta^*}\left|\sum_{m=n+1}^{\infty} \tfrac{F_m'''(z)}{6w^{m+1}}\right|^2 dudv\right)^{\frac{1}{2}} \cdot \\
&\tfrac{3}{\pi}\left(\iint_{\Delta^*} |\varphi(j(\Psi(w)))|^2|\Psi(w) - j(\Psi(w))|^4|\Psi'(w)|^2|j_{\bar{\zeta}}(\Psi(w))|^2 dudv\right)^{\frac{1}{2}} \\
&= \left(\iint_{\Delta^*}\left|\sum_{m=n+1}^{\infty} \tfrac{F_m'''(z)}{6w^{m+1}}\right|^2 dudv\right)^{\frac{1}{2}} \cdot \\
&\tfrac{3}{\pi}\left(\iint_{D^*} |\varphi(j(\zeta))|^2|\zeta - j(\zeta)|^4|j_{\bar{\zeta}}(\zeta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
&\le \tfrac{3\sqrt{M}\|\varphi\|_D}{\pi}\left(\iint_{\Delta^*}\left|\sum_{m=n+1}^{\infty} \tfrac{F_m'''(z)}{6w^{m+1}}\right|^2 dudv\right)^{\frac{1}{2}}.
\end{aligned}$$

Let $1 < r < R < +\infty$.

$$\begin{aligned}
\iint_{r<|w|<R}\left|\sum_{m=n+1}^{\infty} \tfrac{F_m'''(z)}{6w^{m+1}}\right|^2 dudv &= \tfrac{\pi}{36}\sum_{m=n+1}^{\infty} \tfrac{1}{m}\left(\tfrac{1}{r^{2m}} - \tfrac{1}{R^{2m}}\right)|F_m'''(z)|^2 \\
&\le \tfrac{\pi}{9}\sum_{m=n+1}^{\infty} \tfrac{|F_m'''(z)|^2}{m+1},
\end{aligned}$$

and by letting $r \to 1^+$ and $R \to +\infty$ we get

$$\iint_{\Delta^*}\left|\sum_{m=n+1}^{\infty} \frac{F_m'''(z)}{6w^{m+1}}\right|^2 dudv \le \frac{\pi}{9}\sum_{m=n+1}^{\infty} \frac{|F_m'''(z)|^2}{m+1}. \tag{12}$$

Therefore by Lemma 5, we conclude that $\Sigma_{m=3}^{\infty} a_m(\varphi)F_m'''(z)$ converges uniformly to φ on E. □

Corollary 8. *Suppose $P_n(z) \in B_1(D)$ is a polynomial of degree n and $a_m(P_n)$ are its generalized Faber coefficients, then $a_m(P_n) = 0$ for all $m \ge n+4$ and $P_n(z) = \Sigma_{m=3}^{n+3} a_m(P_n)F_m'''(z)$.*

Proof. Let $z \in D$. By Theorem 7, we have

$$P_n(z) = \Sigma_{m=3}^{\infty} a_m(P_n)F_m'''(z).$$

It is obvious that $P_n(z)$ can be written in the form

$$P_n(z) = \Sigma_{v=3}^{n+3} A_v F_v'''(z).$$

Let j be the K-quasiconformal Earle-Nag reflection with respect to ∂D. Since j keeps ∂D pointwise fixed, $\Psi(w) = j(\Psi(w))$ with $|w| = 1$. It is easy to see that

$$\begin{aligned}&[F_v''(j(\Psi(w)))(\Psi(w) - j(\Psi(w)))^2]_{\bar{w}}\\ &= F_v'''(j(\Psi(w)))\overline{\Psi'(w)}j_{\bar{\zeta}}(\Psi(w))(\Psi(w) - j(\Psi(w)))^2\\ &-2F_v''(j(\Psi(w)))\overline{\Psi'(w)}j_{\bar{\zeta}}(\Psi(w))(\Psi(w) - j(\Psi(w))).\end{aligned}$$

and

$$\begin{aligned}&[F_v'(j(\Psi(w)))(\Psi(w) - j(\Psi(w)))]_{\bar{w}}\\ &= F_v''(j(\Psi(w)))\overline{\Psi'(w)}j_{\bar{\zeta}}(\Psi(w))(\Psi(w) - j(\Psi(w)))\\ &-F_v'(j(\Psi(w)))\overline{\Psi'(w)}j_{\bar{\zeta}}(\Psi(w))\\ &= F_v''(j(\Psi(w)))\overline{\Psi'(w)}j_{\bar{\zeta}}(\Psi(w))(\Psi(w) - j(\Psi(w)))\\ &-[F_v(j(\Psi(w)))]_{\bar{w}}.\end{aligned}$$

Hence we get by Green's formula

$$\begin{aligned}a_m(P_n) &= -\tfrac{1}{2\pi}\iint_{\Delta^*} \tfrac{P_n(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2\overline{\Psi'(w)}}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv\\ &= -\sum_{v=3}^{n+3} \tfrac{A_v}{2\pi}\iint_{\Delta^*} \tfrac{F_v'''(j(\Psi(w)))\overline{\Psi'(w)}(\Psi(w)-j(\Psi(w)))^2}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv\\ &= -\sum_{v=3}^{n+3} \tfrac{A_v}{2\pi}\iint_{\Delta^*} \tfrac{[F_v''(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2]_{\bar{w}}}{w^{m+1}} dudv\\ &-\sum_{v=3}^{n+3} \tfrac{A_v}{2\pi}\iint_{\Delta^*} \tfrac{2[F_v'(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))]_{\bar{w}}}{w^{m+1}} dudv\\ &-\sum_{v=3}^{n+3} \tfrac{A_v}{2\pi}\iint_{\Delta^*} \tfrac{2[F_v(j(\Psi(w)))]_{\bar{w}}}{w^{m+1}} dudv\\ &= \sum_{v=3}^{n+3} \tfrac{A_v}{4\pi i}\int_{|w|=1} \tfrac{[F_v''(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2]}{w^{m+1}} dudv\\ &+\sum_{v=3}^{n+3} \tfrac{A_v}{4\pi i}\int_{|w|=1} \tfrac{2[F_v'(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))]}{w^{m+1}} dudv\\ &+\sum_{v=3}^{n+3} \tfrac{A_v}{4\pi i}\int_{|w|=1} \tfrac{2[F_v(j(\Psi(w)))]}{w^{m+1}} dudv\\ &= \sum_{v=3}^{n+3} \tfrac{A_v}{2\pi i}\int_{|w|=1} \tfrac{[F_v(\Psi(w))]}{w^{m+1}} dudv.\end{aligned}$$

Since

$$\frac{1}{2\pi i}\int_{|w|=1} \frac{F_v(\Psi(w))}{w^{m+1}} dudv = \begin{cases} 1, \text{ if } v = m\\ 0, \text{ if } v \neq m, \end{cases} \tag{13}$$

see[10], it follows that $a_m(P_n) = A_m$, for $m = 3, \dots, n+3$, and $a_m(P_n) = 0$, for $m \geq n+4$. So $P_n(z) = \Sigma_{m=3}^{n+3} a_m(P_n) F_m'''(z)$. □

Theorem 9. *Let $\{a_m\}$ be a complex number sequence. If the series $\Sigma_{m=3}^{\infty} a_m F_m'''(z)$ converges to a function $\varphi \in B_1(D)$ in the norm $\|\|_D$, then the a_m are the generalized Faber coefficients of φ.*

Proof. Let j be the K-quasiconformal Earle-Nag reflection with respect to ∂D, and $S_n(z) := \Sigma_{m=3}^{n+3} a_m F_m'''(z)$ be the nth partial sum of $\Sigma_{m=3}^{\infty} a_m F_m'''(z)$. Using (13), it can be shown that

$$\begin{aligned} &\lim_{n\to\infty} -\frac{1}{2\pi} \iint_{\Delta^*} \frac{S_n(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2 \overline{\Psi'(w)}}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv \\ &= a_m, m = 3, 4, \dots, \end{aligned} \tag{14}$$

So, we get by using Hölder inequality

$$\begin{aligned} &|a_m(\varphi) - a_m| \\ &\leq \frac{1}{2\pi} \left| \iint_{\Delta^*} \frac{[\varphi(j(\Psi(w))) - S_n(j(\Psi(w)))](\Psi(w)-j(\Psi(w)))^2 \overline{\Psi'(w)}}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv \right| \\ &+ \left| -\frac{1}{2\pi} \iint_{\Delta^*} \frac{S_n(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2 \overline{\Psi'(w)}}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv - a_m \right| \\ &\leq \frac{1}{2\pi} \left(\iint_{\Delta^*} \frac{dudv}{|w|^{2m+2}} \right)^{\frac{1}{2}} . \\ &\Big(\iint_{\Delta^*} |\varphi(j(\Psi(w))) - S_n(j(\Psi(w)))|^2 |\Psi(w) \\ &- j(\Psi(w))|^4 |\Psi'(w)|^2 |j_{\bar{\zeta}}(\Psi(w))|^2 dudv \Big)^{\frac{1}{2}} \\ &+ \left| -\frac{1}{2\pi} \iint_{\Delta^*} \frac{S_n(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2 \overline{\Psi'(w)}}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv - a_m \right| \\ &\leq \sqrt{\frac{M}{12\pi}} \left(\iint_{D^*} |\varphi(j(\zeta)) - S_n(j(\zeta))|^2 |\zeta - j(\zeta)|^4 |j_{\bar{\zeta}}(\zeta)|^2 dudv \right)^{\frac{1}{2}} \\ &+ \left| -\frac{1}{2\pi} \iint_{\Delta^*} \frac{S_n(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2 \overline{\Psi'(w)}}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv - a_m \right| \\ &\leq \sqrt{\frac{M}{12\pi}} \|\varphi - S_n\|_D \\ &+ \left| -\frac{1}{2\pi} \iint_{\Delta^*} \frac{S_n(j(\Psi(w)))(\Psi(w)-j(\Psi(w)))^2 \overline{\Psi'(w)}}{w^{m+1}} j_{\bar{\zeta}}(\Psi(w))dudv - a_m \right| . \end{aligned}$$

$\lim_{n\to\infty} \|\varphi - S_n\|_D = 0$ and (14) show that $a_m(\varphi) = a_m$, and so the proof is completed. □

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