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APPROXIMATING SOLUTION OF AN INITIAL AND A PERIODIC BOUNDARY VALUE PROBLEM FOR FIRST ORDER QUADRATIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper we prove the algorithms for the existence and approximation of the solutions for an initial and a periodic boundary value problem of nonlinear first order ordinary hybrid quadratic functional differential equations via iteration method embodied in a recent hybrid fixed point principle of Dhage (2014) in a partially ordered normed linear space. A numerical example is also provided to illustrate the abstract theory developed in the paper.

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1. Introduction

Given the real numbers $\delta > 0$ and $T > 0$, consider the closed and bounded

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intervals $I_0 = [-\delta, 0]$ and $I = [0, T]$ in R and let $J = [-\delta, T]$. By $\mathcal{C} = C(I_0, R)$ we denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with he norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{-\delta\le\theta\le 0} |x(\theta)|. \tag{1}$$

Clearly, $\mathcal{C}$ is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question.

For any continuous function $x : J \to R$ and for any $t \in I$, we denote by x_t the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t+\theta), \; -\delta \le \theta \le 0. \tag{2}$$

The differential equations involving the history of the dynamic systems are called functional differential equations and it has been recognized long back the importance of such problems in the theory of differential equations. Since then, several classes of nonlinear functional differential equations have been discussed in the literature for different qualitative properties of the solutions. A special class of functional differential equations has been discussed in Dhage [9], Mule and Ahirrao ([10],[8]) for the existence and approximation of solutions via new Dhage iteration method. Therefore, it is desirable to extend this method to other functional differential equations involving delay. The present paper is also an attempt in this direction.

Given a closed and bounded interval $J = [0, T]$ in the real line R, consider the initial and periodic boundary value problems of first order nonlinear hybrid differential equation (in short QFDE),

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] + \lambda\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x_t), \;\; t \in I,\\ x_0 = \phi, \;\; t \in I_0, \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] + \lambda\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x_t), \;\; t \in J,\\ x(0) = \phi(0) = x(T),\\ x_0 = \phi \end{aligned}\right\} \tag{4}$$

where $\lambda \in R$, $\lambda > 0$ and $f : I \times R \to R - \{0\}$ and $g : I \times \mathcal{C} \to R$ are continuous functions.

By a *solution* of the QFDE (3) or (4) we mean a function $x \in C(J, R)$ such that

(i) the function $t \mapsto \dfrac{x(t)}{f(t, x(t))}$ is differentiable for each $x \in R$, and

(ii) x satisfies the equations in (3) or (4).

The QFDE (3) and (4) is well-known and extensively discussed in the literature for different aspects of the solutions. Functional and neutral functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Kolmanovskii and Myshkis [16], Hale and Verduyn Lunel [15] and Hale [13] and the references therein.

2. Preliminary Notes / Materials and Methods

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in N$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [14] and the references therein.

We need the following definitions (see Dhage [3, 4, 6] and the references therein) in what follows.

Definition 2.1. A mapping $\mathcal{T} : E \to E$ is called **isotone** or **nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either nondecreasing or nonincreasing on E.

Definition 2.2. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly**

partially bounded if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant.

Definition 2.4. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a relatively compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.1. Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

Definition 2.5. The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set R of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space R^n with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6. An upper semi-continuous and monotone nondecreasing function $\psi : R_+ \to R_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{5}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lischitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a partial linear contraction on E with a contraction constant k.

Following applicable hybrid fixed point theorem of Dhage [3] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of a Dhage iteration method is given in Dhage [3, 6] and the references therein.

Theorem 2.1. *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in E. Let $A, B : E \to E$ be two nondecreasing operators such that*

(a) A is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function ψ_A,

(b) B is partially continuous and uniformly partially compact, and

(c) M $\psi_{a<r,r>0}$,where $M = \sup\{\|B(C)\|$ C is chain in x }

(d) there exists an element $x_0 \in E$ such that $x_0 \preceq Ax_0Bx_0$ or $x_0 \succeq Ax_0Bx_0$.

Then the operator equation

$$AxBx = x \tag{6}$$

has a solution x^* in X and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = Ax_nBx_n$, $n = 0, 1, \ldots$, converges monotonically to x^*.

3. Main Results

In this section, we prove an existence and approximation result for the QFDE (3) on a closed and bounded interval $J = [a, b]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the QFDE (3) in the function space $C(J, R)$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, R)$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{7}$$

and

$$x \leq y \iff x(t) \leq y(t) \quad \text{for all} \quad t \in J. \tag{8}$$

Clearly, $C(J, R)$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, R)$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. See Dhage [3, 4, 6] and the references therein. The following useful lemma concerning the $\|\cdot\|$ and $\leq$ are compatible in every partially compact subsets of $C(J, R)$ follows immediately from the Arzelá-Ascoli theorem for compactness.

Lemma 3.1. *Let $\big(C(J,R),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (7) and (8) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(I,R)$.*

Proof. Let S be a partially compact subset of $C(J,R)$ and let $\{x_n\}_{n\in N}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t)\leq x_2(t)\leq x_3(t)\cdots\cdot$$

for each $t\in R_+$ Suppose that a subsequence $\{x_{nk}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ is convergent and converges to a point $x$in S. Then the subsequence $\{x_{nk}\}_{n\in N}$ of the monotone real sequence $\{x_n\}_{n\in N}$ is convergent. By monotone characterization, the whole sequence $\{x_n\}_{n\in N}$ is convergent and converges to a point $x(t)\in R$ for each $t\in R_+$. This shows that the sequence $\{x_n\}_{n\in N}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n\}_{n\in N}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in N}$ is an equicontinuous sequence by Arzella-Ascoli theorem. Hence $\{x_n\}_{n\in N}$ is convergent and converges uniformly to x. As a result $\leq$ and $\|.\|$ are compatible in S. This completes the proof. $\square$

We introduce an order relation $\leq_{\mathcal{C}}$ in $\mathcal{C}$ induced by the order relation $\leq$ defined in $C(J,R)$. Thus, for any $x,y\in\mathcal{C}$, $x\leq_{\mathcal{C}} y$ implies $x(\theta)\leq y(\theta)$ for all $\theta\in I_0$ and vice versa. If $x,y\in C(J,R)$ and $x\leq y$, then $x_t\leq_{\mathcal{C}} y_t$ for all $t\in I$.

3.1. Initial Value Problem

We need the following definition in what follows.

Definition 3.1.1. *A function $u\in C(J,R)$ is called a lower solution of the QFDE (3) on J if the function $t\mapsto\dfrac{u(t)}{f(t,u(t))}$ is differentiable and satisfies the inequalities*

$$\left.\begin{aligned}\frac{d}{dt}\left[\frac{u(t)}{f(t,u(t))}\right]+\lambda\left[\frac{u(t)}{f(t,u(t))}\right]&\leq g(t,u_t),\\ u_0&\leq\phi,\end{aligned}\right\}$$

for all $t\in J$. Similarly, an upper solution $v\in C(J,R)$ of the QFDE (3) on J is defined.

We consider the following basic hypotheses in what follows.

(A$_0$) The map $x\mapsto\dfrac{x}{f(t,x)}$ is increasing for each $t\in I$.

(A_1) There exists a constant $M_f > 0$ such that $0 < f(t,x) \leq M_f$ for all $t \in I$ and $x \in \mathcal{C}$.

(A_2) There exists a $\mathcal{D}$-function φ such that

$$0 \leq f(t,x) - f(t,y) \leq \varphi(x-y)$$

for all $t \in I$ and $x, y \in R$, $x \geq y$.

(A_3) There exists a constant $M_g > 0$ such that $0 < g(t,x) \leq M_g$ for all $t \in J$ and $x \in \mathcal{C}$.

(A_4) The function $g(t,x)$ is monotone nondecreasing in x for each $t \in I$.

(A_5) The QFDE (3) has a solution $u \in C(J,R)$

Lemma 3.1.2. Assume that hypothesis (A_0) holds, $x \in C(J,R)$ is a solution of the QFDE

$$\left.\begin{array}{c} \dfrac{d}{dt}\left[\dfrac{x(t)}{f(t,x(t))}\right] + \lambda\left[\dfrac{x(t)}{f(t,x(t))}\right] = g(t,x_t), \ \ t \in I, \\ x_0 = \phi, \ \ t \in I_0 \end{array}\right\} \tag{9}$$

if and only if x satisfies the hybrid integral equation (QFIE)

$$x(t) = \begin{cases} f(t,x(t))\left(\frac{\phi(0)}{f(0,\phi(0))} + \displaystyle\int_0^t g(s,x_t),ds\right) & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0. \end{cases} \tag{10}$$

Now we are in a position to prove the following existence and approximation theorem for the QFDE (3) on J.

Theorem 3.1.1. *Suppose that hypotheses (A_1)-(A_2) and (A_4) hold. Then the QFDE (3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = \begin{cases} f(t,x(t))\left(\frac{\phi(0)}{f(0,\phi(0))} + \displaystyle\int_0^t g(s,s_t),ds\right) & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0., \end{cases} \tag{11}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J, R)$. Then, in view of Lemma 3.1 every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ are compatible.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \begin{cases} f(t, x(t)) & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \tag{12}$$

and

$$\mathcal{B}x(t) = \begin{cases} \frac{\phi(0)}{f(0,\phi(0))} + \int_0^t g(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{13}$$

From the continuity of the functions f, g and the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to E$. Applying Lemma 3.1, the QFDE (3) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) = x(t),\ t \in J. \tag{14}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \ge y$. Then $x(t) \ge y(t)$ for all $t \in J$ and by hypothesis (A_2), we get

$$\begin{aligned} \mathcal{A}x(t) &= \begin{cases} f(t, x(t)), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \\ &\ge \begin{cases} f(t, y(t)), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \\ &= \mathcal{A}y(t), \end{aligned}$$

for all $t \in J$. This shows that the operator that the operator $\mathcal{A}$ is also nondecreasing on E.

Next, let $x, y \in E$ be such that $x \ge y$. Then $x_t \ge y_t$ for all $t \in I$ and by hypothesis (A_4), we get

$$\mathcal{B}x(t) = \begin{cases} \frac{\phi(0)}{f(0,\phi(0))} + \int_0^t g(s, x_s), ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

$$
\begin{aligned}
&\geq \begin{cases} \frac{\phi(0)}{f(0,\phi(0))} + \int_0^t g(s, y_s), ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\
&= \mathcal{B}y(t),
\end{aligned}
$$

for all $t \in J$. This shows that the operator that the operator $\mathcal{B}$ is also nondecreasing on E.

Step II: $\mathcal{A}$ *is a nonlinear* $\mathcal{D}$*-contraction on* E*.*

Let $x, y \in E$ be any two elements such that $x \geq y$. Then, by hypothesis (A_4),

$$
|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq |f(t, x(t)) - f(t, y(t))| \leq \varphi(|x(t) - y(t)|) \leq \varphi(\|x - y\|) \quad (15)
$$

for all $t \in J$. Taking the supremum over t, we obtain

$$
\|\mathcal{A}x - \mathcal{A}y\| \leq \psi(\|x - y\|)
$$

for all $x, y \in E$, $x \geq y$.

Step III: $\mathcal{B}$ *is partially continuous on* E*.*

Let $\{x_n\}_{n\in N}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the f is continuous, we have

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \begin{cases} \frac{\phi(0)}{f(0,\phi(0))} + \int_0^t \left[\lim_{n\to\infty} g\big(s, x_s^n\big)\right] ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\
&= \begin{cases} \frac{\phi(0)}{f(0,\phi(0))} + \int_0^t g(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\
&= \mathcal{B}x(t)
\end{aligned}
$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now we show that $\{\mathcal{B}x_n\}_{n\in N}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$
\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &\leq \left| \int_0^T |G(t_1, s) - G(t_2, s)| \, \big|g\big(s, x_s^n\big)\big| \, ds \right| \\
&\leq M_g \int_0^T |G(t_1, s) - G(t_2, s)|\, ds \to 0 \quad \text{as} \quad t_2 \to t_1,
\end{aligned}
$$

uniformly for all $n \in N$.

Case II: Let $t_1, t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in N$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(0)| + |\mathcal{B}x_n(0) - \mathcal{B}x_n(t_1)| \to 0 \ \text{as} \ t_2 \to t_1.$$

Thus in all three cases, we obtain

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in N$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and that $\mathcal{B}$ is a partially continuous operator on E into itself.

Step IV: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (A_4)

$$\begin{aligned}
|y(t)| &= |\mathcal{B}x(t)| \\
&\leq \begin{cases} \left|\frac{\phi(0)}{f(0,\phi(0))}\right| + \displaystyle\int_0^t |g(s, x_s)|\, ds, & \text{if } \ t \in I, \\ |\phi(t)|, \quad \text{if } \ t \in I_0, \end{cases} \\
&\leq \|\phi\| + M_g T \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next we show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B} : E \to E$ is a partially compact operator on E into itself.

Step V: *u satisfies the operator inequality inequality $u \leq \mathcal{A}u\mathcal{B}u$.*

By hypothesis (A_5), the QFDE (3) has a lower solution u defined on J. Then we have

$$\left.\begin{array}{c}\frac{d}{dt}\left[\frac{u(t)}{f(t,u(t))}\right] + \lambda\left[\frac{u(t)}{f(t,u(t))}\right] \leq g(t,u_t),\\ u_0 \leq \phi,\end{array}\right\}$$

Integrating the above inequality from 0 to t, we get

$$\begin{aligned} x(t) &= \begin{cases} f(t,x(t))\left(\frac{\phi(0)}{f(0,\phi(0))} + \int_0^t g(s,x_t),ds\right) & \text{if } t \in I,\\ \phi(t), \quad \text{if } t \in I_0. \end{cases}, \\ &= \mathcal{A}u(t)\mathcal{B}u(t) \end{aligned}$$

for all $t \in J$. As a result we have that $u \leq \mathcal{A}u\mathcal{A}u$.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 and so the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution. Consequently the integral equation and that the differential equation (3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) converges monotonically to x^*. This completes the proof. □

Example 3.1.1. Given the closed and bounded intervals $I_0 = [-1,0]$ and $I = [0,1]$, consider the QFDE

$$\left.\begin{array}{c}\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x_t), \quad t \in I,\\ x_0 = \phi, \quad t \in I_0,\end{array}\right\} \tag{16}$$

where $\phi \in \mathcal{C}$ and $f : I \times R \to R$ and $g : I \times \mathcal{C} \to R$ are continuous functions given by

$$\phi(t) = \sin t, \quad t \in [-1,0],$$

$$f(t,x) = \begin{cases} 1 + |x| & \text{if } x > 0,\\ 1 & \text{if } x \leq 0,\end{cases}$$

and

$$g(t,x) = \begin{cases} \frac{1}{4}[2 + \tanh(\|x\|_{\mathcal{C}})] & \text{if } x >_{\mathcal{C}} 0,\\ \phi(t) & \text{if } x \leq_{\mathcal{C}} 0,\end{cases}$$

for all $t \in I$.

Clearly, f is continuous and bounded on $I \times R$ with bound $M_f = 2$. We show that f satisfies the hypothesis (A_2). Let $x, y \in R$ be such that $x \geq y > 0$. Then $|x| \geq |y| > 0$ and therefore, we have

$$0 \leq f(t,x) - f(t,x) = 1 + |x| - 1 + |y| \leq \varphi(|x-y|) = \varphi(x-y)$$

for all $t \in I$, where $\psi(r) = \dfrac{r}{1+r} < r$, $r > 0$. Again, if $x, y \in R$ be such that $x \leq y \leq 0$, then we obtain

$$0 \leq f(t,x) - f(t,x) \leq \varphi(x-y)$$

for all $t \in I$. This shows that the function $f(t,x)$ satisfies the hypothesis (A_2).

Next, g is bounded on $I \times \mathcal{C}$ with $M_g = \frac{3}{4}$. Again, let $x, y \in \mathcal{C}$ be such that $x \geq_{\mathcal{C}} y > 0$. Then $\|x\|_{\mathcal{C}} \geq \|y\|_{\mathcal{C}} > 0$ and therefore, we have

$$g(t,x) = \frac{1}{4}[2 + \tanh(\|x\|_{\mathcal{C}})] \geq \frac{1}{4}[2 + \tanh(\|y\|_{\mathcal{C}})] = g(t,y)$$

for all $t \in I$. Again, if $x, y \in \mathcal{C}$ be such that $x \leq_{\mathcal{C}} y \leq_{\mathcal{C}} 0$, then we obtain

$$g(t,x)\phi(t) = g(t,y)$$

for all $t \in I_0$. This shows that the function $g(t,x)$ is nondecreasing in x for each $t \in I$. Finally,

$$u(t) = \begin{cases} \frac{t}{4}, & \text{if } \ t \in [0,1], \\ \sin t & \text{if } \ t \in [-1,1], \end{cases}$$

is a lower solution of the QFDE (3) defined on J. Thus, f satisfies the hypotheses (A_1), (A_2) and (A_4). Hence we apply Theorem 3.1.1 and conclude that the QFDE (3) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} \frac{t}{4} & \text{if } \ t \in [0,1] \\ \sin t & \text{if } \ t \in [-1,1] \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^t g(s, x_s^n)\,ds, & \text{if } \ t \in [0,1] \\ , \sin t, & \text{if } \ t \in [-1,0], \end{cases}$$

for $n = 0, 1, \ldots,$ converges monotonically to x^*.

3.2. Periodic Boundary Value Problem

we need the following definition in what follows.

A function $u \in C(J,R)$ is called a lower solution of the QFDE (4) on J if the function $t \mapsto \dfrac{u(t)}{f(t,u(t))}$ is differentiable and satisfies the inequalities

$$\left.\begin{array}{c}\dfrac{d}{dt}\left[\dfrac{u(t)}{f(t,u(t))}\right]+\lambda\left[\dfrac{u(t)}{f(t,u(t))}\right]=g(t,u_t),\ \ t\in J,\\ x(0)=\phi(0)=x(T),\\ x_0=\phi\end{array}\right\}$$

for all $t \in J$. Similarly, an upper solution $v \in C(J,R)$ of the QFDE (4) on J is defined.

We consider the following hypotheses in what follows.

(B_0) The map $x \mapsto \dfrac{x}{f(t,x)}$ is increasing for each $t \in J$.

(B_1) There exists a constant $M_f > 0$ such that $0 < f(t,x) \leq M_f$ for all $t \in J$ and $x \in \mathcal{C}$.

(B_2) There exists a $\mathcal{D}$-function φ such that

$$0 \leq f(t,x) - f(t,y) \leq \varphi(x-y)$$

for all $t \in J$ and $x, y \in R$, $x \geq y$.

(B_3) There exists a constant $M_g > 0$ such that $0 < g(t,x) \leq M_g$ for all $t \in I$ and $x \in \mathcal{C}$.

(B_4) The function $g(t,x)$ is monotone nondecreasing in x for each $t \in I$.

(B_5) The QFDE (4) has a solution $u \in C(J,R)$

The following useful lemma is obvious and may be found in Nieto and Lopez [20] the references therein.

Lemma 3.2.1. *For any $h \in L^1(J,R)$ and $\sigma \in L^1(J,R)$, x is a solution to the differential equation*

$$\left.\begin{array}{c}x'(t)+hx(t)=\sigma(t),\ \ t\in J,\\ x(0)=x(T),\end{array}\right\} \tag{17}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_h(t,s)\,\sigma(s)\,ds \tag{18}$$

where,

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)+H(T)}}{e^{H(T)}-1}, & \text{if} \quad 0 \le s \le t \le T, \\ \dfrac{e^{H(s)-H(t)}}{e^{H(T)}-1}, & \text{if} \quad 0 \le t < s \le T. \end{cases} \tag{19}$$

and $H(t) = \int_0^t h(s)\,ds$.

Notice that the Green's function G_h is continuous and nonnegative on $I \times I$ and therefore, the number

$$K_h := \max\{\,|G_h(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J,R)$. For the sake of convenience, we write $G_h(t,s) = G(t,s)$.

An application of above Lemma 3.2.1 we obtain.

Lemma 3.2.2. *Suppose that hypothesis (A_0) hold . Then a function $u \in C(J,R)$ is a solution of the PBVP (4) if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \begin{cases} \left[f(x,x(t))\right]\left(\int_0^T G(t,s)g(s,x_s)\;ds\right) & t \in I, \\ \phi(t) \;\; t \in I_0, \end{cases} \tag{20}$$

for all $t \in J$, where

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda s-\lambda t+\lambda T}}{e^{\lambda T}-1}, & \text{if} \quad 0 \le s \le t \le T, \\ \dfrac{e^{\lambda s-\lambda t}}{e^{\lambda T}-1}, & \text{if} \quad 0 \le t < s \le T. \end{cases} \tag{21}$$

Theorem 3.2.1. *Suppose that hypotheses (B_1)-(B_2) and (B_4) hold. Then the PBVP (4) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive* *approximations defined by*

$$x_{n+1}(t) = \begin{cases} \big[f(x, x_n(t))\big]\left(\displaystyle\int_0^T G(t,s)g(s,x_s)\,ds\right) & \text{if } t \in I, \\ \\ \phi(t), \quad \text{if } t \in I_0, \end{cases} \tag{22}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J, R)$. Then, in view of Lemma 3.2.2, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ are compatible in E.

Define an operator $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \begin{cases} f(x, x(t)), & \text{if } t \in I, \\ \\ 1, \quad \text{if } t \in I_0. \end{cases} \tag{23}$$

and

$$\mathcal{B}x(t) = \begin{cases} \displaystyle\int_0^T G(t,s)g(s,x_s)\,ds & \text{if } t \in I, \\ \\ \phi(t), \quad \text{if } t \in I_0, \end{cases} \tag{24}$$

From the continuity of the integral ,it follows that $\mathcal{A}$ and $\mathcal{B}$ defines the operator $\mathcal{A}, \mathcal{B} : E \to E$. Applying Lemma 3.2.2, the PBVP (4) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) = x(t), \ t \in J.$$

Now, we show that the operators $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 in a series of following steps.

Step I:$\mathcal{A}$ *and* $\mathcal{A}$ are nondecreasing on E.

Let $x, y \in E$ be such that $x \geq y$. Then $x_t \geq_{\mathcal{C}} y_t$ for all $t \in I$ and by hypothesis (B_2), we obtain

$$\mathcal{A}x(t) = \begin{cases} f(x, x(t)), & \text{if } t \in I, \\ \\ 1, \quad \text{if } t \in I_0. \end{cases}$$

$$= \begin{cases} f(y, y(t)), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0. \end{cases}$$

$$= \mathcal{A}y(t),$$

for all $t \in J$. This shows that the operator that the operator $\mathcal{A}$ is nondecreasing on E. similarly,

$$\mathcal{B}x(t) = \begin{cases} \int_0^T G(t,s)g(s,x_s)\ ds & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

$$= \begin{cases} \int_0^T G(t,s)g(s,y_s)\ ds & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

$$= \mathcal{B}y(t),$$

for all $t \in I$. This shows that the operator that the operator $\mathcal{B}$ is also nondecreasing on E.

Step III: *$\mathcal{B}$ is partially continuous on E.*

Let $\{x_n\}_{n \in N}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the g is continuous, we have

$$\lim_{n\to\infty} \mathcal{B}x_n(t) = \begin{cases} \int_0^T G(t,s) \left[\lim_{n\to\infty} g\big(s, x_s^n\big) \right] ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \begin{cases} \int_0^T G(t,s)g(s,x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \mathcal{B}x(t),$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now we show that $\{\mathcal{B}x_n\}_{n\in N}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &= \left| \int_0^T G(t_2,s)g\big(s,x_s^n\big)\,ds - \int_0^T G(t_1,s)g\big(s,x_s^n\big)\,ds \right| \\
&= \left| \int_0^T |G(t_2,s) - G(t_1,s)|\,\big|g\big(s,x_s^n\big)\big|\,ds \right| \\
&\leq M_g \int_0^T |G(t_2,s) - G(t_1,s)|\,ds \\
&\to 0 \quad \text{as} \quad t_2 \to t_1,
\end{aligned}$$

uniformly for all $n \in N$.

Case II: Let $t_1, t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in N$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &\leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(0)| + |\mathcal{B}x_n(0) - \mathcal{B}x_n(t_1)| \\
&\to 0 \quad \text{as} \quad t_2 \to t_1.
\end{aligned}$$

Thus in all three cases, we obtain

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in N$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and that $\mathcal{B}$ is a partially continuous operator on E into itself.

Step IV: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (B_2)

$$\begin{aligned}
|y(t)| &= |\mathcal{B}x(t)| \\
&\leq \begin{cases} \displaystyle\int_0^T G(t,s)|g(s,x_s)|\,ds, & \text{if } \ t \in I, \\ |\phi(t)|, & \text{if } \ t \in I_0. \end{cases}
\end{aligned}$$

$$\leq \|\phi\| + KM_gT = r,$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next we show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B} : E \to E$ is a partially compact operator on E into itself.

Step V: *u satisfies the inequality $u \leq \mathcal{A}u\mathcal{B}u$.*

By hypothesis (B_4), the PBVP (4) has a lower solution u defined on J. Then we have

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{u(t)}{f(t,u(t))}\right] + \lambda\left[\dfrac{u(t)}{f(t,u(t))}\right] = g(t,u_t), \;\; t \in J, \\ u(0) = \phi(0) = u(T), \\ u_0 = \phi. \end{cases} \tag{$*$}$$

Now an application of Lemma 3.2.1, we obtain the inequality

$$u(t) \leq \begin{cases} [f(t,u(t))]\left(\displaystyle\int_0^T G(t,s) f(s,u_s)\,ds,\right) & \text{if } \; t \in I, \\ \phi(t), \quad \text{if } \; t \in I_0. \end{cases}$$

for all $t \in J$. From the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that $u(t) \leq \mathcal{A}u(t)\mathcal{B}u(t)$. Hence $u \leq \mathcal{A}u\mathcal{B}u$

Step VI: *$\mathcal{D}$ function ϕ satisfies the growth condition $M\phi(r) \leq r, r > 0$.*

Finally, $\mathcal{D}$ function ϕ of the operator $\mathcal{A}$ and $\mathcal{B}$ satisfies the inequality given in hypothesis (d) of Theorem 2.1 , it follows that $M\varphi_{\mathcal{A}}(r) \leq KM_gT\phi(r) < r, r > 0$. Thus $\mathcal{A}$ and $\mathcal{B}$ satisfies all the conditions of Theorem 2.1 and so the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution. Consequently the integral equation and the differential equation (4) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (22) converges monotonically to x^*. This completes the proof. □

Example 3.2.1. Given the closed and bounded intervals $I_0 = [-1, 0]$ and $I = [0, 1]$ and given a function $\phi \in \mathcal{C}(I_0, R)$, consider the PBVP

$$\left.\begin{array}{c} \dfrac{d}{dt}\left[\dfrac{x(t)}{f(t,x(t))}\right] + \left[\dfrac{x(t)}{f(t,x(t))}\right] = g(t, x_t), \ \ t \in J, \\ x(0) = \phi(0) = x(T), \\ x_0 = \phi \end{array}\right\} \tag{25}$$

where $\phi \in \mathcal{C}$, and $f : I \times R \to R$ and $g : I \times \mathcal{C} \to R$ are continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in [-1, 0],$$

$$f(t,x) = \begin{cases} 1 + \|x\|, & \text{if } \ x \geq_{\mathcal{C}} 0, \\ \\ 1, & \text{if } \ x \leq_{\mathcal{C}} 0, \end{cases}$$

and

$$g(t,x) = \begin{cases} \tanh(\|x\|_{\mathcal{C}}) + 1, & \text{if } \ x \geq_{\mathcal{C}} 0, \\ \phi(t) & \text{if } \ x \leq_{\mathcal{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly the functions f and g satisfy the hypotheses (B_1) and (B_4) with $M_f = 2 = M_g$. Again, it can be verified that

$$u(t) = \begin{cases} 2\displaystyle\int_0^T G(t,s)\,ds, & \text{if } \ t \in I, \\ \sin t, \ \ \text{if } \ t \in I_0, \end{cases}$$

has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0(t) = \begin{cases} 2\displaystyle\int_0^T G(t,s)\,ds, & \text{if } \ t \in [0,1], \\ \sin t, \ \ \text{if } \ t \in [-1,0], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^T G(t,s) g(s, x_s^n)\,ds, & \text{if } \ t \in [0,1], \\ \\ \sin t, & \text{if } \ t \in [-1,0], \end{cases}$$

converges monotonically to x^*, where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in [-1, 0]$.

References

[1] B.C. Dhage, Periodic boundary value problems of first order Carathéodory and discontinuous differential equations, *Nonlinear Funct. Anal. & Appl.* **13**(2) (2008), 323-352.

[2] B.C. Dhage, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, *Differ. Equ. Appl.* **2** (2010), 465–486.

[3] B.C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ Appl.* **5** (2013), 155-184.

[4] B.C. Dhage, Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.* **45** (2014), 397-426.

[5] B.C. Dhage, Approximation methods in the theory of hybrid differential equations with linear perturbations of second type, *Tamkang J. Math.* **45** (2014), 39-61.

[6] B.C. Dhage, Nonlinear $\mathcal{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations, *Malaya J. Mat.* **3**(1) (2015), 62-85.

[7] B. C. Dhage, S. B. Dhage, D. V. Mule, Local attractivity and stability results for hybrid functional nonlinear fractional integral equations, Nonlinear Funct. Anal. Appl. **19** (2014), 415-433.

[8] Dnyaneshwar V. Mule, Bhimrao R. Ahirrao , *Approximating positive solutions of nonlinear first order ordinary quadratic differential equations with maxima* , Advances in Inequalities and Applications, Vol 2016 (2016), Article ID 11.

[9] B.C. Dhage, *Some generalizations of a hybrid fixed point theorem in a partially ordered metric space and nonlinear functional integral equations*, Differ. Equ Appl. **8** (2016), 77-97.

[10] Dnyaneshwar V. Mule, Bhimrao R. Ahirrao , *Approximating positive solutions of quadratic functional integral equations*, Adv. Fixed Point Theory, 6 (2016), No. 3, 295-307.

[11] B.C. Dhage, V. Lakshmikantham, Basic results on hybrid differential equations, *Nonlinear Analysis: Hybrid Systems* **4** (2010), 414-424.

[12] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer Verlag, 2003.

[13] J.K. Hale,Theory of Functional Differential Equations, Springer-Verlag, New York-Berlin, 1977.

[14] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[15] J. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equa- tions*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[16] V. Kolmanovskii, and A. Myshkis, em Introduction to the Theory and Applications of Functional-Differential Equations , Mathematics and its Applications, 463, Kluwer Academic Publishers, Dordrecht, 1999.

[17] J.J. Nieto, Basic theory for nonresonance impulsive periodic problems of first order, *J. Math. Anal. Appl.* **205** (1997), 423–433.

[18] S. B. Dhage, B. C. Dhage, *Dhage iteration method for approximating positive solutions of nonlinear first order ordinary quadratic differential equations with maxima*, Nonlinear Anal. Forum **16**(1) (2016), 87-100.

[19] E. Zeidler, *Nonlinear Functional Analysis and Its Applications : Part.I*, Springer-Verlag, New York (1985).

[20] J.J. Nieto, R. Rodriguez-Lopez, Existence and application of solution for nonlinear differential equation with peridic boundry conditions,Compt.Math.Appl. *Order* **40** (2000),435-442. 223-239.

[21] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge (1990).

[22] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, In: *Operator Theory: Advances and Applications*, Birkhäuser, Basel (1994), 369-396.

[23] D.S. Moak, The q-analogue of the Laguerre polynomials, *J. Math. Anal. Appl.*, **81** (1981), 20-47.