International Journal of Pure and Applied Mathematics
Volume 113 No. 2 2017, 283-289
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i2.8
ijpam.eu

ON A GENERALIZATION OF SUPPLEMENT SUBMODULES

Celil Nebiyev
Department of Mathematics
Ondokuz Mayımaths University
55270, Kurupelit, Atakum, Samsun, TURKEY

Abstract: In this work, some properties of g-supplement submodules are investigated. Let V be a g-supplement of an essential submodule U in M. Then it is possible to define a bijective map between essential maximal submodules of V and essential maximal submodules of M which contain U. It is also proved that $Rad_g V = V \cap Rad_g M$.

AMS Subject Classification: 16D10, 16D70.
Key Words: g-Small Submodules, Radical, Supplement, Submodules, g-Supplemented Modules.

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a small submodule of M and denoted by $N \ll M$. Let M be an R -module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, N is called a direct summand of M and it is denoted by $M = N \oplus K$. For any module M, we have $M = M \oplus 0$. $RadM$ indicates the radical of M. A submodule N of an R -module M is called an essential submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$.

Received: November 4, 2016
Revised: December 20, 2016
Published: March 19, 2017

The intersection of all essential maximal submodules of an $R-$module M is called the generalized radical of M, denoted by $Rad_g M$ (in [8], it is denoted by $Rad_e M$). If M has not any essential maximal submodule, then we denote $Rad_g M = M$. Let M be an R -module and K be a submodule of M. K is called a generalized small submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [8], it is called an e-small submodule of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let M be an R -module. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V << V$, then V is called a supplement of U in M. M is called a supplemented module if every submodule of M has a supplement in M. Supplemented modules are in [7].

There are some important properties of g-small submodules in [2], [5], [6] and [8].

Lemma 1.1. *Let M be an R -module and K, $N \le M$. Consider the following conditions.* [6, 8]

(1) *If $K \le N$ and N is a generalized small submodule of M, then K is a generalized small submodule of M.*

(2) *If K is contained in N and a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contains N.*

(3) *Let $f : M \to N$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g N$.*

(4) *If $K \ll_g L$ and $N \ll_g T$ for $L, T \le M$, then $K + N \ll_g L + T$.*

Corollary 1.2. *Let M be an R -module and $K \le N \le M$. If $N \ll_g M$, then $N/K \ll_g M/K$.*

Corollary 1.3. *Let M be an R -module, $K \ll_g M$ and $L \le M$. Then $(K + L)/L \ll_g M/L$.*

Lemma 1.4. *Let M be an R -module. Then $Rad_g M = \sum_{L <<_g M} L$.* [2]

Corollary 1.5. *Let M be an R -module and $x \in Rad_g M$. Then $Rx \ll_g M$.* [2]

Definition 1.6. *Let M be an R -module and $U, V \le M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, then V is called a g-supplement of U in M. If every submodule of M has a g-supplement in M, then M is called a g-supplemented module.* [2]

Supplemented modules are g-supplemented.

Definition 1.7. *Let M be an R-module and $V \leq M$. If V is a g-supplement of any submodule of M, then V is called a g-supplement submodule in M.*

Clearly we see that every supplement submodule is g-supplement.

2. Some Properties of G-Supplement Submodules

Lemma 2.1. *Let M be an $R-$module, $U \leq M$ and $V \leq M$. Then V is a g-supplement of U in M if and only if $M = U + V$ and $U \cap V \ll_g V$. (See [2])*

Proposition 2.2. *Let M be an $R-$module, $V \leq M$, U be an essential maximal submodule of M and V be a g-supplement of U in M. Then $U \cap V$ is the unique essential maximal submodule of V. In this case $U \cap V = Rad_g V$.*

Proof. Since $U+V = M$ and U is a maximal submodule of M, then $V \nleq U$. Then by [4] Lemma 2.8, $U \cap V$ is a maximal submodule of V. Since $U \trianglelefteq M$, we clearly see that $U \cap V \trianglelefteq V$. Then $Rad_g V \leq U \cap V$. Since V is a g-supplement of U, $U \cap V \ll_g V$. Then by Lemma 1.4, $U \cap V \leq Rad_g V$. Hence $Rad_g V = U \cap V$ and we clearly see that $U \cap V$ is the unique essential maximal submodule of V. □

Lemma 2.3. *Let M be an $R-$module, $K \leq V \leq M$ and V be a g-supplement of an essential submodule U of M. Then $K \ll_g V$ if and only if $K \ll_g M$.*

Proof. ($\Longrightarrow$) Clear from Lemma 1.1.
($\Longleftarrow$) Let $T \trianglelefteq V$ and $K + T = V$. Since $U + V = M$, $U + K + T = M$. Since $U \trianglelefteq M$, then $(U + T) \trianglelefteq M$. Then by $K \ll_g M$, $U + T = M$. Since V is a g-supplement of U in M, $T = V$. Thus $K \ll_g V$. □

Theorem 2.4. *Let M be an $R-$module, $V \leq M$ and V be a g-supplement of an essential submodule of M. Then $Rad_g V = V \cap Rad_g M$.*

Proof. By Lemma 1.1 and Lemma 1.4, we clearly see that $Rad_g V \leq V \cap Rad_g M$. Let $x \in V \cap Rad_g M$. Then $x \in V$ and $x \in Rad_g M$. Since $x \in Rad_g M$, by Corollary 1.5, $Rx \ll_g M$. Then by Lemma 2.3, $Rx \ll_g V$ and $x \in Rad_g V$. Hence $V \cap Rad_g M \leq Rad_g V$ and since $Rad_g V \leq V \cap Rad_g M$, $Rad_g V = V \cap Rad_g M$. □

Lemma 2.5. *Let V be a g-supplement of U in M, $T \leq V$ and $K \trianglelefteq V$.Then T is a g-supplement of K in V if and only if T is a g-supplement of $U+K$ in M.*

Proof. ($\Longrightarrow$) Let T be a g-supplement of K in V. Then $V = K + T$ and since $M = U + V$, $M = U + K + T$. Let $M = U + K + L$ with $L \trianglelefteq T$. Since $K \trianglelefteq V$ and $L \leq V$, $(K + L) \trianglelefteq V$. Then by V being a g-supplement of U in M, $K + L = V$. Since $L \trianglelefteq T$ and T is a g-supplement of K in V, $L = T$. Hence T is a g-supplement of $U + K$ in M.

($\Longleftarrow$) Let T be a g-supplement of $U + K$ in M. Then $M = U + K + T$. Since $K \trianglelefteq V$ and $L \leq V$, $(K + T) \trianglelefteq V$. Then by V being a g-supplement of U in M, $K + T = V$. Let $K + L = V$ with $L \trianglelefteq T$. Then by $M = U + V$, $M = U + K + L$. Since $L \trianglelefteq T$ and T is a g-supplement of $U + K$ in M, $L = T$. Hence T is a g-supplement of K in V. □

Corollary 2.6. *Let V be a supplement of U in M, $T \leq V$ and $K \trianglelefteq V$.Then T is a g-supplement of K in V if and only if T is a g-supplement of $U+K$ in M.*

Corollary 2.7. *Let $M = U \oplus V$, $T \leq V$ and $K \trianglelefteq V$.Then T is a g-supplement of K in V if and only if T is a g-supplement of $U+K$ in M.*

Lemma 2.8. *Let U and V be mutual g-supplements in M, $S \trianglelefteq U$, $K \trianglelefteq V$, L be a g-supplement of S in U and T be a g-supplement of K in V. Then $L+T$ is a g-supplement of $K + S$ in M.*

Proof. Since $U = S + L$ and $V = K + T$, $M = U + V = S + L + K + T = K + S + L + T$. Since V is a g-supplement of U in M, $K \trianglelefteq V$ and T is a g-supplement of K in V, then by Lemma 2.5, T is a g-supplement of $U + K$ in M. Then $(U + K) \cap T \ll_g T$. Similarly, we can prove that $(V + S) \cap L \ll_g L$. Hence $(K + S) \cap (L + T) \leq (K + S + L) \cap T + (K + S + T) \cap L = (U + K) \cap T + (V + S) \cap L \ll_g T + L = L + T$ and $L + T$ is a g-supplement of $K + S$ in M. □

Corollary 2.9. *Let U and V be mutual supplements in M, $S \trianglelefteq U$, $K \trianglelefteq V$, L be a g-supplement of S in U and T be a g-supplement of K in V. Then $L + T$ is a g-supplement of $K + S$ in M.*

Corollary 2.10. *$M = U \oplus V$, $S \trianglelefteq U$, $K \trianglelefteq V$, L be a g-supplement of S in U and T be a g-supplement of K in V. Then $L + T$ is a g-supplement of $K + S$ in M.*

Lemma 2.11. *Let V be a g-supplement of U in M, $U \trianglelefteq M$ and K be an essential maximal submodule of M. Then $U+K$ is an essential maximal submodule of M. In this case $K = (U+K) \cap V$.*

Proof. Since V is a g-supplement of U in M, $U \cap V \ll_g V$. Then, by K being an essential maximal submodule of V, $U \cap V \leq K$. Hence, by Modular law, $K = U \cap V + K = (U+K) \cap V$. Since $U \trianglelefteq M$, $(U+K) \trianglelefteq M$. Since $\frac{M}{U+K} = \frac{U+V}{U+K} \cong \frac{V}{V \cap (U+K)} = \frac{V}{U \cap V + K} = \frac{V}{K}$ and K is a maximal submodule of V, $U+K$ is a maximal submodule of M. Hence $U+K$ is an essential maximal submodule of M. □

Proposition 2.12. *Let M be an $R-$module, $V \leq M$, $U \trianglelefteq M$ and V be a g-supplement of U in M. Then it is possible to define a bijective map between essential maximal submodules of V and essential maximal submodules of M which contain U.*

Proof. Let $\Gamma = \{K \mid U \leq K, K \textit{ is essential maximal in } M\}$, $\Lambda = \{T \mid T \textit{ is essential maximal in } V\}$. We can define a map $f : \Gamma \to \Lambda$, $K \to f(K) = K \cap V$. Let $K \in \Gamma$. Since $U \leq K$ and K is maximal in M, $V \not\subset K$ and then by [4] Lemma 2.12 $K \cap V$ is a maximal submodule of V. Since $K \trianglelefteq M$, $K \cap V \trianglelefteq V$. That is, f is a function.

Let $T \in \Lambda$. Since T is essential maximal in V, then by Lemma 2.11, $U+T \in \Gamma$ and $f(U+T) = (U+T) \cap V = T$. Thus f is surjective.

Let $f(K) = f(L)$ for $K, L \in \Gamma$. Then $K \cap V = L \cap V$. Since $U \leq K$ and $U \leq L$, then by Modular law $K = M \cap K = (U+V) \cap K = U + V \cap K = U + V \cap L = (U+V) \cap L = M \cap L = L$.

Hence f is bijective. □

Definition 2.13. *Let M be an $R-$module and $U, V \leq M$. If $U+V = M$ and $U \cap V \ll_g M$, then V is called a weak g-supplement of U in M. M is called weakly g-supplemented if every submodule of M has a weak g-supplement in M. (See also* [3])

Definition 2.14. *A submodule L of M is* ***said to g-lie above*** *a submodule N in M if $N \leq L$ and $T = M$ for every $T \leq M$ such that $N \leq T \trianglelefteq M$ and $\frac{L}{N} + \frac{T}{N} = \frac{M}{N}$.*

Lemma 2.15. *Let M be an $R-$module and $N \leq L \leq M$. Then L g-lies above N in M if and only if $N+T = M$ for every $T \trianglelefteq M$ such that $L+T = M$.*

Proof. ($\Longrightarrow$) Let $L+T=M$ with $T \trianglelefteq M$. Then $\frac{L}{N}+\frac{N+T}{N}=\frac{M}{N}$ and since L g-lies above N and $(N+T) \trianglelefteq M$, $N+T=M$.

($\Longleftarrow$) $\frac{L}{N}+\frac{T}{N}=\frac{M}{N}$ with $N \leq T \trianglelefteq M$. Then $L+T=M$ and by hypothesis, $N+T=M$. Since $N \leq T$, $T=M$. Hence L g-lies above N in M. □

Lemma 2.16. *Let $M=U+V$ and $M=T+U\cap V$. Then $M=U+T\cap V=V+T\cap U$.*

Proof. See [1, Lemma 1.24]. □

Theorem 2.17. *Let $L \leq M$, $N \leq L$ and L g-lie above N in M. If L and N have essential weak g-supplements in M, then they have the same essential weak g-supplements in M.*

Proof. Let X be an essential weak g-supplement of L in M. Then $L+X=M$ and by Lemma 2.15, $N+X=M$. Since X is a weak g-supplement of L in M and $N \leq L$, $N\cap X \leq L\cap X \ll_g M$. Thus X is an essential weak g-supplement of N in M.

Let T be an essential weak g-supplement of N in M. Then $N+T=M$ and by $N \leq L$, $L+T=M$. Let $L\cap T+S=M$ with $S \trianglelefteq M$. Then by Lemma 2.16, $L+T\cap S=M$ and by Lemma 2.15, $N+T\cap S=M$. By also Lemma 2.16, $N\cap T+S=M$ and because of $L\cap T \ll_g M$ and $S \trianglelefteq M$, $S=M$. Thus $L\cap T \ll_g M$ and T is an essential weak g-supplement of L in M. □

Theorem 2.18. *Let $L \leq M$, $N \leq L$ and L g-lie above N in M. If L and N have essential g-supplements in M, then they have the same essential g-supplements in M.*

Proof. Let X be an essential g-supplement of L in M. Then $L+X=M$ and by Lemma 2.15, $N+X=M$. Since X is a g-supplement of L in M and $N \leq L$, $N\cap X \leq L\cap X \ll_g X$. Thus X is an essential g-supplement of N in M .

Let T be an essential g-supplement of N in M. Then $N+T=M$ and by $N \leq L$, $L+T=M$. Let $L+S=M$ with $S \trianglelefteq T$. Since $S \trianglelefteq T$ and $T \trianglelefteq M$, $S \trianglelefteq M$. Then by Lemma 2.15, $N+S=M$ and since T is a g-supplement of N in M, $S=T$. Hence T is an essential g-supplement of L in M. □

Theorem 2.19. *Let M be an $R-$module, V be a weak g-supplement of U in M and $L \leq U$. If $M=V+L$, then U g-lies above L in M.*

Proof. Let $U + T = M$ with $T \trianglelefteq M$. Since V is a weak g-supplement of U in M, $M = U + V$ and $U \cap V \ll_g M$. Since $M = V + L$ and $L \leq U$, by Modular law, $U = U \cap V + L$. Then $M = U + T = U \cap V + L + T$ and since $U \cap V \ll_g M$ and $(L + T) \trianglelefteq M$, $L + T = M$. Hence by Lemma 2.15, U g-lies above L in M. □

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