ON A GENERALIZATION OF SUPPLEMENT SUBMODULES

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Abstract: In this work, some properties of g-supplement submodules are investigated. Let $V$ be a g-supplement of an essential submodule $U$ in $M$. Then it is possible to define a bijective map between essential maximal submodules of $V$ and essential maximal submodules of $M$ which contain $U$. It is also proved that $\text{Rad}_g V = V \cap \text{Rad}_g M$.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let $R$ be a ring and $M$ be an $R$–module. We will denote a submodule $N$ of $M$ by $N \leq M$. Let $M$ be an $R$-module and $N \leq M$. If $L = M$ for every submodule $L$ of $M$ such that $M = N + L$, then $N$ is called a small submodule of $M$ and denoted by $N \ll M$. Let $M$ be an $R$-module and $N \leq M$. If there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K = 0$, $N$ is called a direct summand of $M$ and it is denoted by $M = N \oplus K$. For any module $M$, we have $M = M \oplus 0$. $\text{Rad}M$ indicates the radical of $M$. A submodule $N$ of an $R$-module $M$ is called an essential submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. 
The intersection of all essential maximal submodules of an \( R \)-module \( M \) is called the generalized radical of \( M \), denoted by \( \text{Rad}_g M \) (in [8], it is denoted by \( \text{Rad}_e M \)). If \( M \) has not any essential maximal submodule, then we denote \( \text{Rad}_g M = M \). Let \( M \) be an \( R \)-module and \( K \) be a submodule of \( M \). \( K \) is called a generalized small submodule of \( M \) if for every essential submodule \( T \) of \( M \) with the property \( M = K + T \) implies that \( T = M \), then we write \( K \ll_g M \) (in [8], it is called an e-small submodule of \( M \) and denoted by \( K \ll_e M \)). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let \( M \) be an \( R \)-module and \( K \) be a submodule of \( M \). \( K \) is called a generalized small submodule of \( M \) if for every essential submodule \( T \) of \( M \) with the property \( M = K + T \) implies that \( T = M \), then we write \( K \ll_g M \) (in [8], it is called an e-small submodule of \( M \) and denoted by \( K \ll_e M \)).

There are some important properties of g-small submodules in [2], [5], [6] and [8].

**Lemma 1.1.** Let \( M \) be an \( R \)-module and \( K, N \leq M \). Consider the following conditions. [6, 8]

1. If \( K \leq N \) and \( N \) is a generalized small submodule of \( M \), then \( K \) is a generalized small submodule of \( M \).
2. If \( K \) is contained in \( N \) and a generalized small submodule of \( N \), then \( K \) is a generalized small submodule in submodules of \( M \) which contains \( N \).
3. Let \( f : M \to N \) be an \( R \)-module homomorphism. If \( K \ll_g M \), then \( f(K) \ll_g N \).
4. If \( K \ll_g L \) and \( N \ll_g T \) for \( L, T \leq M \), then \( K + N \ll_g L + T \).

**Corollary 1.2.** Let \( M \) be an \( R \)-module and \( K \leq N \leq M \). If \( N \ll_g M \), then \( N/K \ll_g M/K \).

**Corollary 1.3.** Let \( M \) be an \( R \)-module, \( K \ll_g M \) and \( L \leq M \). Then \( (K + L)/L \ll_g M/L \).

**Lemma 1.4.** Let \( M \) be an \( R \)-module. Then \( \text{Rad}_g M = \sum_{L \ll_g M} L \). [2]

**Corollary 1.5.** Let \( M \) be an \( R \)-module and \( x \in \text{Rad}_g M \). Then \( Rx \ll_g M \). [2]

**Definition 1.6.** Let \( M \) be an \( R \)-module and \( U, V \leq M \). If \( M = U + V \) and \( M = U + T \) with \( T \leq V \) implies that \( T = V \), then \( V \) is called a g-supplement of \( U \) in \( M \). If every submodule of \( M \) has a g-supplement in \( M \), then \( M \) is called a g-supplemented module. [2]

Supplemented modules are g-supplemented.
Definition 1.7. Let $M$ be an $R$-module and $V \leq M$. If $V$ is a $g$-supplement of any submodule of $M$, then $V$ is called a $g$-supplement submodule in $M$.

Clearly we see that every supplement submodule is $g$-supplement.

2. Some Properties of $G$-Supplement Submodules

Lemma 2.1. Let $M$ be an $R$-module, $U \leq M$ and $V \leq M$. Then $V$ is a $g$-supplement of $U$ in $M$ if and only if $M = U + V$ and $U \cap V \ll_g V$. (See [2])

Proposition 2.2. Let $M$ be an $R$-module, $V \leq M$, $U$ be an essential maximal submodule of $M$ and $V$ be a $g$-supplement of $U$ in $M$. Then $U \cap V$ is the unique essential maximal submodule of $V$. In this case $U \cap V = \text{Rad}_g V$.

Proof. Since $U + V = M$ and $U$ is a maximal submodule of $M$, then $V \not\leq U$. Then by [4] Lemma 2.8, $U \cap V$ is a maximal submodule of $V$. Since $U \leq M$, we clearly see that $U \cap V \leq V$. Then $\text{Rad}_g V \leq U \cap V$. Since $V$ is a $g$-supplement of $U$, $U \cap V \ll_g V$. Then by Lemma 1.4, $U \cap V \leq \text{Rad}_g V$. Hence $\text{Rad}_g V = U \cap V$ and we clearly see that $U \cap V$ is the unique essential maximal submodule of $V$. □

Lemma 2.3. Let $M$ be an $R$-module, $K \leq V \leq M$ and $V$ be a $g$-supplement of an essential submodule $U$ of $M$. Then $K \ll_g V$ if and only if $K \ll_g M$.

Proof. ($\Rightarrow$) Clear from Lemma 1.1.

($\Leftarrow$) Let $T \leq V$ and $K + T = V$. Since $U + V = M$, $U + K + T = M$. Since $U \leq M$, then $(U + T) \leq M$. Then by $K \ll_g M$, $U + T = M$. Since $V$ is a $g$-supplement of $U$ in $M$, $T = V$. Thus $K \ll_g V$. □

Theorem 2.4. Let $M$ be an $R$-module, $V \leq M$ and $V$ be a $g$-supplement of an essential submodule of $M$. Then $\text{Rad}_g V = V \cap \text{Rad}_g M$.

Proof. By Lemma 1.1 and Lemma 1.4, we clearly see that $\text{Rad}_g V \leq V \cap \text{Rad}_g M$. Let $x \in V \cap \text{Rad}_g M$. Then $x \in V$ and $x \in \text{Rad}_g M$. Since $x \in \text{Rad}_g M$, by Corollary 1.5, $Rx \ll_g M$. Then by Lemma 2.3, $Rx \ll_g V$ and $x \in \text{Rad}_g V$. Hence $V \cap \text{Rad}_g M \leq \text{Rad}_g V$ and since $\text{Rad}_g V \leq V \cap \text{Rad}_g M$, $\text{Rad}_g V = V \cap \text{Rad}_g M$. □
Lemma 2.5. Let $V$ be a $g$-supplement of $U$ in $M$, $T \leq V$ and $K \subseteq V$. Then $T$ is a $g$-supplement of $K$ in $V$ if and only if $T$ is a $g$-supplement of $U + K$ in $M$.

Proof. ($\Rightarrow$) Let $T$ be a $g$-supplement of $K$ in $V$. Then $V = K + T$ and since $M = U + V$, $M = U + K + T$. Let $M = U + K + L$ with $L \leq T$. Since $K \subseteq V$ and $L \leq V$, $(K + L) \leq V$. Then by $V$ being a $g$-supplement of $U$ in $M$, $K + L = V$. Since $L \leq T$ and $T$ is a $g$-supplement of $K$ in $V$, $L = T$. Hence $T$ is a $g$-supplement of $U + K$ in $M$.

($\Leftarrow$) Let $T$ be a $g$-supplement of $U + K$ in $M$. Then $M = U + K + T$. Since $K \subseteq V$ and $L \leq V$, $(K + T) \leq V$. Then by $V$ being a $g$-supplement of $U$ in $M$, $K + T = V$. Let $K + L = V$ with $L \leq T$. Then by $M = U + V$, $M = U + K + L$. Since $L \leq T$ and $T$ is a $g$-supplement of $U + K$ in $M$, $L = T$. Hence $T$ is a $g$-supplement of $K$ in $V$. □

Corollary 2.6. Let $V$ be a supplement of $U$ in $M$, $T \leq V$ and $K \subseteq V$. Then $T$ is a $g$-supplement of $K$ in $V$ if and only if $T$ is a $g$-supplement of $U + K$ in $M$.

Corollary 2.7. Let $M = U \oplus V$, $T \leq V$ and $K \subseteq V$. Then $T$ is a $g$-supplement of $K$ in $V$ if and only if $T$ is a $g$-supplement of $U + K$ in $M$.

Lemma 2.8. Let $U$ and $V$ be mutual $g$-supplements in $M$, $S \subseteq U$, $K \subseteq V$, $L$ be a $g$-supplement of $S$ in $U$ and $T$ be a $g$-supplement of $K$ in $V$. Then $L + T$ is a $g$-supplement of $K + S$ in $M$.

Proof. Since $U = S + L$ and $V = K + T$, $M = U + V = S + L + K + T = K + S + L + T$. Since $V$ is a $g$-supplement of $U$ in $M$, $K \subseteq V$ and $T$ is a $g$-supplement of $K$ in $V$, then by Lemma 2.5, $T$ is a $g$-supplement of $U + K$ in $M$. Then $(U + K) \cap T \leq_g T$. Similarly, we can prove that $(V + S) \cap L \leq_g L$. Hence $(K + S) \cap (L + T) \leq (K + S + L) \cap T + (K + S + T) \cap L = (U + K) \cap T + (V + S) \cap L \leq_g T + L = L + T$ and $L + T$ is a $g$-supplement of $K + S$ in $M$. □

Corollary 2.9. Let $U$ and $V$ be mutual supplements in $M$, $S \subseteq U$, $K \subseteq V$, $L$ be a $g$-supplement of $S$ in $U$ and $T$ be a $g$-supplement of $K$ in $V$. Then $L + T$ is a $g$-supplement of $K + S$ in $M$.

Corollary 2.10. $M = U \oplus V$, $S \subseteq U$, $K \subseteq V$, $L$ be a $g$-supplement of $S$ in $U$ and $T$ be a $g$-supplement of $K$ in $V$ Then $L + T$ is a $g$-supplement of $K + S$ in $M$. 
Lemma 2.11. Let $V$ be a $g$-supplement of $U$ in $M$, $U \triangleleft M$ and $K$ be an essential maximal submodule of $M$. Then $U + K$ is an essential maximal submodule of $M$. In this case $K = (U + K) \cap V$.

Proof. Since $V$ is a $g$-supplement of $U$ in $M$, $U \cap V \ll_g V$. Then, by $K$ being an essential maximal submodule of $V$, $U \cap V \leq K$. Hence, by Modular law, $K = U \cap V + K = (U + K) \cap V$. Since $U \triangleleft M$, $(U + K) \triangleleft M$. Since $M = U + V \Rightarrow V \cap (U + K) = \frac{V}{U + V + K} = \frac{V}{K}$ and $K$ is a maximal submodule of $V$, $U + K$ is a maximal submodule of $M$. Hence $U + K$ is an essential maximal submodule of $M$. □

Proposition 2.12. Let $M$ be an $R$–module, $V \leq M$, $U \triangleleft M$ and $V$ be a $g$-supplement of $U$ in $M$. Then it is possible to define a bijective map between essential maximal submodules of $V$ and essential maximal submodules of $M$ which contain $U$.

Proof. Let $\Gamma = \{ K \mid U \leq K, K$ is essential maximal in $M \}$, $\Lambda = \{ T \mid T$ is essential maximal in $V \}$. We can define a map $f : \Gamma \rightarrow \Lambda$, $K \rightarrow f(K) = K \cap V$. Let $K \in \Gamma$. Since $U \leq K$ and $K$ is maximal in $M$, $V \nsubseteq K$ and then by Lemma 2.12 $K \cap V$ is a maximal submodule of $V$. Since $K \triangleleft M$, $K \cap V \leq V$. That is, $f$ is a function.

Let $T \in \Lambda$. Since $T$ is essential maximal in $V$, then by Lemma 2.11, $U + T \in \Gamma$ and $f(U + T) = (U + T) \cap V = T$. Thus $f$ is surjective.

Let $f(K) = f(L)$ for $K, L \in \Gamma$. Then $K \cap V = L \cap V$. Since $U \leq K$ and $U \leq L$, then by Modular law $K = M \cap K = (U + V) \cap K = U + V \cap K = U + V \cap L = (U + V) \cap L = M \cap L = L$.

Hence $f$ is bijective. □

Definition 2.13. Let $M$ be an $R$–module and $U, V \leq M$. If $U + V = M$ and $U \cap V \ll g M$, then $V$ is called a weak $g$-supplement of $U$ in $M$. $M$ is called weakly $g$-supplemented if every submodule of $M$ has a weak $g$-supplement in $M$. (See also [3])

Definition 2.14. A submodule $L$ of $M$ is said to g-lie above a submodule $N$ in $M$ if $N \leq L$ and $T = M$ for every $T \leq M$ such that $N \leq T \leq M$ and $\frac{L}{N} + \frac{T}{N} = \frac{M}{N}$.

Lemma 2.15. Let $M$ be an $R$–module and $N \leq L \leq M$. Then $L$ g-lies above $N$ in $M$ if and only if $N + T = M$ for every $T \leq M$ such that $L + T = M$. 

Proof. ($\implies$) Let $L + T = M$ with $T \leq M$. Then $\frac{L}{N} + \frac{N + T}{N} = \frac{M}{N}$ and since $L$ g-lies above $N$ and $(N + T) \leq M$, $N + T = M$.

($\impliedby$) $\frac{L}{N} + \frac{T}{N} = \frac{M}{N}$ with $N \leq T \leq M$. Then $L + T = M$ and by hypothesis, $N + T = M$. Since $N \leq T$, $T = M$. Hence $L$ g-lies above $N$ in $M$.

Lemma 2.16. Let $M = U + V$ and $M = T + U \cap V$. Then $M = U + T \cap V = V + T \cap U$.

Proof. See [1, Lemma 1.24].

Theorem 2.17. Let $L \leq M$, $N \leq L$ and $L$ g-lie above $N$ in $M$. If $L$ and $N$ have essential weak $g$-supplements in $M$, then they have the same essential weak $g$-supplements in $M$.

Proof. Let $X$ be an essential weak $g$-supplement of $L$ in $M$. Then $L + X = M$ and by Lemma 2.15, $N + X = M$. Since $X$ is a weak $g$-supplement of $L$ in $M$ and $N \leq L$, $N \cap X \leq L \cap X \ll_g M$. Thus $X$ is an essential weak $g$-supplement of $N$ in $M$.

Let $T$ be an essential weak $g$-supplement of $N$ in $M$. Then $N + T = M$ and by $N \leq L$, $L + T = M$. Let $L \cap T + S = M$ with $S \leq M$. Then by Lemma 2.16, $L + T \cap S = M$ and by Lemma 2.15, $N + T \cap S = M$. By also Lemma 2.16, $N \cap T + S = M$ and because of $L \cap T \ll_g M$ and $S \leq M$, $S = M$. Thus $L \cap T \ll_g M$ and $T$ is an essential weak $g$-supplement of $L$ in $M$.

Theorem 2.18. Let $L \leq M$, $N \leq L$ and $L$ g-lie above $N$ in $M$. If $L$ and $N$ have essential $g$-supplements in $M$, then they have the same essential $g$-supplements in $M$.

Proof. Let $X$ be an essential $g$-supplement of $L$ in $M$. Then $L + X = M$ and by Lemma 2.15, $N + X = M$. Since $X$ is a $g$-supplement of $L$ in $M$ and $N \leq L$, $N \cap X \leq L \cap X \ll_g X$. Thus $X$ is an essential $g$-supplement of $N$ in $M$.

Let $T$ be an essential $g$-supplement of $N$ in $M$. Then $N + T = M$ and by $N \leq L$, $L + T = M$. Let $L + S = M$ with $S \leq T$. Since $S \leq T$ and $T \leq M$, $S \leq M$. Then by Lemma 2.15, $N + S = M$ and since $T$ is a $g$-supplement of $N$ in $M$, $S = T$. Hence $T$ is an essential $g$-supplement of $L$ in $M$.

Theorem 2.19. Let $M$ be an $R$-module, $V$ be a weak $g$-supplement of $U$ in $M$ and $L \leq U$. If $M = V + L$, then $U$ g-lies above $L$ in $M$. 
Proof. Let $U + T = M$ with $T \trianglelefteq M$. Since $V$ is a weak $g$-supplement of $U$ in $M$, $M = U + V$ and $U \cap V \ll_g M$. Since $M = V + L$ and $L \leq U$, by Modular law, $U = U \cap V + L$. Then $M = U + T = U \cap V + L + T$ and since $U \cap V \ll_g M$ and $(L + T) \leq M$, $L + T = M$. Hence by Lemma 2.15, $U$ g-lies above $L$ in $M$.

References


