International Journal of Pure and Applied Mathematics
Volume 113 No. 2 2017, 291-297
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i2.9

ijpam.eu

NEW APPROACHES ABOUT $I-$CONTINUOUS FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

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Abstract: In [2], Özkurt introduced and investigated the new notion $I-$continuity. We introduce the notions $I_{\mathbf{w}}-$continuity, $I_{\boldsymbol{\theta}}-$continuity, $I_{\mathbf{w}^*}-$continuity. In this paper, we investigate relations among continuous function, $\theta-$continuous function and these new continuous function.

AMS Subject Classification: 54C05, 54C08, 54C10, 54E52

Key Words: $I-$continuous function, $I_{\boldsymbol{\theta}}-$continuous function, $I_{\mathbf{w}}-$continuous function, $I_{\mathbf{w}^*}-$continuous function, $\theta-$continuous function.

1. Introduction and Preliminaries

The concept of ideal topological spaces was introduced Kuratowski [5] and Vanidyanathowamy[9]. Throughout the present paper (X, τ) or (Y, φ) will denote a topological space. For a subset A of a topological space $(X, \tau), Cl(A)$ and $Int(A)$ will denote the closure and the interior of A in (X, τ), respectively. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subseteq A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

Received: November 5, 2016
Revised: January 9, 2017
Published: March 19, 2017

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An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subseteq X$, $A^\star(I) = \{x \in X \mid U \cap A \notin I$ for each open neighbourhood U of $x\}$ ise called the local function of A with respect to I and τ[5]. We simply write $A^\star$ instead of $A^\star(I)$ in case there is no change for confusion. For every ideal topological space (X, τ, I), there exists a topology $\tau^\star(I)$ (briefly $\tau^\star$), finer than τ, generated by $\beta(I, \tau) = \{U - A : U \in \tau$ and $A \in I$ $\}$, but in general $\beta(I, \tau)$ is not always a topology [3]. Additionally, $Cl^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator for $\tau^\star(I)$[4].

A notion of $I-$ continuity was introduced by M.E. Abd El-Monsef et al. [6]. Another version of $I-$ continuity was introduced by A. Özkurt [2] below

Definition 1. [2] $f : (X, \tau) \to (Y, \varphi, I)$ is said to be $I-$continuous function at x if, for every neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x in X such that $f(U) - V \in I$. We say that f is $I-$continuous function if f is $I-$continuous function at every point of X.

If $f(U) \subseteq V$ then $f(U) - V = \emptyset \in I$. Therefore, If f is a continuous function then f is $I-$continuous function. But converse need not be true as shown by the example [2]

Definition 2. $f : (X, \tau) \to (Y, \varphi)$ is said to be $\theta-$continuous function [8] (resp. weakly continuous function [7]) at x_0 if for each open neighbourhood V of $f(x_0)$, there is an open neighbourhood U of x_0 such that $f(Cl(U)) \subseteq Cl(V)$ (resp. $f(U) \subseteq Cl(V)$). f is said to be $\theta-$continuous function(resp. $w-$ continuous function) if it is $\theta-$continuous function(resp. $w-$continuous function) at each point of X.

Definition 3. [1] $f : (X, \tau) \to (Y, \varphi, I)$ is said to be weakly-$I-$continuous function (briefly $w - I - c$ function) if for each $x \in X$ and each open neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $f(U) \subseteq Cl^\star(V)$

We will use the following corollary:

Corollary 4. *[2] Let Y be a space and $f : (X, \tau) \to (Y, \varphi, I)$ be an $I-$continuous function and one to one map. If $f(U) \notin I$ for every non-empty open set U in X then f is continuous function.*

2. Main Results

We introduce the notions of I_w - continuity, I_θ - continuity, I_{w^*} - continuity, I_{θ^*} - continuity by following definitions.

Definition 5. $f:(X,\tau)\to(Y,\varphi,I)$ is said to be I_w - continuous function (resp. I_θ - continuous function, $I_{w^*}-$continuous function, $I_{\theta^*}-$continuous function) If for each $x\in X$ and each open neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $f(U)-Cl(V)\in I$ (resp. $f(Cl(U))-Cl(V)\in I$, $f(U)-Cl^{\square}(V)\in I$, $f(Cl(U))-Cl^{\square}V\in I$).

In this paper, we investigate relations among continuity in Definition $2,3,5$ and continuity.

Observe that if f is continuous function then f is I_w-continuous function. On the other hand I_w-continuity does not imply continuity.

Example 6. Let $f:(\mathbb{R},\tau_{cof})\to(\mathbb{R},\tau_{cof},I)$, $f(x)=cosx$, where $\tau_{cof}=\{V:\mathbb{R}-V\text{is finite}\}\cup\{\emptyset\}$ is cofinite topology on $\mathbb{R}$ and I denotes the ideal of finite subsets of $\mathbb{R}$. We were first show that f is I_w-continuous function. Let $x\in\mathbb{R}$ and $f(x)\in V\in\tau_{cof}$. Then $\mathbb{R}-V$ is finite. If we take $x\in U=\mathbb{R}\in\tau_{cof}$, then $f(U)-Cl(V)\subseteq\mathbb{R}-Cl(V)\subseteq\mathbb{R}-V$. Therefore $f(U)-Cl(V)\in I$. Thus f is I_w-continuous function. We were second show that f is not continuous function. Let $V=\mathbb{R}-\{1\}\in\tau_{cof}$. Then $f^{-1}(V)=\mathbb{R}-\{2k\pi:\ k\in\mathbb{Z}\}\notin\tau_{cof}$. Therefore f is not continuous function.

Since $f(U)-Cl(V)\subseteq f(U)-V$, if f is $I-$continuous function then f is I_w- continuous function. But converse need not be true.

Theorem 7. *Let (Y,φ,I) be a regular space. Then a function $f:(X,\tau)\to(Y,\varphi,I)$ is $I-$continuous function if and only if f is I_w-continuous function.*

Proof. The necessity is clear.

Sufficiency. Let $x\in X$ and V be an open set of Y containing $f(x)$. Since (Y,φ,I) be a regular space, there exists an open neighbourhood W of $f(x)$ such that $Cl(W)\subset V$. Since f is I_w-continuous function, there exists open neighbourhood U of x such that $f(U)-Cl(W)\in I$. Since $f(U)-V\subseteq f(U)-Cl(W)$ and I is ideal, hence we obtain $f(U)-V\in I$. Thus, f is $I-$continuous function. □

Theorem 8. *If $f:(X,\tau)\to(Y,\varphi,I)$ is continuous function then f is $I_\theta-$continuous function.*

Proof. Let $x\in X$ and $f(x)\in V\in\varphi$. Since f is continuous function, there exist a $x\in U\in\tau$ such that $f(U)\subseteq V$ and $f(Cl(U))\subseteq Cl(f(U))\subseteq Cl(V)$. Then $f(Cl(U))-Cl(V)=\emptyset\in I$. Therefore f is $I_\theta-$continuous function. □

The following example show that the converse implication does not hold.

Example 9. Let $X = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$ and

$$I = \{\emptyset, \{1\}, \{4\}, \{1, 4\}\}.$$

Define the function $f : (X, \tau) \to (X, I, \tau)$ as $f = \{(1, 1), (2, 4), (3, 3), (4, 2)\}$. Then f is $I_\theta-$continuous function but not f continuous function. Really:

i) Let $1 \in X$ and $\emptyset \neq V \in \tau$ such that $Cl(V) = X$. Then, there exists $U = \{1\} \in \tau$ such that $Cl(U) = X$ and $f(Cl(U)) - Cl(V) = \emptyset \in I$.

ii) Let $2 \in X$ and $V = X$ such that $Cl(V) = X$. Then, there exists $U = \{1, 2\} \in \tau$ such that $Cl(U) = X$ and $f(Cl(U)) - Cl(V) = \emptyset \in I$.

iii) Let $3 \in X$ and $V = \{1, 3\}, V = \{1, 2, 3\}$ or $V = X$ such that $Cl(V) = X$. Then, there exists $U = \{1\} \in \tau$ such that $Cl(U) = X$ and $f(Cl(U)) - Cl(V) = \emptyset \in I$.

iv) Let $4 \in X$ and $V = \{1, 2\}, V = \{1, 2, 3\}$ or $V = X$ such that $Cl(V) = X$. Then, there exists $U = X \in \tau$ such that $Cl(U) = X$ and $f(Cl(U)) - Cl(V) = \emptyset \in I$.

By i),ii),iii) and iv), f is $I_\theta-$continuous function. On the other hand, for $V = \{1, 2\} \in \tau, f^{-1}(V) = \{1, 4\} \notin \tau$. Therefore f is not continuous function.

Since $f(U) - Cl(V) \subseteq f(Cl(U)) - Cl(V)$, if f is $I_\theta-$continuous function then f is I_w-continuous function. On the other hand I_w-continuity does not imply $I_\theta-$continuity.

Example 10. Let

$$X = \{1, 2, 3, 4\},\ \tau = \{\emptyset, X, \{2\}, \{1, 2\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\},$$

$Y = \{a, b, c, d, e\}$, $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and

$$I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}.$$

Define the function $f : (X, \tau) \to (Y, \sigma, I)$ as $f = (1, a), (2, d), (3, e), (4, b)$. Then f is I_w-continuous function but not f, $I_\theta-$continuous function. Really:

i) Let $1 \in X$ and $V = \{a\}$, $V = \{a, b\}$, $V = \{a, c\}$, $V = \{a, b, c\}$ or $V = Y$ such that $\{a, d, e\} \subset Cl(V)$. Then, there exists $U = \{1, 2\} \in \tau$ such that $f(U) = \{a, d\} \subset Cl(V)$ and $f(U) - Cl(V) = \emptyset \in I$.

ii) Let $2 \in X$ and $V = Y$ such that $Cl(V) = Y$. Then, there exists $2 \in U \in \tau$ such that $f(U) \subset Cl(V)$ and $f(U) - Cl(V) = \emptyset \in I$.

iii)Let $3 \in X$ and $V = Y$ such that $Cl(V) = X$. Then, there exists $2 \in U \in \tau$ such that $f(U) \subset Cl(V)$ and $f(U) - Cl(V) = \emptyset \in I$.

iv)Let $4 \in X$ and $V = \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}$ or $V = Y$ such that $\{b,d,e\} \subset Cl(V)$. Coice $U = \{2,4\}$. Then, $f(U) = \{d,b\} \subset Cl(V)$ and $f(U) - Cl(V) = \emptyset \in I$.

By i),ii),iii) and iv), f is I_w continuous function. Now we show that $f : (X,\tau) \to (Y,\sigma,I)$ is not $I_\theta-$continuous function.

Let $1 \in X$ and $V = \{a\} \in \sigma$. But, for every open set $U \subset X$ such that $1 \in U$, $Cl(U) = X$. $f(Cl(U)) - Cl(V) = \{b,c\} \notin I$, Then f, I_w- continuous function.

Therefore f is not $I_\theta-$continuous function.

Theorem 11. *If $f : (X,\tau) \to (Y,\varphi,I)$ is $I_{w^*}-$continuous function then f is I_w-continuous function.*

Proof. Let $x \in X$ and $f(x) \in V \in \varphi$. Then there exist a $x \in U \in \tau$ such that $f(U) - Cl^{\square}(V) \in I$. Since $\varphi \subset \varphi^{\square}$, $Cl^{\square}(V) \subseteq Cl(V)$ for any subset $V \subseteq Y$. Then we obtain $f(U) - Cl(V) \subseteq f(U) - Cl^{\square}(V)$. Since I is ideal, we obtain $f(U) - Cl(V) \in I$. Therefore f is I_w-continuous function. □

Definition 12. [1] An ideal topological space (X,τ,I) is an $RI-$space if, for each $x \in X$ and each open neighbourhood V of x, there exists an open neighbourhood U of x such that $x \in U \subset Cl^{\square}(U) \subset V$.

Theorem 13. *Let Y be an $RI-$space. Then $f : (X,\tau) \to (Y,\varphi,I)$ is I_{w^*}- continuous function if and only if f is $I-$continuous function.*

Proof. Necessity. Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is an $RI-$space, there exists an open set W of Y such that $f(x) \in W \subseteq Cl^{\square}(W) \subseteq V$. Since f is $I_{w^*}-$continuous function, there exists an open set U such that $x \in U$ and $f(U) - Cl^{\square}(W) \in I$. Since $f(U) - V \subseteq f(U) - Cl^{\square}(W)$, we obtain $f(U) - V \in I$. Thus, f is $I-$continuous function.

Sufficiency. Let $x \in X$ and V be an open set of Y containing $f(x)$. Since f is $I-$continuous function, there exists an open set U such that $x \in U$ and $f(U) - V \in I$. Since $f(U) - Cl^{\square}(V) \subseteq f(U) - V$, we obtain $f(U) - Cl^{\square}(V) \in I$. Thus, f is $I_{w^*}-$continuous function. □

Since every regular space is $RI-$space. The following theorem is an immediate consequence of Theorem 13, Theorem 7 and Corollary 4.

Theorem 14. *Let (Y, I, φ) be regular space and $f : (X, \tau) \to (Y, I, \varphi)$ one to one map and $f(U) \notin I$ for every non empty open set U in X. Then the following properties are equivalent:*
a) f continuous function;
b) $f, I-$continuous function;
c) $f, I_\theta-$continuous function;
d) f, I_w-continuous function;
e) $f, I_{w^*}-$continuous function.

Theorem 15. *If a function $f : (X, \tau) \to (Y, \varphi, I)$ is $w - I$ continuous function then f is I_w-continuous function.*

Proof. Let $x \in X$ and $f(x) \in V \in \varphi$. Since f is $w - I$continuous function. Then there exist a $x \in U \in \tau$ such that $f(U) \subseteq Cl^{\square}(V) \subseteq Cl(V)$. Then $f(U) - Cl(V) = \emptyset \in I$. Therefore f is I_w-continuous function. □

On the other hand I_w-continuity does not imply $w - I$ continuity.

Example 16. Let

$$X = \{a, b, c, d\},\ \tau = \{\emptyset, X, \{a\}, \{a, b\}\},$$

$$\varphi = \{\emptyset, X, \{c\}, \{b, c\}\},$$

and $I = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$. Define the function $f : (X, \tau) \to (X, I, \varphi)$ as $f(x) = x$. Then f is I_w-continuous function but not $f, w - I$ continuous function.

Remark 17. Let $f : (X, \tau) \to (Y, \varphi, I)$ is a function. Since $f(U) \subseteq Cl_\varphi(V)$, if f is $w-$ continuous function then $f(U) - Cl(V) = \emptyset \in I$. Therefore $f,\ I_w-$continuous function.

On the other hand I_w-continuity does not imply $w-$continuity.

Example 18. Let

$$X = \{a, b, c\},\ \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\},$$

$$\varphi = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\},$$

and $I = \{\emptyset, \{a\}\}$. Define the function $f : (X, \tau) \to (X, I, \varphi)$ as $f(x) = x$. Then f is I_w-continuous function but not $f,\ w-$continuous function.

Remark 19. Let $f : (X, \tau) \to (Y, \varphi, I)$ is a function. Since $f(Cl(U)) \subseteq Cl(V)$, if f is θ continuous function then $f(Cl(U)) - Cl(V) = \emptyset \in I$. Therefore $f,\ I_\theta-$continuous function.

On the other hand $I_\theta-$continuity does not imply $\theta-$continuity.

Example 20. Let $X = \{1,2,3\}, \tau = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}\}$, $\varphi = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}\}$ and $I = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. Define the function $f : (X,\tau) \to (X, I, \varphi)$ as $f(x) = x$. Then f is $I_\theta-$continuous function but not f, $\theta-$continuous function.

Theorem 21. *If a function $f : (X,\tau) \to (Y,\varphi, I)$ is $\theta-$continuous function then f is I_w-continuous function.*

Proof. Let $x \in X$ and $f(x) \in V \in \varphi$. Since f is $\theta-$continuous function, then there exist a $x \in U \in \tau$ such that $f(U) \subseteq f(Cl(U)) \subseteq Cl(V)$. Then $f(U) - Cl(V) = \emptyset \in I$. Therefore f is I_w-continuous function. □

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