International Journal of Pure and Applied Mathematics
Volume 113 No. 3 2017, 483-487
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i3.10

ijpam.eu

W_1-ELEMENTS IN A COMMUTATIVE RING WITH UNITY

W. Nazeer[1], W. Khalid[2], Shin Min Kang[3 §], W. Ahmad[4]

[1]Division of Science and Technology
University of Education
Lahore, 54000, PAKISTAN

[2,4]Department of Mathematics
The University of Lahore
Pakpattan, 57400, PAKISTAN

[3]Department of Mathematics and RINS
Gyeongsang National University
Jinju, 52828, KOREA

Abstract: In this paper, we introduce and study w_1-elements in a commutative ring with unity R. An element $0 \neq x \in R$ is said to be a w_1-element of R if whenever $xd = x$ for $1 \neq d \in R$, then there exists $0 \neq z \in R$ such that $dz = 0$. We show that if $Ann_{\mathbf{R}}(x) = Ann_{\mathbf{R}}(y)$ then x is a w_1-element of R if and only if y is a w_1-element of R, that every w_1-element of R is also a w_1-element of its polynomial ring $R[x]$ and that if x^2 is a w_1-element of R with $J = Ann_{\mathbf{R}}(x)$, then $x + J$ is a w_1-element of its quotient ring R/J.

AMS Subject Classification: 13A15, 13F05
Key Words: nilpotent element, annihilator

1. Introduction

Let R be a commutative ring with unity. An element $0 \neq x \in R$ is said to be a w_1-element of R if whenever $xd = x$ for $1 \neq d \in R$, then there exists $0 \neq z \in R$ such that $dz = 0$. For example, $\bar{2} \in \mathbb{Z}_6$ is a w_1-element of $\mathbb{Z}_6$ while $\bar{3} \in \mathbb{Z}_6$ is not a w_1-element of $\mathbb{Z}_6$ see Example 2.2.

Received: December 21, 2016
Revised: February 20, 2017
Published: March 31, 2017

© 2017 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

We show that if xy is a w_1-element of R for $0 \neq x, y \in R$, then both x and y are w_1-elements of R (Proposition 2.3), if x^n is a w_1-element of R for $0 \neq x \in R$ and $n \in \mathbb{N}$, then x is a w_1-element of R (Corollary 2.4), if x is not a zero-divisor of R and y is a w_1-element of R, then xy is a w_1-element of R (Proposition 2.6), if zero is the only nilpotent element of R and $0 \neq x \in R$ a w_1-element of R, then x^n is a w_1-element of R for all $n \in \mathbb{N}$ (Proposition 2.7), if $Ann_R(x) = Ann_R(y)$ for $0 \neq x, y \in R$, then x is a w_1-element of R if and only if y is a w_1-element of R (Proposition 2.9), an element $0 \neq u \in R$ is a w_1-element of R if and only if u is a w_1-element of its polynomial ring $R[x]$ (Proposition 2.10), an element $0 \neq u \in R$ is a w_1-element of R if and only if u is a w_1-element of its polynomial ring $R[x_1, x_2, ..., x_n]$ (Corollary 2.11), if $0 \neq x \in R$ and $J = Ann_R(x)$, then $x + J \in R/J$ is a w_1-element of R/J implies x is a w_1-element of R (Proposition 2.12), if $0 \neq x \in R$ and $J = Ann_R(x)$, then $x^2 \in R$ is a w_1-element of R implies $x + J \in R/J$ is a w_1-element of R/J (Proposition 2.13), if φ is an isomorphism between two rings R and R', then an element $0 \neq x \in R$ is a w_1-element of R if and only if $\varphi(x)$ is a w_1-element of R' (Proposition 2.14).

Our standard references for any undefined notation or terminology are [2] and [3].

2. Main results

Definition 2.1. Let R be a commutative ring with unity. An element $0 \neq x \in R$ is said to be a w_1-element of R if whenever $xd = x$ for $1 \neq d \in R$, then there exists $0 \neq z \in R$ such that $dz = 0$.

Example 2.2. $\bar{2} \in \mathbb{Z}_6$ is a w_1-element of $\mathbb{Z}_6$. As $\bar{2} \cdot \bar{4} = \bar{2}$, there exists $\bar{3} \in \mathbb{Z}_6$ such that $\bar{4} \cdot \bar{3} = \bar{0}$. While $\bar{3} \in \mathbb{Z}_6$ is not a w_1-element of $\mathbb{Z}_6$. Indeed, $\bar{3} \cdot \bar{5} = \bar{3}$ but there does not exist $\bar{a} \in \mathbb{Z}_6$ such that $\bar{5} \cdot \bar{a} = \bar{0}$.

Proposition 2.3. Let R be a commutative ring with unity. If xy is a w_1-element of R for $0 \neq x, y \in R$, then both x and y are w_1-elements of R.

Proof. If $xd = x$ for $1 \neq d \in R$, then $xyd = xy$. By hypothesis, there exists $0 \neq z \in R$ such that $dz = 0$. Thus x is a w_1-element of R. Similary, y is also a w_1-element of R. □

Corollary 2.4. Let R be a commutative ring with unity. If x^n is a w_1-element of R for $0 \neq x \in R$ and $n \in \mathbb{N}$, then x is a w_1-element of R.

Proof. If $xd = x$ for $1 \neq d \in R$, then $x^n d = x^n$. By hypothesis, there exists $0 \neq z \in R$ such that $dz = 0$. Thus x is a w_1-element of R. □

Example 2.5. $\bar{2} \cdot \bar{4} = \bar{8} \in \mathbb{Z}_{10}$ is a w_1-element. As $\bar{8} \cdot \bar{6} = \bar{8}$, there exists $\bar{5} \in \mathbb{Z}_{10}$ such that $\bar{8} \cdot \bar{5} = \bar{0}$. By Proposition 2.3, $\bar{2}$ and $\bar{4}$ are w_1-elements of $\mathbb{Z}_{10}$.

Proposition 2.6. Let R be a commutative ring with unity and $0 \neq x, y \in R$. If x is not a zero-divisor of R and y is a w_1-element of R, then xy is a w_1-element of R.

Proof. If $xyd = xy$ for $1 \neq d \in R$, then $yd = y$ because x is not a zero-divisor of R. By hypothesis, there exists $0 \neq z \in R$ such that $dz = 0$. Thus xy is a w_1-element of R. □

Proposition 2.7. Let R be a commutative ring with unity and $0 \neq x \in R$ a w_1-element of R. Further if zero is the only nilpotent element of R, then x^n is a w_1-element of R for all $n \in \mathbb{N}$.

Proof. If $x^n d = x^n$ for $1 \neq d \in R$, then $x^n(d-1) = 0$ implies $(x(d-1))^n = 0$. Since zero is the only nilpotent element of R, it follows that $x(d-1) = 0$ implies $xd = x$ see [2]. By hypothesis, there exists $0 \neq z \in R$ such that $dz = 0$. Thus x^n is a w_1-element of R. □

Definition 2.8. Let R be a commutative ring with unity and $x \in R$. The annihilator of x in R is denoted by $Ann_R(x)$ and is defined as $Ann_R(x) = \{r \in R \mid rx = 0\}$. Note that $Ann_R(x)$ is always an ideal of R.

Proposition 2.9. Let R be a commutative ring with unity and $Ann_R(x) = Ann_R(y)$ for $0 \neq x, y \in R$. Then x is a w_1-element of R if and only if y is a w_1-element of R.

Proof. If x is a w_1-element of R and $yd = y$ for $1 \neq d \in R$. Then $y(d-1) = 0$ implies $d-1 \in Ann_R(y) = Ann_R(x)$. It follows that $x(d-1) = 0$ implies $xd = x$ see [3]. By hypothesis, there exists $0 \neq z \in R$ such that $dz = 0$. Thus y is a w_1-element of R. Conversely, if y is a w_1-element of R and $xd = x$ for $1 \neq d \in R$. Then $x(d-1) = 0$ implies $d - 1 \in Ann_R(x) = Ann_R(y)$. It follows that $y(d-1) = 0$ implies $yd = y$. By hypothesis, there exists $0 \neq z \in R$ such that $dz = 0$. Thus x is a w_1-element of R. □

Proposition 2.10. Let R be a commutative ring with unity. Then $0 \neq u \in R$ is a w_1-element of R if and only if u is a w_1-element of the polynomial ring $R[x]$.

Proof. If $ug(x) = u$ for $1 \neq g(x) = g_0 + g_1x + \cdots + g_nx \in R[x]$, then $ug_0 = u, ug_1 = 0, \cdots, ug_n = 0$ see [1]. Since u is a w_1-element of R, there exists $0 \neq z_0 \in R$ such that $g_0z_0 = 0$. Taking $h(x) = z_0u \in R[x]$, it is clear that $g(x)h(x) = 0$. Hence u is a w_1-element of $R[x]$. Conversely, if $uv = u$ for $1 \neq v \in R \subset R[x]$, then there exists $0 \neq g(x) = g_0 + g_1 x + \cdots + g_n x \in R[x]$ such that $vg(x) = 0$. Thus $vg_i = 0$ for some $0 \neq g_i \in R$; $i = 0, 1, ..., n$. Hence u is a w_1-element of R. □

Corollary 2.11. Let R be a commutative ring with unity. Then $0 \neq u \in R$ is a w_1-element of R if and only if u is a w_1-element of the polynomial ring $R[x_1, x_2, ..., x_n]$.

Proof. Follows from Proposition 2.10 and the fact that

$$R[x_1, x_2, ..., x_n] = R[x_1, x_2, ..., x_{n-1}][x_n].$$ □

Proposition 2.12. Let R be a commutative ring with unity and $J = Ann_R(x)$ for $0 \neq x \in R$. If $x + J \in R/J$ is a w_1-element of R/J, then x is a w_1-element of R.

Proof. If $xd = x$ for $1 \neq d \in R$, then $(x+J)(d+J) = x+J$. Since $x+J$ is a w_1-element of R/J, there exists $J \neq z + J \in R/J$ such that $(d+J)(z+J) = J$. It follows that $dz \in J = Ann_R(x)$ implies $d(zx) = 0$ where $0 \neq zx \in R$. Hence x is a w_1-element of R. □

Proposition 2.13. Let R be a commutative ring with unity and $J = Ann_R(x)$ for $0 \neq x \in R$. If $x^2 \in R$ is a w_1-element of R, then $x + J \in R/J$ is a w_1-element of R/J.

Proof. If $(x + J)(d + J) = x + J$ for $1 + J \neq d + J \in R/J$, then $xd - x \in J = Ann_R(x)$ implies that $x^2 d = x^2$ see [1]. Since x^2 is a w_1-element of R, there exists $0 \neq z \in R$ such that $dz = 0$. Thus $(d + J)(z + J) = J$ where $J \neq z + J \in R/J$. Hence $x + J$ is a w_1-element of R/J. □

Proposition 2.14. Let R and R' be any two commutative rings with unity and φ be an isomorphism from R to R'. An element $0 \neq x \in R$ is a w_1-element of R if and only if $\varphi(x)$ is a w_1-element of R'.

Proof. If $\varphi(x)\varphi(d) = \varphi(x)$ for $1_{R'} \neq \varphi(d) \in R'$, then $xd = x$ for $1 \neq d \in R$. Since x is a w_1-element of R, there exists $0 \neq z \in R$ such that $dz = 0$. It follows that $\varphi(d)\varphi(z) = 0_{R'}$ where $0_{R'} \neq \varphi(z) \in R'$. Hence $\varphi(x)$ is a w_1-element of R'. Conversely, if $xd = x$ for $1 \neq d \in R$, then $\varphi(x)\varphi(d) = \varphi(x)$ for $1_{R'} \neq \varphi(d) \in R'$. Since $\varphi(x)$ is a w_1-element of R', there exists $0_{R'} \neq \varphi(z) \in R'$ such that $\varphi(d)\varphi(z) = 0_{R'}$. It follows that $dz = 0$ where $0 \neq z \in R$. Hence x is a w_1-element of R. □

Acknowledgements

We would like to express our sincere thanks to The University of Lahore, Pakpattan Campus, Pakpattan for their support in this research work.

References

[1] J.A. Gallian, Contemporary Abstract Algebra, *Houghton Mifflin Company* (2006).

[2] R. Gilmer, Multiplicative Ideal Theory, *Pure Appl. Math.*, **12**, *Marcel Dekker, Inc, New York* (1972).

[3] I. Kaplansky, Commutative Rings, *The University of Chicago Press*, Chicago (1974).