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SOME NEW SUM PERFECT SQUARE GRAPHS

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Abstract: A (p, q) graph $G = (V, E)$ is called sum perfect square if for a bijection $f : V(G) \to \{0, 1, 2, \dots, p-1\}$ there exists an injection $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(G)$. Here f is called sum perfect square labeling of G. In this paper we derive several new sum perfect square graphs.

AMS Subject Classification: 05C78.
Key Words: Sum perfect square graph, half wheel graph

1. Introduction

We consider simple, finite, undirected graph $G = (p, q)$ (with p vertices and q edges). The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively. For all other terminology and notations we follow Harary[1].

Sonchhatra and Ghodasara[4] initiated the study of sum perfect square graphs. Due to [4] it becomes possible to construct a graph, whose all edges can be labeled by different perfect square integers. In [4] the authors proved that P_n, C_n, C_n with one chord, C_n with twin chords, tree, $K_{1,n}$, $T_{m,n}$ are sum

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perfect square graphs.

In this paper we prove that half wheel, corona, middle graph, total graph, $K_{1,n}+K_1$, K_2+mK_1 are sum perfect square graphs.

Definition 1.1. Let $G=(p,q)$ be a graph. A bijection $f: V(G) \rightarrow \{0,1,2,\ldots,\ p-1\}$ is called *sum perfect square labeling* of G, if the induced function $f^*: E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv)=(f(u))^2+(f(v))^2+2f(u)\cdot f(v)$ is injective, $\forall uv \in E(G)$.

A graph which admits sum perfect square labeling is called sum perfect square graph.

Definition 1.2. The corona product $G \odot H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G by an edge, $1 \leq i \leq |V(G)|$.

Definition 1.3. The middle graph of a graph G denoted by $M(G)$ is the graph with vertex set $V(G) \cup E(G)$, where two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and other is an edge incident with it.

Definition 1.4. The total graph of a graph G denoted by $T(G)$ is the graph with vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if

(1) $x, y \in V(G)$ are adjacent.

(2) $x, y \in E(G)$ are adjacent.

(3) $x \in V(G)$, $y \in E(G)$ and y is incident to x.

Definition 1.5. Half wheel graph, denoted by HW_n is constructed by the following steps.

Step 1: Consider a star $K_{1,n}$. Let $\{v_1, v_2, \ldots, v_n\}$ be the pendant vertices of $K_{1,n}$ and v be the apex vertex of $K_{1,n}$.

Step 2: Add an edge between v_i and v_{i+1}, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Note that $|V(H(W_n))| = n+1$, $|E(H(W_n))| = n + \lfloor \frac{n}{2} \rfloor$.

Definition 1.6. Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \phi$. The join of G_1 and G_2, denoted by G_1+G_2, is the graph with $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup J$, where $J=\{uv/u \in V(G_1)$ and $v \in V(G_2)\}$.

2. Main Results

Observation 1: If $G = (V, E)$ is not sum perfect square graph, then its supergraph is also not sum perfect square graph, but the converse may not be true.

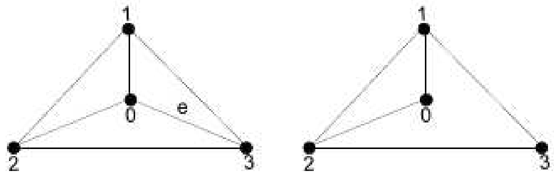

Figure 1: A non sum perfect square graph K_4 with sum perfect square subgraph $K_4 - \{e\}$.

In [4] Sonchhatra and Ghodasara posed the following conjecture.

Conjecture 2.1. An odd simple graph G with $\delta(G) = 3$ is not sum perfect square.

Here we prove this conjecture by using the principle of mathematical induction on number of vertices of the graph.

Theorem 2.2. An odd simple graph G with $\delta(G) = 3$ is not sum perfect square.

Proof. For any graph $G = (V, E)$ with $|V(G)| = n$, since $d(v) \geq 3$, $\forall v \in G$, n must be even and $n \geq 4$. We prove this conjecture by using principle of mathematical induction on number of vertices n of graph G.

Step 1. For $n = 4$, $G = K_4$ is not sum perfect square graph (See [4]).

Step 2. Suppose the result is true for $n = k$, $4 < k < n$.

Step 3. Let $G' = (V', E')$ be the graph with $|V'| = k + 1$. $H = G' - \{v\}$ is the graph with k vertices, where v is any arbitrary vertex of G'. By induction hypothesis, H is not a sum perfect square graph. Since G' is a supergraph of H, it is not a sum perfect square graph (Observation 1). □

Theorem 2.3. $C_n \odot K_1$ is sum perfect square graph, $n \geq 3$.

Proof. Let $V(C_n \odot K_1) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\}$, where $u_1, u_2, \ldots, u_n$ are successive vertices of C_n and $v_1, v_2, \ldots, v_n$ be the successive vertices corresponding to n copies of K_1,

$$E(C_n \odot K_1) = \{e_i^{(1)} = u_i u_{i+1}; 1 \le i \le n-2\} \cup \{e_{n-1}^{(1)} = u_n u_1\} \cup \{e_i^{(2)} = u_i v_i; 1 \le i \le n\}.$$

We note that $|V(C_n \odot K_1)| = 2n$ and $|E(C_n \odot K_1)| = 2n$.

We define a bijection $f : V(C_n \odot K_1) \to \{0, 1, 2, \ldots, 2n-1\}$ as

$$\begin{aligned} f(u_1) &= 0, \\ f(u_i) &= \begin{cases} 4i-6; & 2 \le i \le \lfloor \frac{n}{2} \rfloor + 1. \\ 4n-4i+4; & \lfloor \frac{n}{2} \rfloor + 2 \le i \le n. \end{cases} \\ f(v_i) &= f(u_i)+1, 1 \le i \le n. \end{aligned}$$

Let $f^* : E(C_n \odot K_1) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(C_n \odot K_1)$.

Injectivity for edge labels: For $1 \le i \le \lfloor \frac{n}{2} \rfloor + 1$, $f^*(e_i^{(1)})$ is increasing in terms of $i \Rightarrow f^*(u_i u_{i+1}) < f^*(u_{i+1} u_{i+2})$ and for $\lfloor \frac{n}{2} \rfloor + 2 \le i \le n$, $f^*(e_i^{(1)})$ is decreasing in terms of $i \Rightarrow f^*(u_i u_{i+1}) > f^*(u_{i+1} u_{i+2})$. Similarly $f^*(e_i^{(2)})$ is increasing for $1 \le i \le \lfloor \frac{n}{2} \rfloor + 1$ and decreasing for $\lfloor \frac{n}{2} \rfloor + 2 \le i \le n$.

Claim: $\{f^*(e_i^{(1)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor + 1\} \ne \{f^*(e_i^{(1)}), \lfloor \frac{n}{2} \rfloor + 2 \le i \le n-1\} \ne f^*(e_n^{(1)}) \ne \{f^*(e_i^{(2)}), 1 \le i \le \lfloor \frac{n}{2} \rfloor + 1\} \ne \{f^*(e_i^{(2)}), \lfloor \frac{n}{2} \rfloor + 2 \le i \le n\}$.

We have

$$\begin{aligned} f^*(e_i^{(1)}) &= \begin{cases} (8i-8)^2; 2 \le i \le \lfloor \frac{n}{2} \rfloor. \\ (8n-8i+4)^2; \lfloor \frac{n}{2} \rfloor + 2 \le i \le n-1. \end{cases} \\ f^*(e_{\lfloor \frac{n}{2} \rfloor + 1}^{(1)}) &= (4n-6)^2, f^*(e_1^{(1)}) = 2, f^*(e_n^{(1)}) = 4. \end{aligned}$$

Further

$$f^*(e_i^{(2)}) = \begin{cases} (8i-11)^2; 2 \le i \le \lfloor \frac{n}{2} \rfloor + 1. \\ (8n-8i+9)^2; \lfloor \frac{n}{2} \rfloor + 2 \le i \le n. \end{cases}$$

It is clear that

(1) $f^*(e_1^{(2)}) = 1$, $f^*(e_1^{(1)}) = 2$ and $f^*(e_n^{(1)}) = 4$ are three smallest edge labels among all the edge labels in this graph.

(2) $\{f^*(e_i^{(1)}), 1 \le i \le n\}$ are even, $\{f^*(e_i^{(2)}), 1 \le i \le n\}$ are odd.

(3) $f^*(e_{\lfloor \frac{n}{2} \rfloor+1})$ is larger than the largest edge label of $\{f^*(e_i^{(1)}), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ and smaller than the smallest edge label of $\{f^*(e_i^{(1)}), \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n \}$.

Hence we only need to prove the following.

(1) $\{f^*(e_1^{(1)}); 2 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(1)}), \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n-1\}$.

(2) $\{f^*(e_1^{(2)}); 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(2)}), \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\}$.

Assume if possible $\{f^*(e_1^{(1)}); 2 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \{f^*(e_i^{(1)}), \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n-1\}$, for some i.
$\Longrightarrow 8i - 8 = 8n - 8i + 4$ or $8i - 8 = 8i - 8n - 4$.
$\Longrightarrow i = \frac{2n+3}{4}$ or $n = \frac{1}{2}$, which contradicts with the choice of i and n as $i, n \in \mathbb{N}$.
Assume if possible $\{f^*(e_1^{(2)}); 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \neq \{f^*(e_i^{(2)}), \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\}$, for some i.
$\Longrightarrow 8i - 11 = 8n - 8i + 9$ or $8i - 11 = 8i - 8n - 9$.
$\Longrightarrow i = \frac{2n+5}{4}$ or $n = \frac{1}{4}$, which contradicts with the choice of i as $i \in \mathbb{N}$.

Hence $f^* : E(C_n \odot K_1) \to \mathbb{N}$ is injective. So $C_n \odot K_1$ is sum perfect square graph, $n \geq 3$. □

The below illustration provides better idea about the above defined labeling pattern.

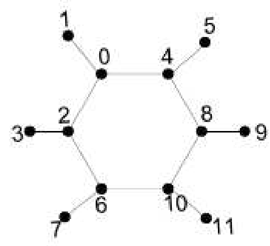

Figure 2 : Sum perfect square labeling of $C_6 \odot K_1$.

Theorem 2.4. $M(P_n)$ is sum perfect square graph, $\forall n \in \mathbb{N}$.

Proof. Let $V(M(P_n)) = \{v_i; 1 \leq i \leq n\} \cup \{v_i'; 1 \leq i \leq n-1\}$, where v_i' is adjacent with v_{i+1}', v_i and v_{i+1}, for $1 \leq i \leq n-1$. Here $E(M(P_n)) = \{e_i = v_i'v_{i+1}'; 1 \leq i \leq n-2\} \cup \{e_i^{(1)} = v_iv_i'; 1 \leq i \leq n-1\} \cup \{e_i^{(2)} = v_{i+1}v_i'; 1 \leq i \leq n-1\}$. We note that $|V(M(P_n))| = 2n - 1$ and $|E(M(P_n))| = 3n - 4$.

We define the bijection $f : V(M(P_n)) \to \{0, 1, 2, \ldots, 2n-2\}$ as
$f(v_i) = 2i - 2$, $1 \leq i \leq n$, $f(v_i') = 2i - 1$, $1 \leq i \leq n-1$.

Let $f^* : E(M(P_n)) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(M(P_n))$.

Injectivity for edge labels: For $1 \leq i \leq n-2$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v'_i v'_{i+1}) < f^*(v'_{i+1} v'_{i+2})$, $1 \leq i \leq n-3$. Similarly $f^*(e_i^{(1)})$ and $f^*(e_i^{(2)})$ are also increasing, $1 \leq i \leq n-1$.

Claim: $\{f^*(e_i); 1 \leq i \leq n-2\} \neq \{f^*(e_i^{(1)}); 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(2)}); 1 \leq i \leq n-1\}$.
We have $f^*(e_i) = (4i)^2$, $1 \leq i \leq n-2$ and $f^*(e_i{}^{(1)}) = (4i-3)^2$, $f^*(e_i{}^{(2)}) = (4i-1)^2$, $1 \leq i \leq n-1$.
$f^*(e_i)$ are even, $1 \leq i \leq n-2$ and $f^*(e_i{}^{(j)})$ are odd, $1 \leq i \leq n-1$, $j = 1, 2$.
Assume if possible $f^*(e_i^{(1)}) = f^*(e_i^{(2)})$, for some i, $1 \leq i \leq n-1$.
$\Longrightarrow 4i - 3 = 4i - 1$ or $4i - 3 = 1 - 4i$
$\Longrightarrow 3 = 1$ or $i = \frac{1}{2}$, which contradicts the choice of i, as $i \in \mathbb{N}$.
So $f^* : E(M(P_n)) \to \mathbb{N}$ is injective. Hence $M(P_n)$ is sum perfect square graph, $\forall n \in \mathbb{N}$. □

The below illustration provides the better idea of the above defined labeling pattern.

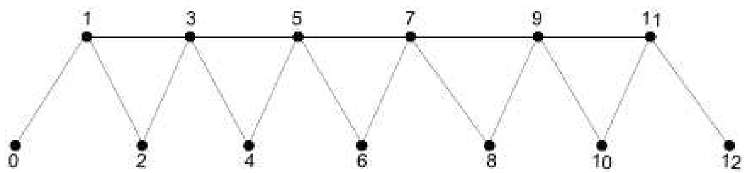

Figure 3 : Sum perfect square labeling of $M(P_7)$.

Theorem 2.5. $T(P_n)$ is sum perfect square graph, $\forall n \in \mathbb{N}$.

Proof. Let $V(T(P_n)) = \{v_i; 1 \leq i \leq n\} \cup \{v'_i; 1 \leq i \leq n-1\}$, where v'_i is adjacent with v'_{i+1}, v_i and v_{i+1}, $1 \leq i \leq n-1$. $E(T(P_n)) = \{e_i = v'_i v'_{i+1}; 1 \leq i \leq n-2\} \cup \{e_i^{(1)} = v_i v'_i; 1 \leq i \leq n-1\}$
$\cup \{e_i^{(2)} = v_{i+1} v'_i; 1 \leq i \leq n-1\} \cup \{e_i^{(3)} = v_i v_{i+1}; 1 \leq i \leq n-1\}$. $|V(T(P_n))| = 2n-1$ and $|E(T(P_n))| = 4n-5$.
We define a bijection $f : V(T(P_n)) \to \{0, 1, 2, \ldots, 2n-2\}$ as
$f(v_i) = 2i-2$, $1 \leq i \leq n$, $f(v'_i) = 2i-1$, $1 \leq i \leq n-1$.
Let $f^* : E(T(P_n)) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(T(P_n))$.

Injectivity for edge labels: For $1 \leq i \leq n-2$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(v'_i v'_{i+1}) < f^*(v'_{i+1} v'_{i+2})$, $1 \leq i \leq n-3$. Similarly $f^*(e_i^{(j)})$ are also increasing, $1 \leq i \leq n-1$, $1 \leq j \leq 3$.

Claim: $\{f^*(e_i); 1 \leq i \leq n-2\} \neq \{f^*(e_i^{(1)}); 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(2)}); 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(3)}); 1 \leq i \leq n-1\}$.
$f^*(e_i) = (4i)^2$, $1 \leq i \leq n-2$. For $1 \leq i \leq n-1$, $f^*({e_i}^{(1)}) = (4i-3)^2$, $f^*({e_i}^{(2)}) = (4i-1)^2$, $f^*({e_i}^{(3)}) = (4i-2)^2$.
$\{f^*(e_i), 1 \leq i \leq n-2\}$, $f^*({e_i}^{(3)})$ are even and $f^*({e_i}^{(j)})$ are odd, for $1 \leq i \leq n-1$, $j = 1, 2, 3$. It is enough to prove the following.

(1) $\{f^*(e_i), 1 \leq i \leq n-2\} \neq \{f^*(e_i^{(3)}), 1 \leq i \leq n-1\}$.

(2) $\{f^*(e_i^{(1)}), 1 \leq i \leq n-1\} \neq \{f^*(e_i^{(2)}), 1 \leq i \leq n-1\}$.

Assume if possible $\{f^*(e_i), 1 \leq i \leq n-2\} = \{f^*(e_i^{(3)}), 1 \leq i \leq n-1\}$, for some i.
$\Longrightarrow 4i = 4i-2$ or $4i = 2-4i$.
$\Longrightarrow 1 = -2$ or $i = \frac{1}{4}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Assume if possible $\{f^*(e_i^{(2)}), 1 \leq i \leq n-1\} = \{f^*(e_i^{(3)}), 1 \leq i \leq n-1\}$, for some i.
$\Longrightarrow 4i-3 = 4i-1$ or $4i-3 = 1-4i$.
$\Longrightarrow -3 = -1$ or $i = \frac{1}{2}$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

So $f^* : E(T(P_n)) \to \mathbb{N}$ is injective. Hence $T(P_n)$ is sum perfect square graph, $\forall n \in \mathbb{N}$. □

The below illustration provides the better idea of the above defined labeling pattern.

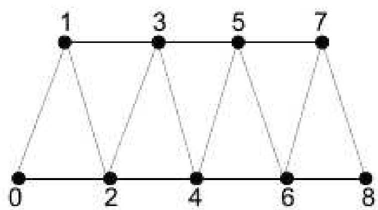

Figure 4 : Sum perfect square labeling of $T(P_5)$.

Theorem 2.6. *HW_n is sum perfect square graph, $\forall n \in \mathbb{N}$.*

Proof. Let $V(HW_n) = \{v\} \cup \{v_i; 1 \leq i \leq n\}$ and $E(HW_n) = \{e_i = vv_i; 1 \leq i \leq n\} \cup \{e_i^{(1)} = v_iv_{i+1}; 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. $|V(HW_n)| = n+1$ and $|E(HW_n)| = n + \lfloor \frac{n}{2} \rfloor$.

We define a bijection $f : V(HW_n) \to \{0, 1, 2, \ldots, n\}$ as $f(v) = n$, $f(v_i) = i - 1$, $1 \leq i \leq n$.

Let $f^* : E(HW_n) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(HW_n)$.

Injectivity for edge labels: For $1 \leq i \leq n$, $f^*(e_i)$ is increasing in terms of $i \Rightarrow f^*(vv_i) < f^*(vv_{i+1})$, $1 \leq i \leq n-1$. Similarly $f^*(e_i^{(1)})$ is also increasing.

The largest edge label of $f^*(e_i^{(1)})$ is smaller than the smallest edge label of $f^*(e_i)$, therefore $\{f^*(e_i); 1 \leq i \leq n\} \neq \{f^*(e_i^{(1)}); 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

Hence the induced edge labeling $f^* : E(H(W_n)) \to \mathbb{N}$ is injective.

So HW_n is sum perfect square graph, $\forall n \in \mathbb{N}$. □

The below illustration gives the better understanding of above defined labeling pattern.

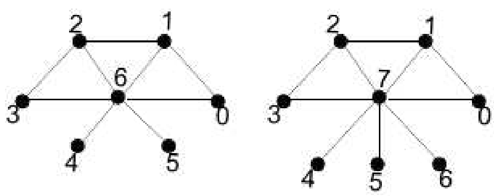

Figure 5 : Sum perfect square labeling of $H(W_6)$ and $H(W_7)$.

Theorem 2.7. $K_{1,n} + K_1$ is sum perfect square graph, $\forall n \in \mathbb{N}$.

Proof. Let $V(K_{1,n}+K_1) = \{v\} \cup \{v_i; 1 \leq i \leq n\} \cup \{w\}$, where $\{v_1, v_2, \ldots, v_n\}$ are the pendant vertices and v is the apex vertex of $K_{1,n}$ and w is the apex vertex corresponding to K_1. Here $E(K_{1,n} + K_1) = \{e_i^{(1)} = vv_i; 1 \leq i \leq n\} \cup \{e_i^{(2)} = wv_i; 1 \leq i \leq n\} \cup \{e = vw\}$. Note that $|V(K_{1,n} + K_1)| = n + 2$ and $|E(K_{1,n} + K_1)| = 2n + 1$.

We define a bijection $f : V(K_{1,n} + K_1) \to \{0, 1, 2, \ldots, n + 1\}$ as $f(v) = 0$, $f(v_i) = i$, $1 \leq i \leq n$, $f(w) = n + 1$.

Let $f^* : E(K_{1,n} + K_1) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(K_{1,n} + K_1)$.

Injectivity for edge labels: For $1 \leq i \leq n$, $f^*(e_i^{(1)})$ is increasing in terms of $i \Rightarrow f^*(vv_i) < f^*(vv_{i+1})$, $1 \leq i \leq n - 1$. Similarly $f^*(e_i^{(1)})$ is also increasing, for $1 \leq i \leq n$.

Claim: $\{f^*(e_i^{(1)}); 1 \leq i \leq n\} \neq \{f^*(e_i^{(2)}); 1 \leq i \leq n\} \neq f^*(e)$.

We have $f^*(e_i^{(1)}) = (i)^2$, $1 \leq i \leq n$, $f^*(e_i{}^{(2)}) = (n+i+1)^2$, $1 \leq i \leq n$ and $f^*(e) = (n+1)^2$.

The largest edge label of $f^*(e_i^{(1)})$ is smaller than the smallest edge label of $f^*(e_i{}^{(2)})$.

If $\{f^*(e_i^{(1)}), 1 \leq i \leq n\} = \{f^*(e)\}$ for some i, then $i = n+1$ or $i = -n-1$, which contradicts with the choice of i, as $i \in \mathbb{N}$.

Further the smallest edge label of $f^*(e_i{}^{(2)})$ is larger than $f^*(e)$. Therefore $\{f^*(e_i^{(2)}); 1 \leq i \leq n\} \neq f^*(e)$.

So $f^* : E(K_{1,n} + K_1) \to \mathbb{N}$ is injective.

Hence $K_{1,n} + K_1$ is sum perfect square graph. □

The below illustration provides the better idea of the above defined labeling pattern.

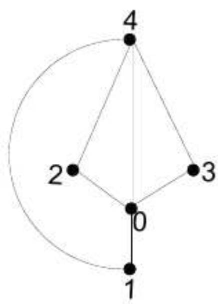

Figure 6 : Sum perfect square labeling of $K_{1,3} + K_1$.

Theorem 2.8. $K_2 + mK_1$ is sum perfect square graph, $\forall m \in \mathbb{N}$.

Proof. Let $V(K_2 + mK_1) = \{u_1, u_2\} \cup \{v_i; 1 \leq i \leq m\}$, where $\{u_1, u_2\}$ be the vetex set of K_2. $E(K_2 + mK_1) = \{e = u_1u_2\} \cup \{e_i^{(1)} = u_1v_i; 1 \leq i \leq m\} \cup \{e_i^{(2)} = u_2v_i; 1 \leq i \leq m\}$. Here $|V(K_2 + mK_1)| = m+2$ and $|E(K_2 + mK_1)| = 2m+1$.

We define the bijection $f : V(K_2 + mK_1) \to \{0, 1, 2, \ldots, m+1\}$ as $f(u_1) = 0$, $f(u_2) = m+1$ and $f(v_i) = i$, $1 \leq i \leq m$.

Let $f^* : E(K_2 + mK_1) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(K_2 + mK_1)$.

Injectivity for edge labels: For $1 \leq i \leq m$, since $f(v_i)$ is increasing in terms of $i \Rightarrow f^*(u_jv_i) < f^*(u_jv_{i+1})$, $j = 1, 2$, $1 \leq i \leq m-1$.

Claim: $f^*(e) \neq \{f^*(e_i^{(1)}); 1 \leq i \leq m\} \neq \{f^*(e_i^{(2)}); 1 \leq i \leq m\}$.

We have $f^*(e) = (m+1)^2$, $f^*(e_i{}^{(1)}) = (i)^2$, $f^*(e_i^{(2)}) = (m+i+1)^2$, $1 \leq i \leq m$.

The largest edge label of $f^*(e_i{}^{(1)})$ is smaller than the smallest edge label of $f^*(e_i{}^{(2)})$. Also $f^*(e)$ is larger than the highest edge label of $f^*(e_i{}^{(1)})$ and smaller than the smallest edge label of $f^*(e_i{}^{(2)})$. Hence the claim is proved. So the induced edge labeling $f^* : E(K_2 + mK_1) \to \mathbb{N}$ is injective. So $K_2 + mK_1$ is sum perfect square graph, $\forall m \in \mathbb{N}$. □

The below illustration provides the better idea of the above defined labeling pattern.

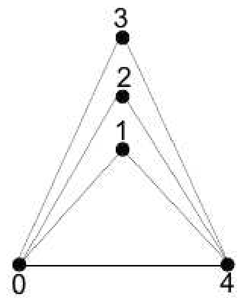

Figure 7 : Sum perfect square labeling of $K_2 + 3K_1$.

3. Conclusion

In this paper a conjecture related to sum perfect square graph have been proved, a new graph called half wheel have been presented and various sum perfect square graphs are found.

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