Abstract: In a strictly two-sided commutative quantale having an order reversing involution, we introduce the notion of $L$-fuzzy $(K, E)$-soft pre-proximity spaces and $L$-fuzzy $(K, E)$-soft pre-uniform spaces. We investigate their properties. In particular, an $L$-fuzzy $(K, E)$-soft pre-uniformity induces an $L$-fuzzy $(K, E)$-soft pre-proximity. We give their examples.

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1. Introduction

In 1999 Molodtsov [14] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. In [15], Molodtsov applied successfully in directions such as, smoothness of func-
tions, game theory, operations research, Riemann-integration, Perron integration, probability and theory of measurement.

Maji et al. [12,13] gave the first practical application of soft sets in decision making problems. In 2003, Maji et al. [13] defined and studied several basic notions of soft set theory. Many researchers have contributed towards the algebraic structure of soft set theory [1-3,7]. In 2011, Shabir and Naz [19] initiated the study of soft topological spaces. They defined soft topology on the collection of soft sets over X and established their several properties. Aygınoglu et.al [4] introduced the concept of soft topology in the sense of Šostak [10]. Çetkin et.al [5] studied soft proximities and discuss their properties.

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices. Höhle [9,10] introduced $L$-fuzzy topologies with algebraic structure $L(cqm, quantales, MV$-algebra). It has developed in many directions [17-19].

The notion of an $L$-fuzzy pre-proximity spaces, $L$ is a strictly two sided commutative quantale lattice having a strong negation was introduced by Kim et al. [11]. Lather, Ramadan et al. [17,18] define the the concept of $L$- fuzzy soft topogenous orders, $L$- fuzzy soft uniform spaces, $L$- fuzzy soft topological spaces in strictly two sided commutative quantales and investigated the relation between them.

The purpose of this paper is to introduce the notion of $L$-fuzzy $(K, E)$-soft pre-proximity spaces and $L$-fuzzy $(K, E)$-soft pre-uniform spaces. We investigate their properties. In particular, an $L$-fuzzy $(K, E)$-soft pre-uniformity induces an $L$-fuzzy $(K, E)$-soft pre-proximity. We give their examples.

2. Preliminaries

Let $L = (L, \le, \lor, \land, 0, 1)$ be a completely distributive lattice with the least element 0 and the greatest element 1 in $L$.

Definition 2.1. [9-11] A complete lattice $(L, \le, \circ)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L, \circ)$ is a commutative semigroup,
(L2) $x = x \circ 1$, for each $x \in L$ and 1 is the universal upper bound,
(L3) $\circ$ is distributive over arbitrary joins, i.e. $(\bigvee_i x_i) \circ y = \bigvee_i (x_i \circ y)$.

There exists a further binary operation $\to$ (called the implication operator or residuated) satisfying the following condition

$$x \to y = \bigvee \{z \in L | x \circ z \le y\}.$$
Then it satisfies Galois correspondence; i.e., \((x \odot z) \leq y\) iff \(z \leq (x \rightarrow y)\).

In this paper, we always assume that \((L, \leq, \odot, \rightarrow, \oplus, *)\) is a stsc-quantales with an order reversing involution \(*\) which is defined by

\[ x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \rightarrow 0 \]

unless otherwise specified.

**Remark 2.2.** Every completely distributive lattice \((L, \leq, \land, \lor, *)\) with order reversing involution \(*\) is a stsc-quantale \((L, \leq, \odot, \oplus, *\) with a strong negation \(*\) where \(\odot = \land\) and \(\oplus = \lor\).

**Lemma 2.3.** [9-11] For each \(x, y, z, x_i, y_i, w \in L\), we have the following properties.

1. \(1 \rightarrow x = x, 0 \odot x = 0\),
2. If \(y \leq z\), then \(x \odot y \leq x \odot z, x \oplus y \leq x \oplus z, x \rightarrow y \leq x \rightarrow z\) and \(z \rightarrow x \leq y \rightarrow x\),
3. \(x \leq y\) iff \(x \rightarrow y = 1\),
4. \((\bigwedge_i y_i)^* = \bigvee_i y_i^*, (\bigvee_i y_i)^* = \bigwedge_i y_i^*\),
5. \(x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)\),
6. \((\bigvee_i x_i) \rightarrow y = \bigvee_i (x_i \rightarrow y)\),
7. \(x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)\),
8. \((\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)\),
9. \((x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)\),
10. \((x \odot y) = (x \rightarrow y)^* + x \oplus y = x^* \rightarrow y\),
11. \((x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)\),
12. \((x \rightarrow y) \leq (x \odot z) \rightarrow (y \odot z)\) and \((x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z\),
13. \((x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)\),
14. \((x \rightarrow y) = y^* \rightarrow x^*\),
15. \((x \lor y) \odot (z \lor w) \leq (x \lor z) \lor (y \lor w) \leq (x \lor z) \odot (y \lor w)\),
16. \(\bigvee_{i \in \Gamma} x_i \rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)\) and \(\bigwedge_{i \in \Gamma} x_i \rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)\),
17. \((x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)\).

Throughout this paper, \(X\) refers to an initial universe, \(E\) and \(K\) are the sets of all parameters for \(X\), and \(L^X\) is the set of all \(L\)-fuzzy sets on \(X\).

**Definition 2.4.** [4] A map \(f\) is called an \(L\)-fuzzy soft set on \(X\), where \(f\) is a mapping from \(E\) into \(L^X\), i.e., \(f_e := f(e)\) is an \(L\)-fuzzy set on \(X\), for each \(e \in E\). The family of all \(L\)-fuzzy soft sets on \(X\) is denoted by \((L^X)^E\). Let \(f\) and \(g\) be two \(L\)-fuzzy soft sets on \(X\).
(1) \( f \) is an \( L \)-fuzzy soft subset of \( g \) and we write \( f \subseteq g \) if \( f_e \leq g_e \), for each \( e \in E \). \( f \) and \( g \) are equal if \( f \subseteq g \) and \( g \subseteq f \).

(2) The intersection of \( f \) and \( g \) is an \( L \)-fuzzy soft set \( h = f \cap g \), where \( h_e = f_e \land g_e \), for each \( e \in E \).

(3) The union of \( f \) and \( g \) is an \( L \)-fuzzy soft set \( h = f \cup g \), where \( h_e = f_e \lor g_e \), for each \( e \in E \).

(4) An \( L \)-fuzzy soft set \( h = f \ob c g \) is defined as \( h_e = f_e \ob c g_e \), for each \( e \in E \).

(5) An \( L \)-fuzzy soft set \( h = f \ob f g \) is defined as \( h_e = f_e \ob f g_e \), for each \( e \in E \).

(6) The complement of an \( L \)-fuzzy soft sets on \( X \) is denoted by \( f^* \), where \( f^* : E \to L^X \) is a mapping given by \( f_e^* = (f_e)^* \), for each \( e \in E \).

(7) \( f \) is called a null \( L \)-fuzzy soft set and is denoted by \( 0_X \), if \( f_e(x) = 0 \), for each \( e \in E \), \( x \in X \).

(8) \( f \) is called an absolute \( L \)-fuzzy soft set and is denoted by \( 1_X \), if \( f_e(x) = 1 \), for each \( e \in E \), \( x \in X \) and \( (1_x)e(x) = 1 \).

**Definition 2.5.** [4] Let \( \varphi : X \to Y \) and \( \psi : E \to K \) be two mappings, where \( E \) and \( K \) are parameters sets for the crisp sets \( X \) and \( Y \), respectively. Then \( \varphi \psi : (X, E) \to (Y, K) \) is called a fuzzy soft mapping. Let \( f \) and \( g \) be two fuzzy soft sets over \( X \) and \( Y \), respectively and let \( \varphi \psi \) be a fuzzy soft mapping from \( (X, E) \) into \( (Y, K) \).

(1) The image of \( f \) under the fuzzy soft mapping \( \varphi \psi \), denoted by \( \varphi \psi(f) \) is the fuzzy soft set on \( Y \) defined by

\[
\varphi(f)k(y) = \begin{cases} 
    \bigvee\varphi(x)=y \left( \bigvee\psi(e)=k \ f_e(x) \right), & \text{if } x \in \varphi^{-1}(y) \\
    0, & \text{otherwise},
\end{cases}
\]

\( \forall k \in K, \forall y \in Y \).

(2) The pre-image of \( g \) under the fuzzy soft mapping \( \varphi \psi \), denoted by \( \varphi^{-1}(g) \) is the fuzzy soft set on \( X \) defined by

\[
\varphi^{-1}(g)e(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.
\]

**Definition 2.6.** [16] An \( L \)-fuzzy \((K, E)\)-soft pre-uniformity is a mapping \( U : K \to L^{(X \times X)} \) which satisfies the following conditions .

(SU1) There exists \( u \in (L^{X \times X})^E \) such that \( U_k(u) = 1 \).

(SU2) If \( v \subseteq u \), then \( U_k(v) \leq U_k(u) \).

(SU3) For every \( u, v \in (L^{X \times X})^E \), \( U_k(u \odot v) \geq U_k(u) \odot U_k(v) \).

(SU4) If \( U_k(u) \neq 0 \) then \( T_\Delta \subseteq u \) where, for each \( e \in E \),

\[
(T_\Delta)e(x, y) = \begin{cases} 
    1, & \text{if } x = y, \\
    0, & \text{if } x \neq y.
\end{cases}
\]
The pair \((X, \mathcal{U})\) is called an \(L\)-fuzzy \((K, E)\)-soft pre-uniform space.

An \(L\)-fuzzy \((K, E)\)-soft pre-uniformity is an \(L\)-fuzzy \((K, E)\)-soft quasi-uniformity if
\[(UQ) \quad \forall u \in (L^{X \times X} E)^K, \quad \mathcal{U}_k(u) \subseteq \bigvee \{ \mathcal{U}_k(v) \cap \mathcal{U}_k(w) \mid v \circ w \subseteq u \}, \]
where
\[v_e \circ w_e(x, z) = \bigvee_{y \in X} v_e(x, y) \cap w_e(y, z),\]

An \(L\)-fuzzy \((K, E)\)-soft quasi-uniform space \((X, \mathcal{U})\) is said to be an \(L\)-fuzzy \((K, E)\)-soft uniform space if
\[(U) \quad \forall u \in (L^{X \times X} E)^K, \quad \mathcal{U}_k(u) \subseteq \mathcal{U}_k(u^{-1}), \quad \text{where} \quad (u^{-1})_e(x, y) = u_e(y, x) \quad \text{for each} \quad k \in K \quad \text{and} \quad u \in (L^{X \times X} E)^K.

Let \((X, \mathcal{U}^1)\) be an \(L\)-fuzzy \((K_1, E_1)\)-soft pre-uniform space and \((Y, \mathcal{U}^2)\) be an \(L\)-fuzzy \((K_2, E_2)\)-soft pre-uniform space. Let \(\varphi : X \to Y\), \(\psi : E_1 \to E_2\) and \(\eta : K_1 \to K_2\) be mappings. Then \(\varphi, \psi, \eta\) from \((X, \mathcal{U}^1)\) into \((Y, \mathcal{U}^2)\) is called \(L\)-fuzzy soft uniformly continuous if
\[\mathcal{U}^2_{\eta(k)}(v) \subseteq \mathcal{U}^1_k((\varphi \times \varphi)^{-1}(v)) \quad \forall v \in (L^{Y \times Y}) E^2, k \in K_1.

**Remark 2.7.** Let \((X, \mathcal{U})\) be an \(L\)-fuzzy \((K, E)\)-soft uniform space.

1. By \((SU1)\) and \((SU2)\), we have \(\mathcal{U}_k(1_{X \times X}) = 1\) because \(u \subseteq 1_{X \times X}\) for all \(u \in (L^{X \times X} E)^K\).
2. Since \(\mathcal{U}_k(u) \subseteq \mathcal{U}_k(u^{-1}) \subseteq \mathcal{U}_k((u^{-1})^{-1}) = \mathcal{U}_k(u)\), then \(\mathcal{U}_k(u) = \mathcal{U}_k(u^{-1})\).

### 3. \(L\)-fuzzy \((K, E)\)-soft pre-proximities induced by \(L\)-fuzzy \((K, E)\)-soft pre-uniformities

**Definition 3.1.** A mapping \(\delta : K \to L^{(L^X) E \times (L^X) E} \delta_k = \delta(k) : (L^X) E \times (L^X) E \to L\) is called an \(L\)-fuzzy \((K, E)\)-soft pre-proximity on \(X\) if it satisfies the following axioms.
\[(SP1) \quad \delta_k(0, 1_X) = \delta_k(1_X, 0) = 0.
\[(SP2) \quad \text{If} \ \delta_k(f, g) \neq 1, \ \text{then} \ f \subseteq g^*.
\[(SP3) \quad \text{If} \ f_1 \subseteq f_2 \text{ and } g_1 \subseteq g_2, \ \text{then} \ \delta_k(f_1, g_1) \leq \delta_k(f_2, g_2).
\[(SP4) \quad \delta_k(f_1 \odot f_2, g_1 \oplus g_2) \leq \delta_k(f_1, g_1) \oplus \delta_k(f_2, g_2).

The pair \((X, \delta)\) is called an \(L\)-fuzzy \((K, E)\)-soft pre-proximity space.

An \(L\)-fuzzy \((K, E)\)-soft pre-proximity is called an \(L\)-fuzzy \((K, E)\)-soft quasi-proximity on \(X\) if
\[(PQ) \quad \delta_k(f, g) \geq \bigwedge_{h} \{ \delta_k(f, h) \oplus \delta_k(h^*, g) \}.\]
An $L$-fuzzy $(K,E)$-soft quasi-proximity is called an $L$-fuzzy $(K,E)$-soft proximity on $X$ if

$$(SP) \quad \delta_k(f, g) = \delta_k(g, f).$$

Let $(X, \delta^1)$ be an $L$-fuzzy $(K_1, E_1)$-soft quasi proximity space and $(Y, \delta^2)$ be an $L$-fuzzy $(K_2, E_2)$-soft pre-proximity space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi, \eta}$ from $(X, \delta^1)$ into $(Y, \delta^2)$ is called $L$-fuzzy soft proximally continuous if

$$\delta_k^1(f, g) \leq \delta^2_{\eta(k)}(\varphi_{\psi}(f), \varphi_{\psi}(g)) \quad \forall f, g \in (L^X)^{E_1}, k \in K_1.$$

or equivalently,

$$\delta_k^1(\varphi_{\psi}^{-1}(f), \varphi_{\psi}^{-1}(g)) \leq \delta^2_{\eta(k)}(f, g) \quad \forall f, g \in (L^Y)^{E_2}, k \in K_1.$$

**Theorem 3.2.** Let $(X, \mathcal{U})$ be an $L$-fuzzy $(K,E)$-soft pre-uniform space. Define a mapping $\delta^U : K \to L^{(L^X)^E \times (L^X)^E}$ by

$$\delta_k^U(f, g) = \bigwedge \{ U_k^*(u) \mid u[f] \subseteq g^* \},$$

where $u_e[f_e](x) = \bigvee_{y \in X} (f_e(y) \circ u_e(y, x))$, $\forall x \in X, \forall e \in E$, $\forall u \in (L^{X \times X})^E$ and $f \in (L^X)^E$. Then we have the following properties.

1. $\delta^U$ is an $L$-fuzzy $(K,E)$-soft pre-proximity space.

2. If $(X, \mathcal{U})$ is an $L$-fuzzy $(K,E)$-soft quasi-uniform space, then $\delta^U$ is $L$-fuzzy $(K,E)$-soft quasi-proximity space.

**Proof.**

(1) (SP1) Since $u[0_X] = 0_X$ and $u[1_X] = 1_X$ for $U_k(u) = 1$, we have $\delta_k^U(0_X, 1_X) = 0$ and $\delta_k^U(0_X, 1_X) = 0$.

(SP2) Let $f \nsubseteq g^*$ be given. Since $f \subseteq u[f]$ for all $U_k(u) > 0$, we have $u[f] \nsubseteq g^*$. By the definition of $\delta_k^U$, we have $\delta_k^U(f, g) = 1$.

(SP3) Easily proved.

(SP4) First we show that $(u \circ v)[f \circ g] \subseteq u[f] \circ v[g]$, from:

$$u_e[f_e](x) \circ v_e[g_e](x) = (\bigvee_{y \in X} (f_e(y) \circ u_e(y, x))) \circ (\bigvee_{z \in X} (g_e(z) \circ v_e(z, x)))$$

$$\geq \bigvee_{y \in X} (f_e(y) \circ u_e(y, x)) \circ (g_e(y) \circ v_e(y, x))$$

$$= \bigvee_{y \in X} ((f_e \circ g_e)(y) \circ (u_e \circ v_e)(y, x))$$

$$= (u_e \circ v_e)[f_e \circ g_e](x).$$
\[ \delta^L_k(f_1, g_1) \oplus \delta^L_k(f_2, g_2) = \bigwedge \{ U^+_k(u) \mid u[f_1] \subseteq g_1 \} \lor \bigwedge \{ U^+_k(v) \mid v[f_2] \subseteq g_2 \} \]
\[ = \bigwedge \{ U^+_k(u) \oplus U^+_k(v) \mid u[f_1] \subseteq g_1, v[f_2] \subseteq g_2 \} \]
\[ \geq \bigwedge \{ U^+_k(u) \mid u[f_1] \subseteq g_1 \} \oplus \bigwedge \{ U^+_k(v) \mid v[f_2] \subseteq g_2 \} \]
\[ \geq \bigwedge \{ U^+_k(u \circ v) \mid (u \circ v)[f_1 \circ f_2] \subseteq (g_1 \oplus g_2)^* \} \]
\[ \geq \bigwedge \{ U^+_k(w) \mid w[f_1 \circ f_2] \subseteq (g_1 \oplus g_2)^* \} \]
\[ = \delta^L_k(f_1 \circ f_2, g_1 \oplus g_2). \]

Hence, \( \delta^L \) is an \( L \)-fuzzy \((K, E)\)-soft pre-proximity space.

(2) (PQ) For each \( u \in (L^{X \times X})^E \) such that \( u[f] \subseteq g^* \), by (UQ), we have
\[ U_k(u) \leq \bigvee \{ U_k(v) \circ U_k(w) \mid v \circ w \subseteq u \}. \]

Thus,
\[ \delta^L_k(f, g) = \bigwedge \{ U^+_k(u) \mid u[f] \subseteq g^* \} \]
\[ \geq \bigwedge \{ U^+_k(v) \bigoplus U^+_k(w) \mid v[w[f]] \subseteq g^* \} \]
\[ = \bigwedge \{ U^+_k(v) \bigoplus U^+_k(w) \mid w[f] = (w[f]^*)^*, v[w[f]] \subseteq g^* \} \]
\[ \geq \bigwedge_{h \in L_X} \{ \bigwedge \{ U^+_k(u) \mid u[h] \subseteq h^* \} \bigoplus \bigwedge \{ U^+_k(v) \mid v[h^*] \subseteq g^* \} \} \]
\[ = \bigwedge_{h \in L_X} (\delta^L_k(f, h) \bigoplus \delta^L_k(h^*, g)). \]

Hence, \( \delta^L_k(f, g) = \bigwedge \{ U^+_k(u) \mid u[f] \subseteq g^* \} \geq \bigwedge_{h \in L_X} (\delta^L_k(f, h) \bigoplus \delta^L_k(h^*, g)). \)

**Theorem 3.3.** Let \((X, U)\) be an \( L \)-fuzzy \((K_1, E_1)\)-soft quasi-uniform space and \((Y, V)\) be an \( L \)-fuzzy \((K_2, E_2)\)-soft quasi-uniform space, \( \phi : X \rightarrow Y \), \( \psi : E_1 \rightarrow E_2 \) and \( \eta : K_1 \rightarrow K_2 \) are functions. If \( \phi_{\psi, \eta} \) from \((X, U)\) into \((X, V)\) is \( L \)-fuzzy soft uniformly continuous, then \( \phi_{\psi, \eta} : (X, \delta^L) \rightarrow (Y, \delta^V) \) is \( L \)-fuzzy soft proximally continuous.

**Proof.** Since
\[ (\phi \times \phi)^{-1}_\psi(v_e) \subseteq ([\phi^{-1}_\psi(f)]_e)(x) \]
\[ = \bigvee_{z \in X} \phi^{-1}_\psi(f)_e(z) \circ (\phi \times \phi)^{-1}_\psi(v_e)(z, x) \]
\[ = \bigvee_{z \in X} f_\psi(e)(\phi(z)) \circ v_\psi(e)(\phi(z), \phi(x)) \]
\[ \leq \bigvee_{y \in Y} f_\psi(e)(y) \circ v_\psi(e)(y, \phi(x)) \]
\[ = \phi^{-1}_\psi(v_\psi(e)[f_\psi(e)])(x), \]
we have
\[
\delta^\gamma_{\eta(k)}(f, h) = \bigwedge \{ \mathcal{V}^*_\eta(k)(v) \mid v[f] \leq h^* \}
\]
\[
\geq \bigwedge \{ \mathcal{U}^*_k((\phi \times \phi)^{-1}_\psi(v)) \mid \phi^{-1}(v)[f] \leq \phi^{-1}(h)^* \}
\]
\[
\geq \bigwedge \{ \mathcal{U}^*_k((\phi \times \phi)^{-1}_\psi(v)) \mid (\phi \times \phi)^{-1}(v)[\phi^{-1}(f)] \leq \phi^{-1}(h)^* \}
\]
\[
\geq \bigwedge \{ \mathcal{U}^*_k(u) \mid u[\phi^{-1}(f)] \leq \phi^{-1}(h)^* \} = \delta^\gamma_k(\phi^{-1}(f), \phi^{-1}(h)).
\]

Hence, \( \phi_{\psi, \eta} : (X, \delta^U) \to (Y, \delta^V) \) is \( L \)-fuzzy soft proximally continuous.

**Example 3.4.** Let \( X = \{ h_i \mid i = \{1, 2, 3\} \} \) with \( h_i = \) house and \( E = \{ e, b \} \) with \( e = \) expensive, \( b = \) beautiful. Define a binary operation \( \odot \) on \([0, 1] \) by

\[
x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}
\]
\[
x \odot y = \min\{1, x + y\}, \quad x^* = 1 - x
\]

Then \( ([0, 1], \wedge, \rightarrow, 0, 1) \) is a stsc-quantle.

(1) Put \( v, v \odot v, w \in ([0, 1]^{X \times X})^E \) as

\[
v_e = \begin{pmatrix}
1 & 0.6 & 0.5 \\
0.3 & 1 & 0.5 \\
0.4 & 0.6 & 1
\end{pmatrix}
\]
\[
v_b = \begin{pmatrix}
0.7 & 1 & 0.5 \\
0.6 & 0.6 & 1
\end{pmatrix}
\]
\[
(v \odot v)_e = \begin{pmatrix}
1 & 0.2 & 0 \\
0 & 1 & 0 \\
0 & 0.2 & 1
\end{pmatrix}
\]
\[
(v \odot v)_b = \begin{pmatrix}
1 & 0 & 0 \\
0.4 & 1 & 0 \\
0.2 & 0.2 & 1
\end{pmatrix}
\]
\[
w_e = \begin{pmatrix}
1 & 0.4 & 0.5 \\
0.4 & 1 & 0.5 \\
0.4 & 0.6 & 1
\end{pmatrix}
\]
\[
w_b = \begin{pmatrix}
1 & 0.5 & 0.3 \\
0.3 & 1 & 0.5 \\
0.2 & 0.3 & 1
\end{pmatrix}
\]

We define \( \mathcal{U} : E \to [0, 1]^{([0,1]^{X \times X})^E} \) as follows:

\[
\mathcal{U}_e(u) = \begin{cases}
1, & \text{if } u = 1_{Y \times Y} \\
0.6, & \text{if } v \subseteq u \neq 1_{Y \times Y}, \\
0.3, & \text{if } v \odot v \subseteq u \nsubseteq v, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
\mathcal{U}_b(u) = \begin{cases}
1, & \text{if } u = 1_{Y \times Y} \\
0.5, & \text{if } w \subseteq u \neq 1_{Y \times Y}, \\
0, & \text{otherwise}.
\end{cases}
\]
Since \( v \circ v = v, w \circ w = w \) and \( (v \circ v) \circ (v \circ v) = (v \circ v) \), \( \mathcal{U} \) is not a \([0, 1]\)-fuzzy \((E, E)\)-soft quasi-uniformity on \( X \) because

\[
0.6 = \mathcal{U}_k(v) \not\leq \bigvee \{ \mathcal{U}_k(u) \circ \mathcal{U}_k(w) \mid u \circ w \sqsubseteq v \} = 0.2.
\]

From Theorem 3.2, we obtain a \([0, 1]\)-fuzzy \((E, E)\)-soft pre-proximity \( \delta^\mathcal{U} : E \rightarrow [0, 1]^{([0,1]^X)^E} \times ([0,1]^X)^E \) as follows

\[
\delta^\mathcal{U}_c(f, g) = \begin{cases} 
0, & \text{if } [1_{X \times X}](f) \sqsubseteq g^* \not\sqsupseteq v[f], \\
0.4, & \text{if } v[f] \sqsubseteq g^* \not\sqsupseteq (v \circ v)[f], \\
0.7, & \text{if } (v \circ v)[f] \sqsubseteq g^*, \\
1, & \text{otherwise,}
\end{cases}
\]

\[
\delta^\mathcal{U}_b(f, g) = \begin{cases} 
0, & \text{if } [1_{X \times X}](f) \sqsubseteq g^* \not\sqsupseteq w[f], \\
0.5, & \text{if } w[f] \sqsubseteq g^*, \\
1, & \text{otherwise,}
\end{cases}
\]

But it is not a \([0, 1]\)-fuzzy \((E, E)\) soft quasi-proximity because, for each \( 0_X \neq f \in ([0,1]^X)^E \) with \( w[f] \leq g^* \neq 1_X \), \( \delta^\mathcal{U}_b(f, h) \neq 0 \) and \( \delta^\mathcal{U}_b(h^*, g) \neq 0 \) imply

\[
0.5 = \delta^\mathcal{U}_b(f, g) \not\leq \bigwedge_{h \in ([0,1]^X)^E} (\delta^\mathcal{U}_b(f, h) \oplus \delta^\mathcal{U}_b(h^*, g)).
\]

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References


