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THE FEKETE-SZEGÖ PROBLEM FOR A SUBCLASS OF ANALYTIC FUNCTIONS RELATED TO SIGMOID FUNCTION

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Abstract: Sigmoid function is an novel concept in the area of univalent function theory. Recently (O. A. Fadipe-Joseph , 2013) introduced and studied the sigmoid function and few classes were discussed in past few years . In this present work, we obtain the first few coefficients of the class and estimates the relevant connection to the famous classical Fekete-Szegö inequality of functions belonging to the class. The authors sincerely hope that this article will revive this concept and encourage other researchers to work in this sigmoid function.

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1. Introduction and preliminaries

The theory of a special function does not have a general definition but it is of eventual importance to scientist and engineers who are concerned with actual mathematical calculations and have a extensive application in specific problems of physics, engineering and computer science.

The theory of special functions has been developed by C. F.Gauss, C. G. J. Jacobi, F. Klein and many others in nineteenth century. However, in the twenti-

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eth century the theory of special functions has been outstripping by other fields like real analysis, functional analysis, topology, algebra, differential equations and so on.

Example of special function is activation function. Activation function acts as a squashing function, such that the output of a neuron in a neural network is between certain values (Usually 0 and 1, or -1 and 1). There are three types of activation function, namely: threshold function, piecewiselinear function and sigmoid function.

The most popular activation function in the hardware implementation of artificial neural networks (ANN) is the sigmoid function. Sigmoid function is often used with gradient descendent type learning algorithms. There are different possibilities for evaluating this function, such as truncated series expansion, look-up tables, or piecewise approximation. The sigmoid function is of the form

$$h(z) = \frac{1}{1 + e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is an injective function.
- It increases monotonically.

From the above properties, it is clear that sigmoid function plays a key role in geometric functions theory.

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$ is of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad z \in \mathbb{U} \tag{1.2}$$

and normalized $f(0) = f'(0) - 1 = 0$.Further, let $\mathcal{S}$ denote the class of analytic,normalized and univalent functions in $\mathbb{U}$.

Lemma 1. [8] If a function p $\in$ P is given by

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \quad (z \in \mathbb{U}),$$

then $|p_k| \leq 1$, $k \in N$ where P is the family of all functions analytic in $\mathbb{U}$ for which $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0, (z \in \mathbb{U})$.

In virtue of Löwners method, Fekete and Szegö [3] proved the striking result, if $f \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0 \\ 1 + 2exp\left(\frac{-2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu \leq 1 \\ 4\mu - 3 & \text{if } \mu \geq 1 \end{cases}$$

holds for function $f \in \mathcal{S}$ and the result is sharp.The problem of finding the sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ of any compact family of functions is popularly known as the Fekete-Szegö problem. Several known authors at different time have applied the classical Fekete-Szegö to various classes to obtain sharp bounds. Keogh and Merkes in 1969 [4] obtained the sharp upper bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for some subclasses of univalent function $\mathcal{S}$. Researchers like [9],[10] and [11] studied several subclasses of functions making use of Fekete-Szegö problem and the very interesting results obtained can be found in many literature.

Recently, Abiodun [1], Murugusundarmoorthy et al [5], Olantunji et al [6] and Olanji [7] have studied sigmoid function for various classes of analytic and univalent functions.

Lemma 2. [2] Let h be the sigmoid function defined in (1.1) and

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m,$$

then $\Phi(z) \in P, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 3. [2] Let

$$\Phi_{n,m} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m$$

then $|\Phi_{n,m}(z)| < 2$.

Lemma 4. [2] If $\Phi(z) \in P$ and it is starlike, then f is is a normalized univalent function of the form (1.2). Taking m = 1, Joseph et al [2] remarked the following:

Remark 1. Let

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$$

where $C_n = \frac{(-1)(-1)^n}{2n!}$ then $|c_n| \leq 2, n = 1, 2, 3 \ldots$ this result is sharp for each n.

Definition 1. For $b \in \mathbb{C}$. Let the class $\mathcal{M}_\lambda(b, \Phi_{n,m})$ denote the subclass of $\mathcal{A}$ consisting of functions f of the form (1.2), and satisfying the following subordination condition

$$1 + \frac{1}{b}\left[\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda)f(z)} - 1\right] > 0.$$

for $0 \le \lambda \le 1$ and $\Phi_{n,m}$ is a simple logistic sigmoid activation function.

Definition 2. For $b \in \mathbb{C}$. Let the class $\mathcal{G}_\lambda(b, \Phi_{n,m})$ denote the subclass of $\mathcal{A}$ consisting of functions f of the form (1.2), and satisfying the following subordination condition

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} + \lambda\frac{z^2 f''(z)}{f(z)} - 1\right] > 0.$$

for $0 \le \lambda \le 1$ and $\Phi_{n,m}$ is a simple logistic sigmoid activation function.

2. Initial Coefficients

The first few coefficient estimates for the classes of $\mathcal{M}_\lambda(b, \Phi_{n,m})$ and $\mathcal{G}_\lambda(b, \Phi_{n,m})$ are obtained in the following theorems.

Theorem 1. Let

$$\Phi_{n,m} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m$$

where $\Phi_{n,m} \in \mathcal{A}$ is a modified logistic sigmoid activation function and $\Phi'_{n,m}(0) > 0$. If $f(z)$ given by (1.2) belongs to the class $\mathcal{M}_\lambda(b, \Phi_{n,m})$ then,

$$|a_2| \le \frac{b}{2(1+\lambda)}$$

$$|a_3| \le \frac{b^2}{8(1+2\lambda)}$$

$$|a_4| \le \frac{b^3}{48(1+3\lambda)}$$

Proof. If $f \in \mathcal{M}_\lambda(b, \Phi_{n,m})$, then

$$1 + \frac{1}{b}\left[\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda)f(z)} - 1\right] = \Phi_{n,m}(z) \tag{2.1}$$

where $\Phi_{n,m}(z)$ is a modified sigmoid function given by

$$\Phi_{n,m}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 \ldots \tag{2.2}$$

In view of (2.1) and (2.2), expanding in series forms we have

$$\begin{aligned} &\frac{1}{b}\left[(1+\lambda)a_2z^2 + 2(1+2\lambda)a_3z^3 + 3(1+3\lambda)a_4z^4\right] + \ldots \\ &= \frac{1}{2}z^2 + \frac{1}{2}(1+\lambda)a_2z^3 + \frac{1}{2}(1+2\lambda)a_3z^4 + \ldots \end{aligned} \tag{2.3}$$

Comparing the coefficients of z^2, z^3 and z^4 in (2.3), we obtain

$$a_2 = \frac{b}{2(1+\lambda)} \tag{2.4}$$

$$a_3 = \frac{b^2}{8(1+2\lambda)} \tag{2.5}$$

$$a_4 = \frac{b^3}{48(1+3\lambda)} \tag{2.6}$$

□

Corollary 1. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{M}_1(b, \Phi_{n,m})$, then

$$|a_2| \leq \frac{b}{4}, \quad |a_3| \leq \frac{b^2}{24}, \quad |a_4| \leq \frac{b^3}{192}$$

Corollary 2. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{M}_0(b, \Phi_{n,m})$, then

$$|a_2| \leq \frac{b}{2}, \quad |a_3| \leq \frac{b^2}{8}, \quad |a_4| \leq \frac{b^3}{48}$$

Theorem 2. Let

$$\Phi_{n,m} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m$$

where $\Phi_{n,m} \in \mathcal{A}$ is a modified logistic sigmoid activation function and $\Phi'_{n,m}(0) > 0$. If $f(z)$ given by (1.2) belongs to the class $\mathcal{G}_\lambda(b, \Phi_{n,m})$, then

$$|a_2| \leq \frac{b}{2(1+2\lambda)}$$

$$|a_3| \leq \frac{b^2}{8(1+2\lambda)(1+3\lambda)}$$

$$|a_4| \leq \frac{b^3}{48(1+2\lambda)(1+3\lambda)(1+4\lambda)}$$

Proof. If $f \in \mathcal{G}_\lambda(b, \Phi_{n,m})$, then

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} - 1\right] = \Phi_{n,m}(z) \tag{2.7}$$

where $\Phi_{n,m}(z)$ is a modified sigmoid function given by

$$\Phi_{n,m}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 \dots \tag{2.8}$$

In view of (2.7) and (2.8), expanding in series forms we have

$$\begin{aligned} &\frac{1}{b}\left[(1+2\lambda)a_2 z^2 + 2(1+3\lambda)a_3 z^3 + 3(1+4\lambda)a_4 z^4\right] + \dots \\ &= \frac{1}{2}z^2 + \frac{1}{2}a_2 z^3 + \frac{1}{2}a_3 z^4 + \dots \end{aligned} \tag{2.9}$$

Comparing the coefficients of z^2, z^3 and z^4 in (2.3), we obtain

$$a_2 = \frac{b}{2(1+2\lambda)} \tag{2.10}$$

$$a_3 = \frac{b^2}{8(1+2\lambda)(1+3\lambda)} \tag{2.11}$$

$$a_4 = \frac{b^3}{48(1+2\lambda)(1+3\lambda)(1+4\lambda)} \tag{2.12}$$

□

Corollary 3. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{G}_1(b, \Phi_{n,m})$, then

$$|a_2| \leq \frac{b}{6}, \quad |a_3| \leq \frac{b^2}{96}, \quad |a_4| \leq \frac{b^3}{2880}$$

Corollary 4. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{G}_0(b, \Phi_{n,m}) = \mathcal{M}_0(b, \Phi_{n,m})$, then

$$|a_2| \leq \frac{b}{2}, \quad |a_3| \leq \frac{b^2}{8}, \quad |a_4| \leq \frac{b^3}{48}$$

3. The Fekete-Szegö Inequality

In this section, we prove the following Fekete-Szegö result for the function in the classes $\mathcal{M}_\lambda(b, \Phi_{n,m})$ and $\mathcal{G}_\lambda(b, \Phi_{n,m})$ with the values of a_2 and a_3.

Theorem 3. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{M}_\lambda(b, \Phi_{n,m})$, then

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(1+\lambda)} \left| \frac{(1+\lambda)}{2(1+2\lambda)} - \mu \right|$$

Proof. From (2.4) and (2.5) we get

$$a_3 - \mu a_2^2 = \frac{b^2}{8(1+2\lambda)} - \mu \left(\frac{b}{2(1+\lambda)} \right)^2$$

By simple calculation we get

$$a_3 - \mu a_2^2 = \frac{b^2}{4(1+\lambda)} \left[\frac{(1+\lambda)}{2(1+2\lambda)} - \mu \right]$$

Hence, we have

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(1+\lambda)} \left| \frac{(1+\lambda)}{2(1+2\lambda)} - \mu \right|$$

which completes the proof. □

For taking $\mu = 1$, we get

Corollary 5. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{M}_\lambda(b, \Phi_{n,m})$, then

$$|a_3 - a_2^2| \leq \frac{b^2}{4(1+\lambda)} \left| \frac{1+3\lambda}{2(1+2\lambda)} \right|$$

Theorem 4. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{G}_\lambda(b, \Phi_{n,m})$, then

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(1+2\lambda)^2} \left| \frac{(1+2\lambda)}{2(1+3\lambda)} - \mu \right|$$

Proof. From (2.10) and (2.11) we get

$$a_3 - \mu a_2^2 = \frac{b^2}{8(1+2\lambda)} - \mu \left(\frac{b}{2(1+2\lambda)} \right)^2$$

By simple calculation we get

$$a_3 - \mu a_2^2 = \frac{b^2}{4(1+2\lambda)^2} \left[\frac{(1+2\lambda)}{2(1+3\lambda)} - \mu \right]$$

Hence, we have

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(1+2\lambda)^2}\left|\frac{(1+2\lambda)}{2(1+3\lambda)} - \mu\right|$$

which completes the proof. □

For taking $\mu = 1$, we get

Corollary 6. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{G}_\lambda(b, \Phi_{n,m})$, then

$$|a_3 - a_2^2| \leq \frac{b^2}{4(1+2\lambda)^2}\left|\frac{1+4\lambda}{2(1+3\lambda)}\right|$$

4. Hankel Determinant

Theorem 5. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{M}_\lambda(b, \Phi_{n,m})$, then

$$|a_2a_4 - \mu a_3^2| \leq \frac{b^4}{32}\left|\frac{1}{3(1+\lambda)(1+3\lambda)} - \mu\frac{1}{2(1+2\lambda)^2}\right|$$

Proof. From (2.4),(2.5) and (2.6) we get

$$a_2a_4 = \frac{b^4}{96(1+\lambda)(1+3\lambda)}$$

$$a_2a_4 - \mu a_3^2 = \frac{b^4}{96(1+\lambda)(1+3\lambda)} - \mu\frac{b^4}{64(1+2\lambda)^2},$$

then

$$a_2a_4 - \mu a_3^2 = \frac{b^4}{32}\left[\frac{1}{3(1+\lambda)(1+3\lambda)} - \mu\frac{1}{2(1+2\lambda)^2}\right]$$

which completes the proof. □

For taking $\mu = 1$, we get

Corollary 7. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{M}_\lambda(b, \Phi_{n,m})$, then

$$|a_2a_4 - \mu a_3^2| = \frac{b^4}{32}\left|\frac{1}{3(1+\lambda)(1+3\lambda)} - \frac{1}{2(1+2\lambda)^2}\right|$$

Theorem 6. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{G}_\lambda(b, \Phi_{n,m})$, then

$$|a_2a_4 - \mu a_3^2| \leq \frac{b^4}{16(1+2\lambda)^2(1+3\lambda)^2}\left|\frac{1+3\lambda}{6(1+4\lambda)} - \frac{\mu}{4}\right|$$

Proof. From (2.10),(2.11) and (2.12) we get

$$a_2a_4 = \frac{b^4}{96(1+2\lambda)^2(1+3\lambda)(+4\lambda)}$$

$$a_2a_4 - \mu a_3^2 = \frac{b^4}{96(1+\lambda)(1+3\lambda)} - \mu\frac{b^4}{64(1+2\lambda)^2(1+3\lambda)^2},$$

then

$$|a_2a_4 - \mu a_3^2| \leq \frac{b^4}{16(1+2\lambda)^2(1+3\lambda)^2}\left[\frac{1+3\lambda}{6(1+4\lambda)} - \frac{\mu}{4}\right]$$

which completes the proof. ☐

For taking $\mu = 1$, we get

Corollary 8. If $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{G}_\lambda(b, \Phi_{n,m})$, then

$$|a_2a_4 - \mu a_3^2| \leq \frac{b^4}{16(1+2\lambda)^2(1+3\lambda)^2}\left|\frac{1+3\lambda}{6(1+4\lambda)} - \frac{1}{4}\right|$$

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