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STABILITY OF AN ADDITIVE-QUARTIC FUNCTIONAL EQUATION IN ORTHOGONALITY SPACES

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Abstract: Using the fixed point method, we prove the Hyers-Ulam stability of the following additive-quartic functional equation

$$f(x+2y)+f(x-2y)=f(2x+y)+f(2x-y)-f(2x)-7[f(x)+f(-x)] +15[f(y)+f(-y)], \quad \forall x,y \text{ with } x\perp y, \tag{0.1}$$

in orthogonality spaces. Here $\perp$ is the orthogonality in the sense of Rätz.

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1. Introduction

Assume that X is a real inner product space and $f : X \to R$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x)+f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus

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orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

The stability problem of functional equations originated from a question of Ulam [37] in 1940, concerning the stability of group homomorphisms in metric groups. Let $(G_1, \cdot)$ be a group and let $(G_2, \circ)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) \circ (y)) < \epsilon$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \delta$ for all $x \in G_1$? The case of approximately additive functions was solved by Hyers [14] under the assumption that G_1 and G_2 are Banach spaces. In 1951 and in 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Aoki [1] and Rassias [33]. The stability problem of functional equations has been extensively investigated by some mathematicians (see [5, 17, 24, 25, 30, 34]).

There are several orthogonality notations on a real normed space are available. But here, we present the orthogonality concept introduced by R*ä*tz [35].

Definition 1.1. ([35]) A real vector space X is called an orthogonality vector space if there is a relation $x \perp y$ on X such that

(i) $x \perp 0$, $0 \perp x$ for all $x \in X$;

(ii) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x \perp y$, $ax \perp by$ for all $a, b \in \mathbb{R}$;

(iv) if P is a two-dimensional subspace of X, then
(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$,
(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0$, $0 \perp x$ for all x, and define $x \perp y$ if and only if x, y are linearly independent for nonzero vectors x, y. The relation $\perp$ is called symmetic if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$.

The pair $(X, \perp)$ is called an orthogonality space. It becomes an orthogonality normed space when the orthogonality space is equipped with a norm.

The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), x \perp y \tag{1.1}$$

in which $\perp$ is an abstract orthogonally was first investigated by Gudder and Strawther [13]. Ger and Sikorska discussed the orthogonal stability of the equation (1.1) in [12].

The orthogonally quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), x\perp y$$

was first investigated by F. Vajzovic [38] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljevic [7], Fochi [11], Moslehian [22, 23] and Szab [36] generalized this result.

Let X be a set. A function $d : X \to X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) = d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.2. ([2, 6]) *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \quad \forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point $y*$ of J;

(3) $y*$ is the unique fixed point of J in the set $Y = \{y \in X/d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [15] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors ([3, 4, 8, 9, 10, 18, 21, 26, 27, 28, 29, 31, 32]).

In [16], Jun and Kim considered the following additive functional equation

$$f(x+2y)+f(x-2y)=f(2x+y)+f(2x-y)-f(2x). \tag{1.2}$$

It is easy to show that the function $f(x) = x$ satisfies the functional equation (1.2), which is called an additive functional equation and every solution of the additive functional equation is said to be an additive mapping.

In [19], Lee et al. considered the following quartic functional equation

$$f(x+2y)+f(x-2y)=f(2x+y)+f(2x-y)-f(2x)-14f(x)+30f(y). \tag{1.3}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.3) , which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the orthogonally additive-quartic functional equation (0.1) in orthogonality spaces for an odd mapping.

In Section 3, we prove the Hyers-Ulam stability of the orthogonally additive-quartic functional equation (0.1) in orthogonality spaces for an even mapping.

Throughout this paper, assume that $(X,\perp)$ is an orthogonality space and that $(Y,\|\cdot\|_Y)$ is a real Banach space.

2. Stability of the Orthogonally Additive-Quartic Functional Equation: An Odd Mapping Case

In this section, we deal with the stability problem for the orthogonally additive-quartic functional equation

$$\begin{aligned} Df(x,y) := f(x+2y)+f(x-2y)-f(2x+y)-f(2x-y)+f(2x) \\ +7[f(x)+f(-x)]-15[f(y)+f(-y)] \end{aligned}$$

for all $x,y\in X$ with $x\perp y$: an odd mapping case.

Definition 2.1. An odd mapping $f:X\to Y$ is called an orthogonally additive mapping if

$$f(x+2y)+f(x-2y)=f(2x+y)+f(2x-y)-f(2x)$$

for all $x,y\in X$ with $x\perp y$.

Theorem 2.2. *Let $\phi:X^2\to[0,\infty)$ be a function such that there exists an $\alpha<1$ with*

$$\varphi(x,y)\le 2\alpha\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{2.1}$$

for all $x,y\in X$ with $x\perp y$. Let $f:X\to Y$ be an odd mapping satisfying

$$\|Df(x,y)\|_Y\le\varphi(x,y) \tag{2.2}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ Such that

$$\|f(x) - A(x)\|_Y \leq \frac{1}{2-2\alpha}\varphi(x, 0) \tag{2.3}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.2), we get

$$\|2f(x) - f(2x)\|_Y \leq \varphi(x, 0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| f(x) - \frac{1}{2}f(2x) \right\|_Y \leq \frac{1}{2}\varphi(x, 0) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = inf\left\{\mu \in R_+ : \|g(x) - h(x)\|_Y \leq \mu\varphi(x, 0), \quad \forall x \in X\right\},$$

where, as usual, inf $\varphi = +\infty$. It is easy to show that (S, d) is complete (see [20]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \epsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \varphi(x, 0)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\|_Y \leq \alpha\varphi(x, 0)$$

for all $x \in X$. So $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq \alpha\epsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that $d(f, Jf) \leq \frac{1}{2}$.

By Theorem 1.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(2x) = 2A(x) \tag{2.6}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(h, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\|_Y \leq \mu\varphi(x, 0)$$

for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{2 - 2\alpha}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}\|DA(x, y)\|_Y &= \lim_{n\to\infty} \frac{1}{2^n} \|Df(2^n x, 2^n y)\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) \leq \lim_{n\to\infty} \frac{2^n \alpha^n}{2^n} \varphi(x, y) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$DA(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$. Since f is odd, A is odd. Hence $A : X \to Y$ is an orthogonally additive mapping, i.e.,

$$A(x + 2y) + A(x - 2y) = A(2x + y) + A(2x - y) - A(2x)$$

for all $x, y \in X$ with $x \perp y$. Thus $A : X \to Y$ is a unique orthogonally additive mapping satisfying (2.3), as desired. □

From now on, in corollaries, assume that $(X, \perp)$ is an orthogonality normed space.

Corollary 2.3. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\|_Y \leq \theta \left(\|x\|^p + \|y\|^p\right) \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{\theta}{2 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking

$$\phi(x, y) = \theta \left(\|x\|^p + \|y\|^p\right),$$

for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{p-1}$ and we get the desired result. □

Theorem 2.4. *Let $f : X \to Y$ bea an odd mapping satisfying* (2.2) *for which there exists a function $\phi : X^2 \to [0, \infty)$ such that*

$$\varphi(x, y) \leq \frac{\alpha}{2} \varphi\left(2x, 2x\right)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{\alpha}{2 - 2\alpha} \varphi(x, 0) \tag{2.8}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $d(f, Jf) \leq \frac{\alpha}{2}$. So

$$d(f, A) \leq \frac{\alpha}{2 - 2\alpha}$$

. Thus we obtain the inequality (2.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.7) *Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_Y \leq \frac{\theta}{2^p - 2} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\phi(x, y) = \theta \left(\|x\|^p + \|y\|^p \right),$$

for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{1-p}$ and we get the desired result. □

3. Stability of the Orthogonally Additive-Quartic Functional Equation: An Even Mapping Case

In this section, we deal with the stability problem for the orthogonally additive-quartic functional equation given in the previous section: An even mapping case.

Definition 3.1. An even mapping $f : X \to Y$ is called an orthogonally quartic mapping if

$$f(x + 2y) + f(x - 2y) = f(2x + y) + f(2x - y) - f(2x) - 14f(x) + 30f(y)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 3.2. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq 16\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2)*. Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{16 - 16\alpha} \varphi(x, 0)$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.2), we get

$$\|16f(x) - f(2x)\|_Y \leq \varphi(x,0) \tag{3.1}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| f(x) - \frac{1}{16} f(2x) \right\|_Y \leq \frac{1}{16}\varphi(x,0)$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16} g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let θ be a positive real number and p a real number with $0 < p < 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7) Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{\theta}{16 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\phi(x,y) = \theta\left(\|x\|^p + \|y\|^p\right)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{p-4}$ and we get the desired result. □

Theorem 3.4. *Let $f : X \to Y$ bea an even mapping satisfying* (2.2) *and $f(0) = 0$ for which there exists a function $\phi : X^2 \to [0, \infty)$ such that*

$$\varphi(x,y) \leq \frac{\alpha}{16}\varphi\left(2x, 2y\right)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{\alpha}{16 - 16\alpha}\varphi(x,0) \tag{3.2}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2. Now we consider the linear mapping $J : S \to S$ such that

$$J_g(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.1) that $d(f, Jf) \leq \frac{\alpha}{16}$. So we obtain the inequality (3.2).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let θ be a positive real number and p a real number with $p > 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{\theta}{2^p - 16} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\phi(x, y) = \theta\left(\|x\|^p + \|y\|^p\right)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{4-p}$ and we get the desired result. □

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_o is an odd mapping and f_e is an even mapping such that $f = f_o + f_e$.

The above corollaries can be summarized as follows:

Theorem 3.6. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 1$ or $p > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exist an orthogonally additive mapping $A : X \to Y$ and an orthogonally quartic mapping $Q : X \to Y$ such that*

$$\|f(x) - A(x) - Q(x)\|_Y \leq \theta\left(\frac{1}{|2 - 2^p|} + \frac{1}{|16 - 2^p|}\right)\|x\|^p$$

for all $x \in X$.

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