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SOLVABILITY CONDITIONS OF THE SECOND ORDER DIFFERENTIAL EQUATION WITH DRIFT

Kordan N. Ospanov[1], Danagul R. Beisenova[2]

[1]Institute of Mathematics and Mathematical Modeling
Almaty, KAZAKHSTAN
[1]L.N. Gumilyov Eurasian National University
Astana, KAZAKHSTAN
[2]L.N. Gumilyov Eurasian National University
Astana, KAZAKHSTAN

Abstract: We proved the unique solvability in Hilbert space of the second order differential equation with complex and unbounded intermediate coefficients.

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1. Introduction and Main Result

We consider the following equation

$$ly = -y'' + r(x)y' + q(x)\bar{y'} + s(x)y + p(x)\bar{y} = f, \tag{1.1}$$

where $f \in L_2 := L_2(\mathbb{R})$, and r and q are continuously differentiable, and s and p are continuous functions, $y = y_1 + iy_2$ and $\bar{y} = y_1 - iy_2$.

When r and q are unbounded functions, properties of (1.1) are very different from the properties of the Sturm-Liouville equation. So, the study of (1.1) has a theoretical value. On the other hand, a number of problems of stochastic

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processes related to the description of particle motion, and wave propagation in a compressible medium and in the medium with resistance lead to the equation (1.1) (see [1-5]). In the case $s = p = 0$ (1.1) was studied in [6]. When $q = p = 0$, r and s are real-valued, and $f \in L_1(\mathbb{R})$ (this is a more simple case) it was investigated in [7]. For more details on the singular differential equations, see [8-13].

Let g and h be given functions. We put

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,\,t)} \left\|h^{-1}\right\|_{L_2(\,t,\,+\infty)} \quad (t > 0),$$

$$\beta_{g,h}(\tau) = \|g\|_{L_2(\tau,\,0)} \left\|h^{-1}\right\|_{L_2(-\infty,\,\tau)} \, (\tau < 0),$$

$$\gamma_{g,h} = \max\left(\sup_{t>0} \alpha_{g,h}(t),\, \sup_{\tau<0} \beta_{g,h}(\tau)\right).$$

We consider the operator

$$ly = -y'' + r(x)y' + q(x)\bar{y}' + s(x)y + p(x)\bar{y},$$

which is defined on the set $C_0^{(2)}(\mathbb{R})$ of twice continuously differentiable functions with compact support.

Definition 1. We say that $y \in L_2$ is a solution of (1.1), if there exists a sequence $\{y_n\}_{n=1}^{+\infty} \subset C_0^{(2)}(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$ and $\|ly_n - f\|_2 \to 0$ as $n \to +\infty$.

Theorem 2. *Let r and q be continuously differentiable, and let s, p be continuous functions with the following properties:*

$$\sqrt{|Re\, r|} - \omega(|Im\, r| + |q|) \geq 1 \;\; (1 < \omega < 2) \tag{1.2}$$

and

$$\gamma_{1+|s|+|p|,\sqrt{|Re\, r|}} < \infty. \tag{1.3}$$

Then for any $f \in L_2$ there exists a unique solution y of the equation (1.1).

2. Auxillary Statements

Lemma 1. [6] *Let functions g and h satisfy $\gamma_{g,h} < \infty$. Then for $y \in C_0^{(1)}(\mathbb{R})$ the following inequality holds:*

$$\int_{\mathbb{R}} |g(x)y(x)|^2\, dx \leq c_1 \int_{\mathbb{R}} \left|h(x)y'(x)\right|^2 dx. \tag{2.1}$$

Moreover, if c_1 is the smallest constant satisfying (2.1), then

$$\gamma_{g,h} \leq c_1 \leq 2\gamma_{g,h}.$$

Let $r_0 = Re\, r$ and $l_0 y = -y'' + r_0(x)y'$ be defined on $C_0^{(2)}(\mathbb{R})$.

Using Lemma 1, we prove the following result.

Lemma 2. *Let r be continuously differentiable function and*

$$\gamma_{1,\sqrt{r_0}} < \infty. \tag{2.2}$$

Then for any $y \in C_0^{(2)}(\mathbb{R})$ the following estimate holds:

$$\|\sqrt{|r_0|}y'\|_2 + \|y\|_2 \leq c_2\|l_0 y\|_2. \tag{2.3}$$

Proof. Let $y \in C_0^{(2)}(\mathbb{R})$. Using integration by parts, we have

$$(l_0 y, y') = \int\limits_{\mathbb{R}} r_0(x)|y'|^2 dx.$$

By Holder inequality,

$$\left\|\sqrt{|r_0|}y'\right\|_2 \leq \left\|\frac{1}{\sqrt{|r_0|}} l_0 y\right\|_2 .$$

Then using (2.2) and Lemma 2.1, we find that

$$\|\sqrt{r_0}y'\|_2 + \|y\|_2 \leq \left(1 + 2\gamma_{1,\sqrt{|r_0|}}\right)\|l_0 y\|_2.$$

From this follows (2.3), where $c_2 = 1 + 2\gamma_{1,\sqrt{|r_0|}}$.

□

If (2.2) holds, then using Lemma 2, we can prove that the operator l_0 is closable in the space L_2. We denote its closure by L_0. It is easy to see that for any $y \in D(L_0)$ the following inequality holds:

$$\|\sqrt{|r_0|}y'\|_2 + \|y\|_2 \leq c_2\|L_0 y\|_2. \tag{2.4}$$

Lemma 3. *Assume that the conditions of Lemma 2.2 are fulfilled. Then $R(L_0) = L_2$.*

Proof. From (2.4) follows that L_0 has the inverse L_0^{-1} . Let $R(L_0) \neq L_2$. Then there exists nonzero $z_0 \in L_2$ which is orthogonal to $R(L_0)$. Hence

$$(z_0' + r_0 z_0)' = 0,$$

or

$$z_0' + r_0 z_0 = c_3,$$

where c_3 is const. It is easy to prove that z_0 is the continuously differentiable function and the following equality holds:

$$\left(z_0 \exp \int_a^x r_0(t)dt \right)' = c_3 \exp \int_a^x r_0(t)dt. \tag{2.5}$$

1. If $c_3 \neq 0$, then without loss of generality, we can put $c_3 = -1$. Hence

$$\left(z_0 \exp \int_a^x r_0(t)dt \right)' < 0.$$

Then for x_1, x_2 with $x_1 < x_2$,

$$z_0(x_1) > z_0(x_2) \exp \int_{x_1}^{x_2} r_0(t)dt.$$

So

$$z_0(x_1) > z_0(x_2)$$

for all $x_1, x_2 \in \mathbb{R}$ satisfying $x_1 < x_2$. Therefore $z_0(x) \notin L_2(R)$.

2. Let $c_3 = 0$. Then from (2.5) we have

$$z_0(x) = c_4 \exp \left[- \int_a^x r_0(t)dt \right],$$

hence if $c_4 \neq 0$, then $|z_0(x_0)| \geq |c_4|$ for all $x_0 < a$. Therefore, $z_0(x) \notin L_2(R)$.

This is a contradiction. □

3. Proof of Theorem 1.1

Let $x = at$ $(a > 1)$, then (1.1) becomes that

$$L_a z = -z'' + a^{-1} r_1(t) z' + a^{-1} q_1(t) \bar{z}' + a^{-2} s_1(t) z + \\ + a^{-2} p_1(t) \bar{z} = \tilde{f}(t), \tag{3.1}$$

where $z = z(t) = y(at)$, $r_1(t) = r(at)$, $q_1(t) = q(at)$, $s_1(t) = s(at)$, $p_1(t) = p(at)$ and $\tilde{f}(t) = a^{-2} f(at)$.

Let $l_{0a} z = -z'' + a^{-1} Re\ r_1\ z'$, $z \in C_0^{(2)}(\mathbb{R})$. By Lemma 2, the operator l_{0a} is closable in L_2. We denote by l_a the closure of l_{0a}. Since $a^{-1} Re\ r_1(t)$ satisfies the conditions of Lemma 3, it follows that l_a is continuously invertible and for any $z \in D(l_a)$, the following inequality holds:

$$\|\sqrt{a^{-1} Re\ r_1}\ z'\|_2 \leq \|l_a z\|_2.$$

This estimate by (1.2) implies

$$\|a^{-1}[|i\ Im\ r_1 z'| + |q_1 \bar{z}'|]\|_2 \leq \frac{1}{a\omega} \|l_a z\|_2, \forall z \in D(l_a).$$

Hence by Theorem 1.16 in Chapter 4 of [14], we have that the operator $\tilde{l}_a z = -y'' + a^{-1} r_1(t) z' + a^{-1} q_1(t) \bar{z}'$ is invertible and $R\left(\tilde{l}_a\right) = L_2$. Moreover, it is easy to calculate that

$$\|l_a z\|_2 \leq \frac{\omega}{\omega - 1} \|\tilde{l}_a z\|_2, \quad \forall z \in D(\tilde{l}_a). \tag{3.2}$$

Futher by condition (1.3) and Lemma 1, for any $z \in D(l_a)$ the following inequalinies hold:

$$\|a^{-2} s_1 z\|_2 \leq 2a^{-3/2} \gamma_{s_1, \sqrt{|Re\ r_1|}} \|\sqrt{a^{-1} Re\ r_1}\ z'\|_2, \tag{3.3}$$

$$\|a^{-2} p_1 \bar{z}\|_2 \leq 2a^{-3/2} \gamma_{p_1, \sqrt{|Re\ r_1|}} \|\sqrt{a^{-1} Re\ r_1}\ z'\|_2. \tag{3.4}$$

We choose

$$a = \omega \left[\frac{2\omega}{\omega - 1} \left(\gamma_{s_1, \sqrt{|Re\ r_1|}} + \gamma_{p_1, \sqrt{|Re\ r_1|}}\right)\right]^{2/3}.$$

Then by (3.2), (3.3) and (3.4), we have

$$\|a^{-2} s_1\ z + a^{-2} p_1 \bar{z}\|_2 \leq \frac{1}{\omega} \|\tilde{l}_a z\|_2, \quad \forall\, z \in D(\tilde{l}_a). \tag{3.5}$$

The estimate (3.5), by Theorem 1.16 in Chapter 4 of [14], implies that the operator $L_a = \tilde{l}_a + a^{-2}s_1(t)E + a^{-2}p_1(t)\bar{E}$ corresponding to (3.1) is invertible and $R\,(L_a) = L_2$. Here $\bar{E}z = \bar{z}$. Taking $t = x/a$, we get that $R\,(L) = L_2$. Then by Definition 1.1, for any $f \in L_2$, (1.1) has a unique solution y. $\square$

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