International Journal of Pure and Applied Mathematics
Volume 113 No. 4 2017, 139-149
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i4.13

ijpam.eu

AN INVERSE PROBLEM OF FINDING THE TIME-DEPENDENT HEAT TRANSFER COEFFICIENT FROM AN INTEGRAL CONDITION

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Abstract: We consider the inverse problem of determining the time-dependent diffusivity in one-dimensional heat equation with periodic boundary conditions and nonlocal over-specified data. The problem is highly nonlinear and it serves as a mathematical model for the technological process of external guttering applied in cleaning admixtures from silicon chips. The well-posedness conditions for the existence, uniqueness and continuous dependence upon the data of the classical solution of the problem are established.

AMS Subject Classification: 35K15, 35P10, 35R30
Key Words: heat transfer, heat equation, inverse problem, thermal diffusivity, integral condition

1. Introduction

Parameter identification from over-specified data plays an important role in applied mathematics, physics and engineering. The problem of identifying the diffusivity was investigated by many researchers under various boundary and over-determination conditions, [1-5]. It is important to note that in [6], the

Received: December 15, 2016
Revised: January 30, 2017
Published: April 6, 2017

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time-dependent diffusion coefficient has been determined from different over-determination conditions in the case of self-adjoint auxiliary spectral problems. In the present work, a nonlocal over-specified data is used together with periodic boundary conditions for the determination of the time-dependent diffusivity. The mathematical formulation of the inverse problem under investigation is given in Section 2. In Section 3, the existence, uniqueness and continuous dependence upon the data of the classical solution of the inverse problem for some small parameters are established by using the generalized Fourier method.

2. Mathematical Formulation

In the rectangle $Q_T = \{(x,t)|0 < x < 1, 0 < t \leq T\} = (0,1) \times (0,T]$, we consider the inverse problem given by the heat equation

$$\frac{\partial u}{\partial t}(x,t) = k(t)\frac{\partial^2 u}{\partial x^2}(x,t), \ \ (x,t) \in Q_T, \tag{1}$$

with unknown concentration/temperature $u(x,t)$ and unknown time-dependent diffusivity $k(t) > 0$, subject to the initial condition

$$u(x,0) = \varphi(x), \ \ 0 \leq x \leq 1, \tag{2}$$

where φ is a given function, the periodic and heat flux boundary conditions

$$u(0,t) = u(1,t), \ \ t \in (0,T], \tag{3}$$

$$au_x(0,t) = u_x(1,t), \ \ t \in (0,T], \tag{4}$$

and the over-determination condition, [7, 8],

$$p(t)u(0,t) + \int_0^1 u(x,t)dx = E(t), \ \ t \in [0,T], \tag{5}$$

with $p(t) = \alpha + \beta k^{-\gamma}(t)$, where $\alpha, \beta, \gamma > 0$ are segregation coefficients.

This problem for $a = 0$ for the first time has been considered in [9]. This problem arises in the mathematical modeling of the technological process of external guttering applied, for example (if $a = 0$), in cleaning admixtures from silicon chips, [8]. In this case, $\varphi(x)$ is the distribution of admixture in the chip for $x \in (0,1)$ at the initial time $t = 0$, while $u(x,t)$ is its distribution at time t. Condition (3) means that the admixtures in the left and right boundaries of the chip are the same. For $a = 0$ the adiabatic condition (4) means that the right boundary $x = 1$ of the chip is perfectly insulated. For $a \neq 0$ the adiabatic condition (4) is the proportional property to the flow on the opposite points of the boundary. Therefore, reasonably assume that $a \geq 0$. Condition (5) means that part of the substance is concentrated (segregated) on the left side $x = 0$ of the chip, [7, 8].

When $\alpha = \beta = 0$, then the resulting inverse problem has been previously investigated in [1], and it is the purpose of this paper to investigate the non-trivial case when α and β are non-zero.

3. Existence and Uniqueness

The pair $(k(t), u(x,t))$ from the class $C[0,T] \times (C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q}_T))$ for which conditions (1)-(5) are satisfied and $k(t) > 0$ on the interval $[0,T]$ is called *the classical solution* of the inverse problem (1)-(5).

The analysis is similar to that of [10] for the identification of the time-dependent blood perfusion coefficient in the bio-heat equation. Consider the spectral problem

$$-X^{''}(x) = \lambda X(x), \ \ 0 \leq x \leq 1, \tag{6}$$

$$aX^{'}(0) = X^{'}(1), \ \ X(0) = X(1). \tag{7}$$

This problem if $a = 0$ is well-known in [11], as the auxiliary spectral problem for solving a nonlocal boundary value problem for heat equation by the Fourier method.

The case $a = 1$ is simpler and we will not go into details. When $a = -1$ the boundary conditions (7) are irregular. This case will not be considered. Suppose that $a \neq \pm 1$. In this case the problem (6) - (7) has double eigenvalues $\lambda_k = (2k\pi)^2$ (except for the first $\lambda_0 = 0$). Eigenfunctions of the problem are the following:

$$\begin{aligned} &X_0(x) = 2, \lambda_0 = 0; \\ &X_{2k-1}(x) = 4\cos(2\pi kx), \lambda_k = (2\pi k)^2, \ \ k = 1, 2, ... \end{aligned} \tag{8}$$

To avoid the problem of choosing the associated functions [12, 13] for their construction we use the equation:

$$\begin{aligned} &-X''(x) = \lambda_k X(x) + \sqrt{\lambda_k} X_{2k-1}(x), \ \ 0 < x < 1; \\ &\alpha X'(0) = X'(1), \ \ X(0) = X(1). \end{aligned} \tag{9}$$

Then associated functions are in the form:

$$X_{2k}(x) = 2\frac{1-(1-\alpha)\,x}{1-\alpha}\sin(2\pi kx), \ \ k = 1, 2, ... \tag{10}$$

This system of functions $\{X_0(x), X_{2k-1}(x), X_{2k}(x)\}$ form a Riesz basis in $L_2(0,1)$ [14]. System biorthogonal to it is the following system:

$$Y_0(x) = \frac{\alpha + (1-\alpha)\,x}{1+\alpha}, \;\; Y_{2k-1}(x) = \frac{\alpha + (1-\alpha)\,x}{1+\alpha}\cos(2\pi kx),$$
$$Y_{2k}(x) = 2\frac{1-\alpha}{1+\alpha}\sin(2\pi kx), \;\; k = 1, 2, ... \tag{11}$$

The following lemmas are important for the mathematical analysis of the inverse problem.

Lemma 1. *If $\phi(x) \in C^3[0,1]$ satisfies the conditions $\phi(0) = \phi(1)$, $a\phi^{'}(0) = \phi^{'}(1)$, $\phi^{''}(0) = \phi^{''}(1)$, then the inequalities*

$$\sum_{n=1}^{\infty} n^2|\phi_{2n}| \le C_1\|\phi\|_{C^3[0,1]}, \;\; \sum_{n=1}^{\infty} n|\phi_{2n-1}| \le C_2\|\phi\|_{C^3[0,1]} \tag{12}$$

hold, where C_1 and C_2 are constants, $\phi_n = \int_0^1 \phi(x)Y_n(x)dx$.

Proof. Because $\phi(0) = \phi(1)$, $\phi^{''}(0) = \phi^{''}(1)$, the equality

$$\phi_{2n} = \int_0^1 \phi(x)\sin(2\pi nx)dx = -\frac{1}{8\pi^3 n^3}\int_0^1 \phi^{'''}(x)\cos(2\pi nx)dx$$

holds by three times integrating by parts. Analogously, by integrating by parts twice and using that $\phi(0) = \phi(1)$, $a\phi^{'}(0) = \phi^{'}(1)$, we obtain that

$$\phi_{2n-1} = \int_0^1 \phi(x)x\cos(2\pi nx)dx = -\int_0^1 \frac{x\phi^{''} + 2\phi^{'}}{4\pi^2 n^2}\cos(2\pi nx)dx.$$

From the earlier discussion, by using the Schwarz and Bessel inequalities, we obtain

$$\sum_{n=1}^{\infty} n^2|\phi_{2n}| \le \frac{1}{8\pi^3}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]^{\frac{1}{2}}\left[\sum_{n=1}^{\infty}\left(\int_0^1 \phi^{'''}(x)\cos(2\pi nx)dx\right)^2\right]^{\frac{1}{2}} \le$$
$$\le C_1\|\phi^{'''}\|_{L_2[0,1]} \le C_1\|\phi\|_{C^3[0,1]}$$

and

$$\sum_{n=1}^{\infty} n|\phi_{2n-1}| \le \left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]^{\frac{1}{2}}\left[\sum_{n=1}^{\infty}\left(\int_0^1 \frac{x\phi^{''} + 2\phi^{'}}{4\pi^2}\cos(2\pi nx)dx\right)^2\right]^{\frac{1}{2}} \le$$
$$\le \frac{C_2}{2}\|x\phi^{''} + 2\phi^{'}\|_{L_2[0,1]} \le C_2\|\phi\|_{C^3[0,1]}$$

for some constants C_1 and C_2. $\square$

Lemma 2. *If $k_m(t) \in C[0,T]$ satisfies the condition $0 < a \le k_m(t),\ m = 1, 2,$ then for $\forall n \in \mathbb{N}$ and $\forall t \in [0,T]$, the inequality*

$$\left| e^{-n\int_0^t k_1(s)ds} - e^{-n\int_0^t k_2(s)ds} \right| \le \frac{1}{ae} \|k_1 - k_2\|_{C[0,T]} \tag{13}$$

holds.

Proof. For arbitrary fixed $t \in [0,T]$ and $n \in \mathbb{N}$, by using the mean value theorem for the function e^{-x}, we obtain that there exists θ between $n\int_0^t k_1(s)ds$ and $n\int_0^t k_2(s)ds$ such that

$$\left| e^{-n\int_0^t k_1(s)ds} - e^{-n\int_0^t k_2(s)ds} \right| = e^{-\theta} \left| n\int_0^t k_1(s)ds - n\int_0^t k_2(s)ds \right|.$$

By using that $xe^{-bx} \le \frac{1}{be};\ x \ge 0, b = const > 0$, we obtain that

$$\frac{n}{e^\theta} \left| \int_0^t (k_1(s) - k_2(s))ds \right| \le \frac{nt}{e^{nat}} \|k_1 - k_2\|_{C[0,T]} \le \frac{1}{ae} \|k_1 - k_2\|_{C[0,T]},$$

and this proves (13). ☐

The main result of Section 3.1 is in the following theorem.

Theorem 3. *Let the functions $\varphi(x) \in C^3[0,1], E(t) \in C[0,T]$ satisfy the conditions*

$$\varphi(0) = \varphi(1),\ a\varphi'(0) = \varphi'(1),\ \varphi''(0) = \varphi''(1), \tag{14}$$

$$\varphi_{2k} \ge 0,\ \varphi_{2k-1} \le 0,\ \varphi_0 + 2\varphi_1 < 0,\ E(t) < 2\varphi_0,\ \forall t \in [0,T], \tag{15}$$

where $\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx$ for $k = 0, 1, 2, \ldots$. Then, there exist positive numbers α_0 and γ_0 such that the inverse problem given by (1)-(5) with the parameters $\alpha < \alpha_0,\ \gamma > \gamma_0$ has a unique solution, where the numbers α_0 and γ_0 are determined by the data of the problem.

Proof. For arbitrary positive $k(t) \in C[0,T]$, using that $\varphi \in C^3[0,1]$ satisfies condition (14), by applying the standard procedure of the Fourier series method, we obtain the solution of the direct problem given by (1)-(4) in the following form:

$$u(x,t) = \varphi_0 X_0(x) + \sum_{n=1}^{\infty} \varphi_{2n} e^{-(2\pi n)^2 \int_0^t k(s)ds} X_{2n}(x) + \tag{16}$$

$$+ \sum_{n=1}^{\infty} (\varphi_{2n-1} - 4\pi n \varphi_{2n} t) e^{-(2\pi n)^2 \int_0^t k(s)ds} X_{2n-1}(x).$$

The series in (16) and its x-partial derivative are uniformly convergent in $\overline{Q}_T$ because their majorizing sums are absolutely convergent by Lemma 1. Therefore, their sums involved in expressing $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{Q}_T$. Because the majorizing sum $\sum_{n=1}^{\infty} n^3 e^{-K(2\pi n)^2\varepsilon} (K = const > 0)$ is convergent, the t-partial derivative and the xx-second-order partial derivative series of (16) are uniformly convergent for $t \geq \varepsilon > 0$ (ε is an arbitrary positive number). Thus, we have $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q}_T)$, which satisfies conditions (1)-(4) for arbitrary positive $k(t) \in C[0,T]$.

Applying the over-determination condition (5), we obtain

$$p(t) = F[p(t)], \tag{17}$$

where

$$F[p(t)] = \frac{2\varphi_0 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\varphi_{2n} e^{-(2\pi n)^2 \int_0^t k(s)ds} - E(t)}{-2\varphi_0 + 4\sum_{n=1}^{\infty}[4\pi n\varphi_{2n} t - \varphi_{2n-1}] e^{-(2\pi n)^2 \int_0^t k(s)ds}}, \tag{18}$$

$$k(t) = \left[\frac{\beta}{p(t) - \alpha}\right]^{\frac{1}{\gamma}}.$$

Denote

$$\alpha_0 = \frac{2\varphi_0 - E_{\max}}{-2\varphi_0 + 4\sum_{n=1}^{\infty}(4\pi n\varphi_{2n} t - \varphi_{2n-1})}, \tag{19}$$

$$\alpha_1 = \frac{2\varphi_0 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\varphi_{2n} - E_{\min}}{-2\varphi_0 - 4\varphi_1},$$

where $E_{\max} = \max_{t\in[0,T]} E(t)$, $E_{\min} = \min_{t\in[0,T]} E(t)$. Then, from (15), (17), and (18), it follows that

$$0 < \alpha_0 \leq p(t) \leq \alpha_1, \ \ t \in [0,T]. \tag{20}$$

Under condition $\alpha_0 > \alpha_1$, the inequalities

$$0 < \left[\frac{\beta}{\alpha_1 - \alpha}\right]^{\frac{1}{\gamma}} \leq k(t) \leq \left[\frac{\beta}{\alpha_0 - \alpha}\right]^{\frac{1}{\gamma}} \tag{21}$$

hold. $\square$

Let us denote

$$C_{\alpha_0,\alpha_1}[0,T] := \{p(t) \in C[0,T] | \alpha_0 \leq p(t) \leq \alpha_1, \forall t \in [0,T]\}.$$

It is easy to verify that

$$F : C_{\alpha_0,\alpha_1}[0,T] \to C_{\alpha_0,\alpha_1}[0,T].$$

Let us show that F is a contraction mapping in $C_{\alpha_0,\alpha_1}[0,T]$ for small α and large γ. Indeed, $\forall p_1(t), p_2(t) \in C_{\alpha_0,\alpha_1}[0,T]$, we have

$$F[p_1(t)] - F[p_2(t)] = \frac{1}{-2\varphi_0 + \alpha_{1,2}(t)} \times \tag{22}$$

$$\times \left(\frac{2\varphi_0 + \alpha_{0,1}(t) - E(t)}{-2\varphi_0 + \alpha_{1,1}(t)} \left(\alpha_{1,2}(t) - \alpha_{1,1}(t)\right) - \left(\alpha_{0,2}(t) - \alpha_{0,1}(t)\right) \right),$$

where

$$\alpha_{0,m}(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \varphi_{2n} e^{-(2\pi n)^2 \int_0^t k_m(s)ds},$$

$$\alpha_{1,m}(t) = 4 \sum_{n=1}^{\infty} (4\pi n \varphi_{2n} t - \varphi_{2n-1}) e^{-(2\pi n)^2 \int_0^t k_m(s)ds},$$

$$k_m = \left[\frac{\beta}{p_m(t) - \alpha} \right]^{\frac{1}{\gamma}}, \ m = 1, 2.$$

Lemma 2 and inequalities (21) imply

$$\left| e^{-(2\pi n)^2 \int_0^t k_1(s)ds} - e^{-(2\pi n)^2 \int_0^t k_2(s)ds} \right| \le \frac{(\alpha_1 - \alpha)^{\frac{1}{\gamma}}}{\beta^{\frac{1}{\gamma}} e} \|k_1 - k_2\|_{C[0,T]}.$$

Then, we obtain

$$|\alpha_{0,2}(t) - \alpha_{0,1}(t)| \le \frac{(\alpha_1 - \alpha)^{\frac{1}{\gamma}}}{\beta^{\frac{1}{\gamma}} e} \left(\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \varphi_{2n} \right) \|k_1 - k_2\|_{C[0,T]},$$

$$|\alpha_{1,2} - \alpha_{1,1}| \le \frac{(\alpha_1 - \alpha)^{\frac{1}{\gamma}}}{\beta^{\frac{1}{\gamma}} e} \left(16\pi T \sum_{n=1}^{\infty} n\varphi_{2n} - 4 \sum_{n=1}^{\infty} \varphi_{2n-1} \right)$$
$$\|k_1 - k_2\|_{C[0,T]}.$$

From these inequalities and (22), we obtain

$$\max_{0 \le t \le T} \left| F[p_1(t)] - F[p_2(t)] \right| \le \frac{(\alpha_1 - \alpha)^{\frac{1}{\gamma}}}{\beta^{\frac{1}{\gamma}}} \delta \|k_1 - k_2\|_{C[0,T]}, \tag{23}$$

where

$$\delta = \frac{2}{\pi e}\frac{(8\pi^2 T\alpha_1 + 1)\sum_{n=1}^{\infty} n\varphi_{2n} - 2\pi\alpha_1 \sum_{n=1}^{\infty}\varphi_{2n-1}}{-2\varphi_0 - 4\varphi_1}. \tag{24}$$

By using the mean value theorem and (21), it is easy to show that

$$|k_1(t) - k_2(t)| \le \frac{\beta^{\frac{1}{\gamma}}}{\gamma(\alpha_0 - \alpha)^{1+\frac{1}{\gamma}}}|p_1(t) - p_2(t)|. \tag{25}$$

Thus, from (23) and (26), we obtain

$$\|F[p_1] - F[p_2]\|_{C[0,T]} \le \frac{\delta}{(\alpha_0 - \alpha)}\frac{1}{\gamma}\left(\frac{\alpha_1 - \alpha}{\alpha_0 - \alpha}\right)^{\frac{1}{\gamma}} \|p_1 - p_2\|_{C[0,T]}.$$

Let us fix a sufficiently large number $\gamma_0 > 0$ such that

$$K := \frac{\delta}{(\alpha_0 - \alpha)}\frac{1}{\gamma_0}\left(\frac{\alpha_1 - \alpha}{\alpha_0 - \alpha}\right)^{\frac{1}{\gamma_0}} \le 1. \tag{26}$$

Thus, in the case $\gamma > \gamma_0$, Equation (17) has a unique solution $k(t) \in C_{\alpha_0,\alpha_1}[0,T]$ by the Banach fixed point theorem.

We therefore obtain a unique positive function $k(t)$, continuous on $[0,T]$, which together with the solution of (1)-(4) given by the Fourier series (16), forms the unique solution of the inverse problem given by (1)-(5). This concludes the proof of the theorem.

4. Continuous Dependence upon the Data

The following result on continuous dependence on the data of the solution of (1)-(5) holds.

Theorem 4. *Consider the (input) data in the form of $\Phi = \{\varphi, E\}$, which satisfy the assumptions of Theorem 1 with*

$$2\varphi_0 - E_{\max} \ge N_1 > 0,\ \ \varphi_0 + 2\varphi_1 \le -N_2 < 0 \tag{27}$$

and let

$$\|\varphi\|_{C^3[0,1]} \le N_3,\ \ \|E\|_{C[0,T]} \le N_4 \tag{28}$$

for some positive numbers N_1, N_2, N_3, and N_4. Then the solution $(k(t), u(x,t))$ of the inverse problem (1)-(5) depends continuously upon the data for sufficiently small α and large γ.

Proof. Let $\Phi = \{\varphi, E\}$ and $\overline{\Phi} = \{\overline{\varphi}, \overline{E}\}$ be two sets of the data, which satisfy the conditions of Theorem1. Let us denote $\|\Phi\| := \|\varphi\|_{C^3[0,1]} + \|E\|_{C[0,T]}$.

Let (k, u) and $(\bar{k}, \bar{u})$ be solutions of (1)-(5) corresponding to the data Φ and $\overline{\Phi}$, respectively. According to (18),

$$p(t) = \frac{2\varphi_0 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\varphi_{2n} e^{-(2\pi n)^2 \int_0^t k(s)ds} - E(t)}{-2\varphi_0 + 4\sum_{n=1}^{\infty}[4\pi n\varphi_{2n}t - \varphi_{2n-1}]e^{-(2\pi n)^2 \int_0^t k(s)ds}},$$

$$\bar{p}(t) = \frac{2\overline{\varphi}_0 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\overline{\varphi}_{2n} e^{-(2\pi n)^2 \int_0^t \bar{k}(s)ds} - \overline{E}(t)}{-2\overline{\varphi}_0 + 4\sum_{n=1}^{\infty}[4\pi n\overline{\varphi}_{2n}t - \overline{\varphi}_{2n-1}]e^{-(2\pi n)^2 \int_0^t \bar{k}(s)ds}},$$

$$k(t) = \left[\frac{\beta}{p(t) - \alpha}\right]^{\frac{1}{\gamma}}, \ \bar{k}(t) = \left[\frac{\beta}{\bar{p}(t) - \alpha}\right]^{\frac{1}{\gamma}}.$$

First, let us estimate the difference $p - \bar{p}$. Using (12), (13), and (28), we obtain

$$\left|\sum_{n=1}^{\infty} \frac{1}{n}\varphi_{2n} e^{-(2\pi n)^2 \int_0^t k(s)ds}\right| \leq C\|\varphi\|_{C^3[0,1]} \leq CN_3,$$

$$\left|\sum_{n=1}^{\infty}[4\pi n\varphi_{2n}t - \varphi_{2n-1}]e^{-(2\pi n)^2 \int_0^t k(s)ds}\right| \leq 4\pi C(1+T)N_3,$$

$$\left|\sum_{n=1}^{\infty} \frac{1}{n}\varphi_{2n} e^{-(2\pi n)^2 \int_0^t k(s)ds} - \sum_{n=1}^{\infty} \frac{1}{n}\overline{\varphi}_{2n} e^{-(2\pi n)^2 \int_0^t \bar{k}(s)ds}\right| \leq$$

$$\leq M_1\|\varphi - \overline{\varphi}\|_{C^3[0,1]} + M_2\|k - \overline{k}\|_{C[0,T]},$$

$$\left|\sum_{n=1}^{\infty}[4\pi n\varphi_{2n}t - \varphi_{2n-1}]e^{-(2\pi n)^2 \int_0^t k(s)ds} - \right.$$

$$\left. - \sum_{n=1}^{\infty}[4\pi n\overline{\varphi}_{2n}t - \overline{\varphi}_{2n-1}]e^{-(2\pi n)^2 \int_0^t \bar{k}(s)ds}\right| \leq$$

$$\leq M_3\|\varphi - \overline{\varphi}\|_{C^3[0,1]} + M_4\|k - \overline{k}\|_{C[0,T]},$$

where $M_k, \ k = \overline{1,4}$ are some positive constants. By using these inequalities, simple manipulations yield the estimate

$$|p(t) - \bar{p}(t)| \leq \frac{M_5\|\varphi - \overline{\varphi}\|_{C^3[0,1]} + M_6\|k - \overline{k}\|_{C[0,T]} + M_7\|E - \overline{E}\|_{C[0,T]}}{4N_2^2}, \quad (29)$$

where $M_k,\ k = \overline{5,7}$ are some constants that are determined by C_1, C_2 and $N_k,\ k = \overline{1,4}$.

It is known from (24) that, for $\alpha < \alpha_0$,

$$|k(t) - \bar{k}(t)| \leq \frac{\beta^{\frac{1}{\gamma}}}{\gamma(\alpha_0 - \alpha)^{1+\frac{1}{\gamma}}} |p(t) - \bar{p}(t)|, \tag{30}$$

with $\alpha_0 \geq \frac{2\varphi_0 - E_{\max}}{M_8 \|\varphi\|_{C^3[0,1]}} \geq \frac{N_1}{M_8 N_3}$, for some positive constant M_8. If α is sufficiently small such that $\alpha < \frac{N_1}{M_8 N_3}$, using (30) in (29), we obtain

$$(1 - M_9)\|p - \bar{p}\|_{C[0,T]} \leq M_{10} \left(\|\varphi - \overline{\varphi}\|_{C^3[0,1]} + \|E - \overline{E}\|_{C[0,T]} \right), \tag{31}$$

for some positive constants M_{10} and $M_9 := \frac{M_6}{4N_2^2} \frac{\beta^{\frac{1}{\gamma}}}{\gamma\left(\frac{N_1}{M_8 N_3} - \alpha\right)^{1+\frac{1}{\gamma}}}$.

The inequality $M_9 < 1$ holds for sufficiently large γ. This means that p continuously depends upon the data. Then, the equality $k(t) = \left[\frac{\beta}{p(t) - \alpha}\right]^{\frac{1}{\gamma}}$ implies the continuous dependence of k upon the data. Similarly, we can prove that u, which is given in (16), depends continuously upon the data. This concludes the proof of the theorem. □

Brief abstract of this work has been published in the Materials of the workshop "Differential operators and modeling of complex systems" (April 7-8, 2017, Almaty, Kazakhstan) [15].

Acknowledgements

This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 0824/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

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