International Journal of Pure and Applied Mathematics
Volume 113 No. 4 2017, 31-45
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i4.4
ijpam.eu

ON THE EXISTENCE OF A NONTRIVIAL SOLUTION OF THE HOMOGENEOUS BOUNDARY VALUE PROBLEM FOR THE BURGERS EQUATION

Meiramkul M. Amangaliyeva[1], Muvasharkhan T. Jenaliyev[2],
Murat I. Ramazanov[3]
[1,2]Institute of Mathematics and Mathematical Modeling
Almaty, KAZAKHSTAN
[3]Institute of Applied Mathematics
Karaganda, KAZAKHSTAN
[3]Buketov Karaganda Stated University
Karaganda, KAZAKHSTAN

Abstract: Research of the Burgers equation has a long history. In work of Y. Benia, B.-K. Sadallah in the Sobolev classes there are results on the existence, uniqueness and regularity for the solution to the Burgers equation in non-cylindrical (non-degenerating) domain, that can be converted into a rectangular domain by a regular replacement of the independent variables. The authors point out that the development of the results of Y. Benia, B.-K. Sadallah for the case of the degenerating domain will be considered in the future. The goal of our work is: to show that the homogeneous boundary value problem for the Burgers equation in the angular (degenerating) domain along with a trivial solution may have a nontrivial solution.

AMS Subject Classification: 35K05, 35K20, 35Q35
Key Words: Burgers equation, heat equation, boundary value problems, nontrivial solution

1. Preliminary Points and Statement of the Problem

Consider the following homogeneous boundary value problem for the Burgers

Received: December 15, 2016
Revised: January 30, 2017
Published: April 6, 2017

equation:

$$\begin{cases} u_t + uu_x - a^2 u_{xx} = 0, \ 0 < x < t, \, t > 0, \\ u\big|_{x=0} = 0, \quad u\big|_{x=t} = 0. \end{cases} \tag{1}$$

By means of transformation of the Hopf-Cole

$$u(x,t) = -2a^2 \cdot \frac{w_x(x,t)}{w(x,t)} \tag{2}$$

boundary value problem (1) is reduced to the following auxiliary boundary problem

$$\begin{cases} w_t - a^2 w_{xx} = 0, \ 0 < x < t, \, t > 0, \\ w_x\big|_{x=0} = 0, \quad w_x\big|_{x=t} = 0. \end{cases} \tag{3}$$

In fact, substituting the function (2) into equation (1), we obtain

$$\frac{\partial}{\partial x}\left[\frac{w_t(x,t) - a^2 w_{xx}(x,t)}{w(x,t)}\right] = 0, \ 0 < x < t, \, t > 0, \tag{4}$$

that is

$$w_t(x,t) - a^2 w_{xx}(x,t) = c(t) w(x,t), \tag{5}$$

where $c(t)$ is an arbitrary function.

The inverse transformation to (2) is the following transformation

$$w(x,t) = \exp\left\{-\frac{1}{2a^2}\int_0^x u(\xi,t)d\,\xi + b(t)\right\}, \ 0 < x < t, \ t > 0, \tag{6}$$

from which it follows

$$w_x(x,t) = -\frac{u(x,t)}{2a^2}\exp\left\{-\frac{1}{2a^2}\int_0^x u(\xi,t)d\,\xi + b(t)\right\}, \tag{7}$$

where $b(t)$ is an arbitrary bounded function on $(0,+\infty)$. Based on formulas (2), (4)–(7), boundary value problem (3) follows from (1).

Thus we obtain that one solution of the Burgers equation from (1) corresponds to each solution of the equation

$$w_t(x,t) - a^2 w_{xx}(x,t) = 0; \tag{8}$$

conversely, to any solution of the Burgers equation from (1) there is a family of solutions to equation (5), determined by arbitrary (for example, integrable in its definition domain) functions $c(t)$. Obviously, the elements of this family differ from each other by the exponential factor $\exp\{-\int\limits_0^t c(\tau)d\tau\}$.

Note that the problem (3) has a solution $w(x,t) \equiv \text{const}$, which according to (2) corresponds to the trivial solution $u(x,t) \equiv 0$ of the problem (1). However, we are interested in the question: whether the boundary value problem (1) has a non-trivial solution?

2. Reducing the Auxiliary Problem to an Integral Equation

Solution of the problem (3) we are looking as the sum of the single-layer potentials [4, p.476–479]:

$$w(x,t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{1}{(t-\tau)^{1/2}} \left[\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau) + \right. \tag{9}$$

$$\left. + \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\right] d\tau,$$

which satisfies the equation (3) for any functions $\nu(t)$ and $\varphi(t)$, that are yet unknown and should be defined.

Solution (9) we satisfy to the first of the boundary conditions (3). To do this, at first we calculate the derivative with respect to x from (9). We have:

$$\frac{\partial w(x,t)}{\partial x} = -\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \times \tag{10}$$

$$\times \nu(\tau)d\tau - \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau.$$

Now we satisfy derivative (10) to the first of the boundary conditions (3). We get:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = -\frac{\nu(t)}{2a^2} +$$

$$+\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau = 0.$$

From this we express the function $\nu(t)$ in terms $\varphi(t)$:

$$\nu(t)=\frac{1}{2a\sqrt{\pi}}\int\limits_0^t\frac{\tau}{(t-\tau)^{1/2}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau. \tag{11}$$

Further we satisfy derivative (10) to the second of the boundary conditions (3). We have:

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=t}=-\frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t\frac{t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}\nu(\tau)d\tau+$$

$$+\frac{\varphi(t)}{2a^2}-\frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)d\tau=0.$$

From here, taking into account equality (11), we obtain an integral equation for the unknown function $\varphi(t)$:

$$\varphi(t)-\frac{1}{2a\sqrt{\pi}}\int\limits_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\,\tau- \tag{12}$$

$$-\frac{1}{2a\sqrt{\pi}}\int\limits_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)d\,\tau.$$

The following calculations are used here. We introduce the following notation:

$$J(t)=\frac{1}{4a^2\,\pi}\int\limits_0^t\frac{t}{(t-\tau)^{\frac{3}{2}}}\,\exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}\times \tag{13}$$

$$\times\left(\int\limits_0^\tau\frac{\tau_1}{(\tau-\tau_1)^{\frac{3}{2}}}\,\exp\left\{-\frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right\}\,\varphi\left(\tau_1\right)d\tau_1\right)\,d\tau.$$

By changing the order of integration in the integral expression (13), we obtain:

$$J(t)=\frac{1}{4a^2\,\pi}\int\limits_0^t t\,\tau_1\varphi\left(\tau_1\right)\cdot I(t,\tau_1)\,d\tau_1, \tag{14}$$

where

$$I(t,\tau_1)=\int\limits_{\tau_1}^{t}\frac{1}{(t-\tau)^{\frac{3}{2}}(\tau-\tau_1)^{\frac{3}{2}}}\exp\left\{-\frac{t^2}{4a^2(t-\tau)}-\frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right\}d\tau,$$

replacement of the form: $z=\sqrt{\frac{t-\tau}{\tau-\tau_1}}$ and the known formula [5, p.321, 3.325] lead to the result:

$$I(t,\tau_1)=2a\sqrt{\pi}\frac{t+\tau_1}{t\tau_1\,(t-\tau_1)^{\frac{3}{2}}}\exp\left\{-\frac{(t+\tau_1)^2}{4a^2(\tau-\tau_1)}\right\}.$$

Substituting the last expression into (14), we have:

$$J(t)=\frac{1}{2a\sqrt{\pi}}\int\limits_0^t\frac{t+\tau_1}{(t-\tau_1)^{\frac{3}{2}}}\exp\left\{-\frac{(t+\tau_1)^2}{4a^2(\tau-\tau_1)}\right\}\varphi\,(\tau_1)\;d\tau_1. \tag{15}$$

Taking into account (13)–(15), we obtain equation (12).

3. Properties of Integral Equation (12)

Introducing the notation:

$$K(t,\,\tau)=\frac{1}{2a\,\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right)+\right.$$

$$\left.+\frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\},$$

we write equation (12) in the form:

$$\varphi(t)-\int\limits_0^t K(t,\,\tau)\;\varphi(\tau)\,d\tau=0. \tag{16}$$

In the equation (16) the kernel K(t, τ) has the properties:

1^0. $K(t,\,\tau)\geq 0$ is continuous at $0<\tau\leq t\leq +\infty$;

2^0. $\lim\limits_{t\to t_0}\int\limits_{t_0}^{t}K(t,\,\tau)\,d\tau=0,\ t_0\geq\varepsilon>0$; 3^0. $\lim\limits_{t\to +0}\int\limits_0^t K(t,\,\tau)\,d\tau=1.$

4. The Characteristic Integral Equation

Singularity of equation (16) consisting in property 3^0 of the kernel $K(t,\,\tau)$ is expressed in the fact that the corresponding homogeneous equation can not be solved by the method of successive approximations.

In integral equation (16) we transform its kernel $K(t,\tau)$. We obtain:

$$K(t,\tau) = k(t,\tau)\exp\left\{-\frac{t-\tau}{4a^2}\right\}, \tag{17}$$

where

$$k(t,\tau) = \frac{1}{a\sqrt{\pi}}\frac{t}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}+ \tag{18}$$

$$+\frac{1}{2a\sqrt{\pi}}\frac{1}{(t-\tau)^{\frac{1}{2}}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

It is known that to solve equation (16) it is sufficient to find a solution of the "simplified" equation:

$$\tilde{\varphi}(t) - \int_0^t k(t,\tau)\,\tilde{\varphi}(\tau)\,d\tau = 0, \tag{19}$$

where $\tilde{\varphi}(t) = \exp\{t/(4a^2)\}\varphi(t)$. To study integral equation (19) we allocate its characteristic part, namely:

$$\tilde{\varphi}(t) - \int_0^t k_0(t,\tau)\,\tilde{\varphi}(\tau)\,d\tau = f_1(t), \tag{20}$$

where

$$k_0(t,\tau) = \frac{t}{a\sqrt{\pi}\,(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}, \tag{21}$$

$$f_1(t) = \int_0^t k_1(t,\tau)\;\tilde{\varphi}(\tau)\,d\tau,$$

$$k_1(t,\tau) = \frac{1}{2a\sqrt{\pi}\,(t-\tau)^{\frac{1}{2}}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

Equation (20) is the characteristic equation for (19), since:

$$\lim_{t\to+0}\int_0^t k_0(t,\,\tau)\,d\tau = 1; \quad \lim_{t\to+0}\int_0^t k_1(t,\,\tau)\,d\tau = 0.$$

5. Solving the Characteristic Integral Equation

Assuming that the right side of equation (20) is known, we will find its solution, that is the solution to the characteristic equation.

Similarly, as in the works [6–9] equation (20) will be reduced to an equation with a difference kernel. For this purpose, producing the substitutions therein:

$$t = \frac{1}{y},\ \tau = \frac{1}{x},\ \psi(y) = \frac{1}{\sqrt{y}}\,\tilde{\varphi}\left(\frac{1}{y}\right),\ f_2(y) = \frac{1}{\sqrt{y}}\,f_1\left(\frac{1}{y}\right), \tag{22}$$

we get: $\forall\, y > 0$,

$$\psi\,(y) - \int_y^\infty \frac{1}{a\,\sqrt{\pi}(x-y)^{\frac{3}{2}}}\ \exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi\,(x)\,d\,x = f_2(y). \tag{23}$$

Note that integral equations of the second kind of this kind with infinite upper and lower limits of the variables of integration in the case of a difference kernel appear in a number of applied problems.

The solution of equation (23) can be found by operational method [10–12] or by reducing it to the Riemann boundary value problem [13]. It is proved that the homogeneous equation:

$$\psi\,(y) - \int_y^\infty \frac{1}{a\,\sqrt{\pi}(x-y)^{\frac{3}{2}}}\ \exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi\,(x)\ d\,x\ = 0,$$

corresponding to (23), has the unique solution $\psi(y) = C_1 = \text{const}$, and the solution of the nonhomogeneous equation (23) is the function:

$$\psi(y) = f_2(y) + \int_y^\infty r_-(y-x)\,f_2(x)\,dx + C_1\ (C_1 = \text{const}), \tag{24}$$

where (see [10, p.86, according to formula (2.1.56)] when $\lambda = 1$):

$$r_{-}(-\theta) = \frac{1}{a\sqrt{\pi}\,\theta^{\frac{3}{2}}} \sum_{n=1}^{\infty} n \cdot \exp\left\{-\frac{n^2}{a^2\theta}\right\}, \ \theta > 0.$$

After substitutions inverse to (22), we obtain the solution to nonhomogeneous equation (20):

$$\tilde{\varphi}(t) = f_1(t) + \int_0^t r(t,\tau)\ f_1(\tau)\,d\tau + \frac{C_1}{\sqrt{t}}, \tag{25}$$

where

$$r(t,\tau) = \frac{t}{a\sqrt{\pi}\ (t-\tau)^{\frac{3}{2}}} \sum_{n=1}^{\infty} n \cdot \exp\left\{-n^2 \frac{t\tau}{a^2(t-\tau)}\right\}. \tag{26}$$

6. Reducing to the Abelian Equation

Now we proceed to solving the equation (19), i.e. "simplified" variant of initial equation (16).

Using formulas (25)–(26) for the solution of the characteristic equation (20), and taking into account relations (21) for the function $f_1(t)$, we obtain:

$$\tilde{\varphi}(t) = \int_0^t k_1(t,\tau)\,\tilde{\varphi}(\tau)\,d\tau + \int_0^t r(t,\tau) \int_0^{\tau} k_1(\tau,\tau_1)\,\tilde{\varphi}(\tau_1)\,d\tau_1\,d\tau + \frac{C_1}{\sqrt{t}},$$

where

$$k_1(t,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\left(1 - \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

Changing the order of integration on the right side of this equation, and interchanging the roles for τ and τ_1, we get:

$$\tilde{\varphi}(t) = \int_0^t \left\{ k_1(t,\tau) + \int_{\tau}^t r(t,\tau_1)k_1(\tau_1,\tau)\,d\tau_1 \right\} \tilde{\varphi}(\tau)\,d\tau + \frac{C_1}{\sqrt{t}}. \tag{27}$$

Using the expression for the resolvent $r(t,\tau)$ (26), we calculate the inner integral in (27):

$$J(t,\tau) = \int_{\tau}^t r(t,\tau_1)\,k_1(\tau_1,\tau)\,d\tau_1 = \tag{28}$$

$$= \frac{t}{2a^2\pi} \sum_{n=1}^{\infty} n \cdot \int_{\tau}^{t} \frac{1}{(t-\tau_1)^{\frac{3}{2}} \sqrt{\tau_1 - \tau}} \exp\left\{-n^2 \frac{t\tau_1}{a^2(t-\tau_1)}\right\} \times$$

$$\times \left(1 - \exp\left\{-\frac{\tau_1 \tau}{a^2(\tau_1 - \tau)}\right\}\right) d\tau_1 = \frac{t}{2a^2\pi} \sum_{n=1}^{\infty} n \cdot I_n(t,\tau)\,,$$

where

$$I_n(t,\,\tau) = \int_{\tau}^{t} \frac{1}{(t-\tau_1)^{\frac{3}{2}} \sqrt{\tau_1 - \tau}} \exp\left\{-n^2 \frac{t\tau_1}{a^2(t-\tau_1)}\right\} \times$$

$$\times \left(1 - \exp\left\{-\frac{\tau_1 \tau}{a^2(\tau_1 - \tau)}\right\}\right) d\tau_1 = I_n^{(1)}(t,\,\tau) - I_n^{(2)}(t,\,\tau),$$

and

$$I_n^{(1)}(t,\,\tau) = \int_{\tau}^{t} \frac{1}{(t-\tau_1)^{\frac{3}{2}} (\tau_1 - \tau)^{\frac{1}{2}}} \exp\left\{-\frac{n^2 t\,\tau_1}{a^2(t-\tau_1)}\right\}\, d\tau_1\,,$$

$$I_n^{(2)}(t,\,\tau) = \int_{\tau}^{t} \frac{1}{(t-\tau_1)^{\frac{3}{2}} (\tau_1 - \tau)^{\frac{1}{2}}} \times$$

$$\times \exp\left\{-\left(\frac{n^2 t\,\tau_1}{a^2(t-\tau_1)} + \frac{\tau_1 \tau}{a^2(\tau_1 - \tau)}\right)\right\}\, d\tau_1.$$

We calculate the integrals $I_n^{(1)}(t;\,\tau)$ $I_n^{(2)}(t;\,\tau)$. We make the substitution: $z = \sqrt{\frac{\tau_1 - \tau}{t - \tau_1}}$, $\quad \tau_1 = \frac{t\,z^2 + \tau}{1 + z^2}$. After substituting integrals take the form:

$$I_n^{(1)}(t;\,\tau) = \frac{2}{t-\tau} \exp\left\{-\frac{n^2 t\,\tau}{a^2(t-\tau)}\right\} \int_{0}^{\infty} \exp\left\{-\frac{n^2 t^2\, z^2}{a^2(t-\tau)}\right\} dz,$$

$$I_n^{(2)}(t;\,\tau) = \frac{2}{t-\tau} \exp\left\{-\frac{(n^2+1)t\,\tau}{a^2(t-\tau)}\right\} \times$$

$$\times \int_{0}^{\infty} \exp\left\{-\frac{n^2 t^2}{a^2(t-\tau)} \cdot z^2 - \frac{\tau^2}{a^2(t-\tau)} \cdot \frac{1}{z^2}\right\} dz.$$

Now, using the replacement $\xi = \frac{n\,t\,z}{a\sqrt{t-\tau}}$ for the first integral, we obtain:

$$I_n^{(1)}(t;\,\tau) = \frac{2}{t-\tau} \exp\left\{-\frac{n^2 t\,\tau}{a^2(t-\tau)}\right\} \frac{a\sqrt{t-\tau}}{n\,t} \times$$

$$\times \int\limits_0^\infty \exp\left\{-\xi^2\right\}\, d\xi = \frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}} \exp\left\{-\frac{n^2 t\,\tau}{a^2(t-\tau)}\right\}.$$

For the second integral using the well-known formula from [5, p.321, 3.325], we get:

$$I_n^{(2)}(t;\,\tau) = \frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}} \exp\left\{-\frac{(n^2+1)t\,\tau}{a^2(t-\tau)}\right\} \exp\left\{-2\cdot\frac{n\,t\tau}{a^2(t-\tau)}\right\} =$$

$$= \frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}} \exp\left\{-\frac{(n+1)^2 t\,\tau}{a^2(t-\tau)}\right\}.$$

So we have

$$I_n(t;\,\tau) = I_n^{(1)}(t;\,\tau) - I_n^{(2)}(t;\,\tau) =$$

$$= \frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}} \left(\exp\left\{-\frac{n^2 t\,\tau}{a^2(t-\tau)}\right\} - \exp\left\{-\frac{(n+1)^2 t\,\tau}{a^2(t-\tau)}\right\}\right).$$

Substituting into (28), we obtain:

$$J\,(t,\tau) = \frac{1}{2a\,\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{t\,\tau}{a^2(t-\tau)}\right\}.$$

Thus, equation (27) takes the form:

$$\tilde{\varphi}(t) = \int\limits_0^t \left[k_1(t,\tau) + J(t,\tau)\right]\, \tilde{\varphi}(\tau)\, d\tau + \frac{C_1}{\sqrt{t}}.$$

Finally, we have:

$$\tilde{\varphi}(t) - \frac{1}{2a\,\sqrt{\pi}} \int\limits_0^t \frac{\tilde{\varphi}(\tau)}{\sqrt{t-\tau}} d\tau = \frac{C_1}{\sqrt{t}}. \qquad (29)$$

Thus, "simplified" integral equation (19) is reduced to equation (29) which is the nonhomogeneous Abelian integral equation of the second kind.

7. Solving the Abelian Equation (29)

We will find a solution $\tilde{\varphi}_h(t)$ of Abelian equation (29), corresponding homogeneous equation (19) (for simplicity, we assume that a constant C_1 is equal to one).

Under these conditions, the equation (29) takes the form:

$$\tilde{\varphi}_h(t) - \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\tilde{\varphi}_h(\tau)}{\sqrt{t-\tau}} d\tau = \frac{1}{\sqrt{t}}, \ t > 0,$$

which can be rewritten as:

$$\frac{1}{2a} \left(I_{0+}^{1/2} \tilde{\varphi}_h \right)(t) \equiv \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\tilde{\varphi}_h(\tau)}{\sqrt{t-\tau}} d\tau = \tilde{\varphi}_h(t) - \frac{1}{\sqrt{t}}, \ t > 0, \tag{30}$$

where the left expression is written by using the fractional integration operator of Riemann-Liouville $I_{0+}^{1/2}$ of the order 1/2 [14, p.38–39, 41–43, 84–86]. Considering that the right side of (30) is temporarily known and using the fractional integration operator of Riemann-Liouville $\mathscr{D}_{0+}^{1/2}$ of the order 1/2 [14]:

$$(\mathscr{D}_{0+}^{1/2} \psi)(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\psi(\tau) d\tau}{\sqrt{t-\tau}},$$

we find as a solution to the Abelian equation of the first kind (30) [14, p.38–39, 50, 96, 105]:

$$\frac{1}{2a} \tilde{\varphi}_h(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\tilde{\varphi}_h(\tau) d\tau}{\sqrt{t-\tau}}, \tag{31}$$

since $(\mathscr{D}_{0+}^{1/2} \psi)(t) = 0$ for $\psi(t) = t^{-1/2}$.

Further, differentiating once equation (30) with respect to t , we get:

$$\frac{1}{2a\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\tilde{\varphi}_h(\tau) d\tau}{\sqrt{t-\tau}} = \tilde{\varphi}_h'(t) + \frac{1}{2\, t^{3/2}}, \ t > 0. \tag{32}$$

Now, multiplying by $1/(2a)$ equation (31) and adding the obtained left and right sides of equations (31) and (32), we obtain the differential equation:

$$\tilde{\varphi}_h'(t) - \frac{1}{4a^2} \tilde{\varphi}_h(t) = -\frac{1}{2\, t^{3/2}}, \ t > 0, \tag{33}$$

with terminal condition:

$$\lim_{t\to\infty} \tilde{\varphi}_h(t) \exp\left\{-\frac{t}{4a^2}\right\} = \frac{\sqrt{\pi}}{a}. \tag{34}$$

The solution of problem (33)–(34) has the form:

$$\tilde{\varphi}_h(t) = \frac{1}{\sqrt{t}} + \frac{\sqrt{\pi}}{2a} \exp\left\{\frac{t}{4a^2}\right\} \left[1 + \operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right)\right]. \tag{35}$$

This is the solution to the Abelian equation (29), corresponding to homogeneous "simplified" equation (19). We note that after the multiplication of equality (35) by $\exp\{-t/(4a^2)\}$ (i.e. taking into account the substitution after equation (19)), we obtain the solution $\varphi_h(t)$ of the homogeneous equation corresponding to initial equation (12) (or, equivalently, to equation (16)):

$$\varphi_h(t) = \frac{1}{\sqrt{t}} \exp\left\{-\frac{t}{4a^2}\right\} + \frac{\sqrt{\pi}}{2a} \left[1 + \operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right)\right]. \tag{36}$$

Note that the solution (35) can be obtained directly, using a representation of the solution of the Abelian equation by using the convolution of the fundamental solution in the form of the Mittag-Leffler function [14, p.33, formula (1.90)].

8. Solving the Boundary Value Problems (3) and (1)

Now we can find the solution of the boundary value problem (3) for homogeneous equation of heat conductivity in a degenerating domain, which has a nonzero solution $w(x,t)$, defined by the formula:

$$w(x,t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \left[\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \nu_h(\tau) + \right. \tag{37}$$

$$\left. + \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} \varphi_h(\tau)\right] d\tau,$$

where

$$\nu_h(t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\} \varphi_h(\tau)\, d\tau, \tag{38}$$

and the function $\varphi_h(t)$ is determined according to formula (36).

To find the solution to the boundary problem we calculate the derivative of the solution (37) by x. We have:

$$\frac{\partial w(x,t)}{\partial x} = -\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \nu_h(\tau)d\tau -$$

$$-\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} \varphi_h(\tau)d\tau. \tag{39}$$

From (37)–(39) we obtain

$$w(x,t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{1}{(t-\tau)^{1/2}} \left[\exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\} + \right. \tag{40}$$

$$\left. + \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\right] \varphi_h(\tau)d\tau,$$

$$\frac{\partial w(x,t)}{\partial x} = -\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \left[\frac{x+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\} + \right. \tag{41}$$

$$\left. + \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\right] \varphi_h(\tau)d\tau.$$

From (2), (40) and (41) we obtain the solution of the boundary problem (1) in the form:

$$u(x,t) = \int\limits_0^t \left[\frac{x+\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\} + \right. \tag{42}$$

$$\left. + \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\right] \varphi_h(\tau)d\tau \times$$

$$\times \left(\int\limits_0^t \frac{1}{(t-\tau)^{1/2}} \left[\exp\left\{-\frac{(x+\tau)^2}{4a^2(t-\tau)}\right\} + \right.\right.$$

$$\left.\left. + \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\right] \varphi_h(\tau)d\tau\right)^{-1}.$$

Solution (42) is estimated from below as follows:

$$u(x,t) \geq C \cdot \int_0^t \left[\frac{x+\tau}{(t-\tau)^{3/2}} \exp\left\{ -\frac{(x+\tau)^2}{4a^2(t-\tau)} \right\} + \right. \tag{43}$$

$$\left. + \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{ -\frac{(x-\tau)^2}{4a^2(t-\tau)} \right\} \right] \varphi_h(\tau) d\tau,$$

$0 < x < t,\ t > 0,\ C = \text{const} > 0.$

By virtue of the non-negativity of the integrand on the right side of estimate (43) we obtain the existence of a nontrivial solution (42) to the boundary problem (1) for Burgers equation.

Thus, we have established the following result.

Theorem 1. *Homogeneous boundary value problem (1) for Burgers equation has only one non-trivial solution (42).*

Brief abstract of this work has been published in the Materials of the workshop "Differential operators and modeling of complex systems" (April 7-8, 2017, Almaty, Kazakhstan) [15].

Acknowledgements

This research is financially supported by a grants from the Ministry of Science and Education of the Republic of Kazakhstan (Grants No. 0823/GF4 and 1164/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

[1] Y. Benia, B.-K. Sadallah, Existence of solutions to Burgers equations in domains that can be transformed into rectangles, *Electronic Journal of Differential Equations*, No. 157 (2016), 1–13.

[2] J.M. Burgers, *The nonlinear diffusion equation. Asymptotic solutions and statistical problems*, D.Reidel Publishing Company, Dordrecht-Holland / Boston USA (1974).

[3] M.I. Vishik, A.V. Fursikov, *Mathematical problems of statistical hydrodynamics* , Nauka, Moscow (1980) (in Russian).

[4] A.N. Tihonov, A.A. Samarskij, *Equations of the mathematical physics* , Nauka, Moscow (1977) (in Russian).

[5] I.S. Gradstejn, I.M. Ryzhik, *Tables of integrals, series and products* , Fizmatgiz, Moscow (1971) (in Russian).

[6] M.M. Amangaliyeva, M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, On one homogeneous problem for the heat equation in an infinite angular domain, *Siberian Mathematical Journal*, **56**, No. 6 (2015), 982–995.

[7] M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, On a boundary value problem for the spectrally loaded heat conduction operator. 1 , *Differential Equations*, **43**, No. 4 (2007), 498–508 (in Russian).

[8] M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, On a boundary value problem for the spectrally loaded heat conduction operator. 2 , *Differential Equations*, **43**, No. 6 (2007), 788–794 (in Russian).

[9] M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, On a boundary value problem for the spectrally loaded heat conduction operator, *Siberian Mathematical Journal*, **47**, No. 3 (2006), 527–547.

[10] M.T. Jenaliyev, M.I. Ramazanov, *The loaded equations as perturbations of differential equations* , Gylym, Almaty (2010) (in Russian).

[11] D.M. Akhmanova, M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, On a particular second kind Volterra integral equation with a spectral parameter, *Siberian Mathematical Journal*, **52**, No. 1 (2011), 3–14 (in Russian).

[12] M.M. Amangaliyeva, D.M. Akhmanova, M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity , *Differential Equations*, **47**, No. 2 (2011), 231–243 (in Russian).

[13] A.D. Polyanin, A.V. Manzhirov, *Handbook of integral equations*, Fizmatgiz, Moscow (2003) (in Russian).

[14] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Integrals and derivatives of fractional order, and some applications* , Nauka i Tekhnika, Minsk (1987) (in Russian).

[15] M.M. Amangaliyeva, M.T. Jenaliyev, M.I. Ramazanov, On the existence of a nontrivial solution of the homogeneous boundary value problem for the Burgers equation, *Kazakh Mathematical Journal*, **17**, No. 1 (2017), 26–27.