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GENERALIZED SOLUTION OF A BOUNDARY VALUE PROBLEM UNDER POINT EXPOSURE OF EXTERNAL FORCES

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Abstract: In the paper, we study a problem of constructing the generalized solutions of the boundary value problem with wave equation under point exposure of external forces, when the wave process is described by Fredholm integral-differential equation. We have developed an algorithm for constructing the generalized solution of the boundary value problem, which together with its generalized derivative are elements of a Hilbert space. Sufficient conditions for the existence of a unique generalized solution are found. The presence of an integral term in equation stipulate the construction of two types of approximations of the generalized solution of the boundary problem. We prove the convergence of these approximations to the solution of the boundary value problem.

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1. Problem Formulation

Consider the wave process described by boundary value problem

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$$V_{tt} = V_{xx} + \lambda \int_0^T K(t,\tau)V(\tau,x)d\tau + \sum_{k=1}^{m} g_k(x)\delta(x-x_k)f_k[t,u_k(t)],$$
$$V(0,x) = \psi_1(x),\ V_t(0,x) = \psi_2(x),\ 0 < x < 1, \tag{1}$$
$$V_x(t,0) = 0,\ V_x(t,1) + \alpha V(t,1) = 0,\ 0 < t \le T,$$

where the function $V(t,x)$ describes the state of the elastic thread with length equal to one, which fluctuates under the influence of external disturbing forces $g_k(x)f_k[t,u_k(t)],\ k = 1,2,3,...,m$, attached accordingly to the interior points $x_1, x_2, ..., x_m$ of the interval (0.1); $\psi_1(x) \in H_1(0,1), \psi_2(x) \in H(0,1), g_k(x) \in H(0,1), f_k[t,u_k(t)] \in H(0,T)$ are given functions and functions $f_k[t,u_k(t)]$ are nonlinear functions with respect to functional variable $u_k(t)$ and they are monotone functions, i.e.

$$\frac{\partial f_k[t,u_k(t)]}{\partial u_k(t)} \ne 0, \forall t \in [0,T]. \tag{2}$$

Kernel $K(t,\tau)$ in the domain $D = \{0 \le t \le T, 0 \le \tau \le T\}$ is defined and is an element of space $H(D)$, i.e.

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty. \tag{3}$$

$H(Y)$ is a Hilbert space of square-integrable functions defined on the set Y; $H_1(Y)$ is a Sobolev spaces of the first order. The given functions $g_k(x)$ are continuous for each fixed $k = 1,2,3,...,m$, have continuous derivatives $g'_k(x)$ and satisfy the conditions $g_k(1) = 0$; $\delta(x-x_k), x_k \in (0,1)$ is a singular generalized function of Dirac; T is a fixed moment of time, $\alpha > 0$ is a positive constant; λ is a parameter.

2. Solution

The solution of problem (1) we will seek in the form of Fourier series:

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t)z_n(x),\ V_n(t) = \int_0^1 V(t,x)z_n(x)dx, \tag{4}$$

where $z_n(x)$ are defined as the solution of boundary problem $z''_n(x) + \lambda_n^2 z_n(x) = 0,\quad z'_n(0) = 0,\quad z'_n(1) + \alpha z_n(1) = 0$, and form an orthonormal system of eigenfunctions in the space $H(0,1)$, and the corresponding eigenvalues λ_n satisfy the following conditions $\lambda_n < \lambda_{n+1},\ \lim_{n\to\infty} \lambda_n = \infty$, $(n-1)\pi < \lambda_n < \frac{\pi}{2}(2n-1), n =$

$1, 2, 3, ..., ;$ The Fourier coefficients $V_n(t)$ are defined as the solution of the following Cauchy problem

$$V_n''(x) + \lambda_n^2 V_n(x) = \lambda \int_0^T K(t,\tau) V_n(\tau) d\tau +$$
$$+ \sum_{k=1}^{m} g_k(x_k) z_n(x_k) f_k[t, u_k(t)], \tag{5}$$
$$V_n(0) = \psi_{1n},\ V_n'(0) = \psi_{2n}.$$

And have the form of (6)

$$V_n(t) = \psi_{1n} \cos \lambda_n t + \frac{1}{\lambda_n} \sin \lambda_n t +$$
$$+ \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-\tau) \Big(\lambda \int_0^T K(t,s) V_n(s) ds +$$
$$+ \sum_{k=1}^{m} g_k(x_k) z_n(x_k) f_k[\tau, u_k(\tau)] \Big) d\tau, \tag{6}$$

i.e. defined as the solution of the Fredholm integral equation of the second kind. This equation can be rewritten as

$$V_n(t) = \int_0^T K_n(t,s) V_n(s) ds + a_n(t), \tag{7}$$

where

$$K_n(t,s) = \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-\tau) K(\tau, s) d\tau,\ K(0,s) = 0,$$

$$a_n(t) = \psi_{1n} \cos \lambda_n t + \frac{1}{\lambda_n} \psi_{2n} \sin \lambda_n t +$$
$$+ \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-\tau) \sum_{k=1}^{m} g_k(x_k) z_n(x_k) f_k[\tau, u_k(\tau)] d\tau.$$

Definition 1. The function of form (4), which belongs to the Hilbert space $H(Q),\ Q = (0,1) \times (0,T)$, is called a generalized solution of the boundary value problem (1).

The solution of the integral equation (8) we find by the following formulas [11].

$$V_n(t) = \lambda \int_0^T R_n(t,s,\lambda) a_n(s) ds + a_n(t), \tag{8}$$

where

$$R_n(t,s,\lambda)=\sum_{i=1}^{\infty}\lambda_n^{i-1}K_{n,i}(t,s),\ n=1,2,3,... \tag{9}$$

is a resolvent of the kernel $K_{n,1}(t,s) = K_n(t,s)$, and the iterated kernels $K_{n,i}(t,s)$ are defined by formulas

$$K_{n,i+1}(t,s)=\int_0^T K_n(t,\eta)K_{n,i}(\eta,s)d\eta,\ i=1,2,3,...$$

and the following estimate holds

$$\int_0^T R_n^2(t,s,\lambda)ds\leq\frac{K_0T}{(\lambda_n-|\lambda|\sqrt{K_0T^2})^2}, \tag{10}$$

where the radius of convergence of the Neumann series (9) for the parameter λ and each fixed $n=1,2,3,...$ is determined by the inequality

$$|\lambda|<\frac{\lambda_n}{\sqrt{K_0T^2}}. \tag{11}$$

From (11) follows that the radius of convergence of the Neumann series increases when n is growing. It is not difficult to notice that the Neumann series converges for any $n = 1,2,3,...$, for values of the parameter λ satisfying the following inequality

$$|\lambda|<\frac{\lambda_1}{\sqrt{K_0T^2}}. \tag{12}$$

We note that this interval can be extended by reducing the value of K_0, which closely connected with the kernel $K(t,\tau)$. According to formulas (4), (8), we find the solution of problem (1) by the formula

$$V(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds+a_n(t)\right)z_n(x)=$$

$$=\sum_{n=1}^{\infty}\Big\{\psi_n(t,\lambda)+$$

$$+\frac{1}{\lambda_n}\int_0^T\varepsilon_n(t,\eta,\lambda)\sum_{k=1}^{m}g_k(x_k)z_n(x_k)f_k[\eta,u_k(\eta)]d\eta\Big\}z_n(x); \tag{13}$$

where

$$\psi_n(t,\lambda)=\psi_{1n}\left[\cos\lambda_nt+\lambda\int_0^T R_n(t,s,\lambda)\cos\lambda_nsds\right]+$$
$$+\frac{\psi_{2n}}{\lambda_n}\left[\sin\lambda_nt+\lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_nsds\right]; \tag{14}$$

$$\varepsilon_n(t,\eta,\lambda) = \begin{cases} \sin\lambda_n(t-\eta) + \lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, \\ 0 \le \eta \le t \\ \lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, \ t \le \eta \le T. \end{cases}$$

Note that, function $\varepsilon_n(t,\eta,\lambda)$ is continuous when $t=\eta$.

Lemma 1. *Solution of boundary value problem (1), which is defined by the formulas (13) - (14), is an element of the Hilbert space $H(Q)$.*

Proof. The assertion of the lemma is verified by direct computation and follows from the relation

$$\int_0^T\int_0^1 V^2(t,x)dxdt \le \int_0^T\sum_{n=1}^{\infty}V_n^2(t)dt \le$$

$$\le 2\int_0^T\sum_{n=1}^{\infty}\Bigg\{\psi_n^2(t,\lambda)+$$

$$+\frac{1}{\lambda_n^2}\int_0^T\varepsilon_n^2(t,\eta,\lambda)d\eta\int_0^T\left(\sum_{k=1}^{m}g_k(x_k)z_n(x_k)f_k[\eta,u_k(\eta)]\right)^2 d\eta\Bigg\}dt \le$$

$$\le 2\int_0^T\sum_{n=1}^{\infty}\Bigg\{\psi_n^2(t,\lambda)+$$

$$+\frac{1}{\lambda_n^2}\int_0^T\varepsilon_n^2(t,\eta,\lambda)d\eta\int_0^T m\sum_{k=1}^{m}g_k^2(x_k)z_n^2(x_k)f_k^2[\eta,u_k(\eta)]d\eta\Bigg\}dt \le$$

$$\le 8T\left[1+\frac{\lambda^2K_0T^2}{(\lambda_1-|\lambda|\sqrt{K_0T^2})^2}\right]\Bigg\{\|\psi_1(x)\|^2_{H(0,1)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(0,1)}+$$

$$+mT\sum_{k=1}^{m}g_k^2(x_k)\|f_k[t,u_k(t)]\|^2_{H(0,T)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\Bigg\} \le$$

$$\le 8T\left[1+\frac{\lambda^2K_0T^2}{(\lambda_1-|\lambda|\sqrt{K_0T^2})^2}\right]\Bigg\{\|\psi_1(x)\|^2_{H(0,1)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(0,1)}+$$

$$+mT\sum_{k=1}^{m}g_k^2(x_k)\|f_k[t,u_k(t)]\|^2_{H(0,T)}\left(\frac{1}{\lambda_1^2}+\frac{1}{6}\right)\Bigg\} < \infty.$$

This relation was obtained with taking into account the following inequalities

$$\int_0^T \sum_{n=1}^{\infty} \psi_n^2(t,\lambda)dt \le 4T\left[1+\frac{\lambda^2 K_0 T^2}{(\lambda_1-|\lambda|\sqrt{K_0T^2})^2}\right]\sum_{n=1}^{\infty}\left(\psi_{1n}^2+\frac{1}{\lambda_n^2}\psi_{2n}^2\right);$$

$$\int_0^T \varepsilon_n^2(t,\eta,\lambda)d\eta \le 2T\left(1+\frac{K_0T}{(\lambda_n-|\lambda|\sqrt{K_0T^2})^2}\right).$$

□

Lemma 2. *The solution of the boundary value problem (1), which is defined by formulas (13) - (14), has a generalized derivative and this generalized derivative belongs to the Hilbert space $H(Q)$.*

Proof. Differentiating the series (13), we obtain the function of the following form

$$V_t(t,x) = \sum_{n=1}^{\infty}\Big\{\psi_{nt}(t,\lambda)+ \tag{15}$$

$$+\frac{1}{\lambda_n}\int_0^T \varepsilon_{nt}(t,\eta,\lambda)\sum_{k=1}^{m} g_k(x_k)z_n(x_k)f_k[\eta,u_k(\eta)]d\eta\Big\}z_n(x), \tag{16}$$

where $\psi_{nt}(t,\lambda)$ and $\varepsilon_{nt}(t,\eta,\lambda)$ are derivatives with respect to t variable and are defined by formulas

$$\psi_{nt}(t,\lambda) = \psi_{1n}\left[-\lambda_n \sin\lambda_n t + \lambda\int_0^T R_{nt}(t,s,\lambda)\cos\lambda_n s ds\right]+$$
$$+\frac{\psi_{2n}}{\lambda_n}\left[\lambda_n\cos\lambda_n t+\lambda\int_0^T R_{nt}(t,s,\lambda)\sin\lambda_n s ds\right];$$

$$\varepsilon_{nt}(t,\eta,\lambda) = \begin{cases} \lambda_n\cos\lambda_n(t-\eta)+\lambda\int_0^T R_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, \\ \ 0\le\eta\le t \\ \lambda\int_0^T R_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, \ \ t\le\eta\le T; \end{cases}$$

Note that, when $t=\eta$ function $\varepsilon_{nt}(t,\eta,\lambda)$ has discontinuity equal to the value λ_n.

Assertion of the lemma is verified by direct computation and follows from the following relation

$$\int_0^T\int_0^1 V_t^2(t,x)dxdt = \int_0^T\sum_{n=1}^{\infty} V_{nt}^2 dt \le$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty}\Big\{\psi_{nt}^2(t,\lambda)+$$

$$\frac{1}{\lambda_n^2}\int_0^T \varepsilon_{nt}^2(t,\eta,\lambda)d\eta\int_0^T m\sum_{k=1}^{m} g_k^2(x_k)z_n^2(x_k)f_k^2[\eta,u_k(\eta)]d\eta\Big\}dt\leq$$

$$\leq 8T\left[1+\frac{\lambda^2K_0T^2}{(\lambda_1-|\lambda|\sqrt{K_0T^2})^2}\right]\Big\{\sum_{n=1}^{\infty}(\lambda_n^2\psi_{1n}^2+\psi_{2n}^2)+$$

$$+Tm\sum_{k=2}^{m} g_k^2\|f_k[t,u_k(t)]\|^2_{H(0,T)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\Big\}\leq$$

$$\leq 8T\left[1+\frac{\lambda^2K_0T^2}{(\lambda_1-|\lambda|\sqrt{K_0T^2})^2}\right]\Big\{\|\psi_1(x)\|^2_{H(0,1)}+\|\psi_2(x)\|^2_{H(0,1)}+$$

$$Tm\sum_{n=1}^{\infty}\bar{q}_k^2(x_k)\|f_k[t,u_k(t)]\|^2_{H(0,T)}\left(\frac{1}{6}+\frac{1}{\lambda_1^2}\right)\Big\}<\infty.$$

This relation was obtained with taking account the following inequalities

$$\int_0^T R_{nt}^2(t,s,\lambda)ds\leq\frac{\lambda_n^2K_0T}{(\lambda_n-|\lambda|\sqrt{K_0T^2})^2},$$

$$g_k(x_k)z_n(x_k)=\int_0^1 g_k(x)z_n(x)\delta(x-x_k)dx\leq$$

$$\leq\sqrt{2}\int_0^1 g_k(x)cos\lambda_n x\delta(x-x_k)dx=\frac{\sqrt{2}}{\lambda_n}\bar{g}_k,$$

$$\bar{g}_k=\left|\int_0^1 (g_k(x)\delta(x-x_k))_x\, sin\lambda_n xdx\right|.$$

□

Theorem 2. *Suppose that the conditions (2), (12) hold and functions* $f_k[t,u_k(t)]\in H(0,T)$ *for each* $u_k(t)\in H(0,T)$. *Then the boundary value problem (1) has a unique generalized solution* $V(t,x)$ *and this generalized solution together with the generalized derivative* $V_t(t,x)$ *are elements of the Hilbert space* $H(Q)$.

Proof. The assertion of theorem 1 follows from Lemmas 1 and 2. □

3. Approximate Solutions and their Convergence

Solution of boundary value problem is defined as the sum of the infinite series and it is not always possible to find their explicitly. Therefore, in practice we construct the approximations of the exact solution and prove their convergence. In the boundary value problem (1) the existence of the integral term in equation stipulate the construction of two types of approximations of the generalized solution of the boundary problem.

3.1. Approximation of the Solution of Boundary Value Problem with Respect to the Resolvent and its Convergence

The resolvent of the integral equation (7) is defined as the sum of infinite series (9) and it is not always possible to find explicitly. Therefore, we consider the approximation of the resolvent in the form of the following finite sum

$$R_n^N(t,s,\lambda)=\sum_{i=1}^{N}\lambda^{i-1}K_{n,i}(t,s),$$

and the corresponding solutions of boundary problem (1) we find by formula

$$V^N(t,x)=\sum_{n=1}^{\infty}\Big\{\psi_n^N(t,\lambda)+$$
$$+\frac{1}{\lambda_n}\int_0^T\varepsilon_n^N(t,\eta,\lambda)\sum_{k=1}^{m}g_k(x_k)f_k[\eta,u_k(\eta)]d\eta\Big\}z_n(x);$$

where

$$\begin{aligned}\psi_n^N(t,\lambda)=\psi_{1n}\left[\cos\lambda_n t+\lambda\int_0^T R_n^N(t,s,\lambda)\cos\lambda_n s ds\right]+\\ +\frac{\psi_{1n}}{\lambda_n}\left[\sin\lambda_n t+\lambda\int_0^T R_n^N(t,s,\lambda)\sin\lambda_n s ds\right];\end{aligned}\tag{17}$$

$$\varepsilon_n^N(t,\nu,\lambda)=\begin{cases}\sin\lambda_n(t-\eta)+\lambda\int_0^T R_n^N(t,s,\lambda)\sin\lambda_n(s-\eta)ds,\\ \quad 0\le\eta\le t,\\ \lambda\int_0^T R_n^N(t,s,\lambda)\sin\lambda_n(s-\eta)ds,\quad t\le\eta\le T;\end{cases}$$

$$N=1,2,3,...,.$$

Function $V^N(t,x)$ is called the N-th approximation of the generalized solution (13) of the boundary value problem (1).

Lemma 3. *N-th approximation of the generalized solution of boundary value problem (1) converges to the exact solution in the norm of the Hilbert space $H(Q)$.*

Proof. The assertion of lemma 3 is proved by direct computation i.e. by inequality $|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1} < 1$, the assertion of lemma 3 follows from the following relations

$$\|V(t,x) - V^N(t,x)\|^2_{H(Q)} = \int_0^T \sum_{n=1}^{\infty}\Big(V_n(t) - V_n^N(t)\Big)^2 dt =$$

$$= \int_0^T \sum_{n=1}^{\infty}\Big\{\psi_n(t,\lambda) - \psi_n^N(t,\lambda) +$$

$$+\lambda \int_0^T (\varepsilon_n(t,\eta,\lambda) - \varepsilon_n^N(t,\eta,\lambda)A_n(\eta)d\eta)\Big\}^2 dt \leq$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty}\Big\{(\psi_n(t,\lambda) - \psi_n^N(t,\lambda))^2 +$$

$$+\lambda^2 \int_0^T ((\varepsilon_n(t,\eta,\lambda) - \varepsilon_n^N(t,\eta,\lambda))^2 d\eta \int_0^T A_n^2(\eta)d\eta)\Big\}dt \leq$$

$$\leq 2\lambda^2 T K_0 \left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1}\right)^{2N} \times$$

$$\times\left(1 - \frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}}\right)^2 \int_0^T \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\Big\{2T\Big(\psi_{1n}^2 + \frac{1}{\lambda_n^2}\psi_{2n}^2\Big) +$$

$$+\lambda^2 \int_0^T \Big(\sum_{k=1}^{m} g_k(x_k)z_n(x_k)f_k[\eta,u_k(\eta)]^2\Big)d\eta\Big\}dt \leq$$

$$\leq 2\lambda^2 T^2 K_0 \left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1}\right)^{2N}\left(1 - \frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}}\right)^2 \times$$

$$\times \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\Big\{2T\Big(\|\psi_1(x)\|^2_{H(0,1)} + \frac{1}{\lambda_n^2}\|\psi_2(x)\|^2_{H(0,1)}\Big) +$$

$$+\lambda^2 m \sum_{k=1}^{m} g_k^2(x_k)z_n^2(x_k)\int_0^T f_k^2[\eta,u_k(\eta)]d\eta\Big\} \leq$$

$$\leq 4\lambda^2T^2K_0\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1}\right)^{2N}\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}}\right)^2\left(\frac{1}{\lambda_1^2}+\frac{1}{6}\right)\times$$

$$\times\left\{T\left(\|\psi_1(x)\|^2_{H(0,1)}+\frac{1}{\lambda_n^2}\|\psi_2(x)\|^2_{H(0,1)}\right)+\right.$$

$$\left.+\lambda^2m\sum_{k=1}^{m}g_k^2(x_k)\|f_k[\eta,u_k(\eta)]\|^2_{H(0,T)}\right\}\to 0$$

$$[\psi_n(t,\lambda)-\psi_n^N(t,\lambda)]^2\leq$$

$$\leq 2T\left(\psi_{1n}^2+\frac{1}{\lambda_n^2}\psi_{2n}^2\right)\frac{\lambda^2TK_0}{\lambda_n^2}\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right)^{2N}\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2,$$

$$\int_0^T\left(\varepsilon_n(t,\eta,\lambda)-\varepsilon_n^N(t,\eta,\lambda)\right)^2d\eta\leq$$

$$\left\{\int_0^t\lambda\int_\eta^T\left(R_n(t,s,\lambda)-R_n^N(t,s,\lambda)\right)sin\lambda_n(s-\eta)dsd\eta+\right.$$

$$\left.+\int_t^T\lambda\int_\eta^T\left(R_n(t,s,\lambda)-R_n^N(t,s,\lambda)\right)sin\lambda_n(\eta-s)dsd\eta\right\}^2\leq$$

$$\leq\int_0^T\left(\lambda\int_\eta^T\left(R_n(t,s,\lambda)-R_n^N(t,s,\lambda)\right)sin\lambda_n(\eta-s)ds\right)^2d\eta\leq$$

$$\leq\lambda^2T^2\int_0^T\left(R_n(t,s,\lambda)-R_n^N(t,s,\lambda)\right)^2ds\leq$$

$$\leq\lambda^2\frac{TK_0}{\lambda_n^2}\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right)^{2N}\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2.$$

Thus, the followng relation holds

$$\|V(t,x)-V^N(t,x)\|_{H(Q)}\leq C_1(\lambda)\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1}\right)^N\to 0, \qquad (18)$$

where

$$C_1(\lambda)=2\lambda\sqrt{K_0T^2}\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}}\right)\left\{\left(\frac{1}{\lambda_1^2}+\frac{1}{6}\right)\cdot\right.$$

$$\cdot\Big[T\Big(\|\psi_1(x)\|^2_{H(0,1)}+\frac{1}{\lambda_n^2}\|\psi_2(x)\|^2_{H(0,1)}\Big)+$$
$$+\lambda^2 m\sum_{k=1}^{m} g_k^2(x_k)\|f_k[\eta,u_k(\eta)]\|^2_{H(0,T)}\Big]\Big\}^{1/2}.$$

□

4. The Convergence of the Approximations of Generalized Derivative of the Boundary Value Problem Solution

According to (17), N-th approximation of the generalized derivative we find by formula

$$V_t^N(t,x)=\sum_{n=1}^{\infty}\Big\{\psi_{nt}^N(t,\lambda)+\frac{1}{\lambda_n}\int_0^T\varepsilon_{nt}^N(t,\eta,\lambda)-A_n(\eta)d\eta\Big\}z_n(x), \tag{19}$$

where

$$A_n(\eta)=\sum_{k=1}^{m} g_k(x_k)z_n(x_k)f_k[\eta,u_k(\eta)]d\eta.$$

Lemma 4. *N-th approximation of the generalized derivative of the boundary value problem (1) solution converges to function $V_t(t,x)$ in the norm of Hilbert space $H(Q)$.*

Proof. Assertion of lemma 4 is proved by direct computation, it follows from

$$\|V_t(t,x)-V_t^N(t,x)\|^2_{H(Q)}=\int_0^T\sum_{n=1}^{\infty}\Big\{\psi_{nt}(t,\lambda)-\psi_{nt}^N(t,\lambda)+$$
$$\frac{1}{\lambda_n}\int_0^T(\varepsilon_{nt}(t,\eta,\lambda)-\varepsilon_{nt}^N(t,\eta,\lambda)A_n(\eta)d\eta)\Big\}^2dt\le$$
$$\le 2\int_0^T\sum_{n=1}^{\infty}\Big\{(\psi_{nt}(t,\lambda)-\psi_{nt}^N(t,\lambda))^2+$$
$$+\frac{1}{\lambda_n^2}\int_0^T((\varepsilon_{nt}(t,\eta,\lambda)-\varepsilon_{nt}^N(t,\eta,\lambda))^2d\eta\int_0^T A_n^2(\eta)d\eta)\Big\}dt\le$$
$$\le 2\int_0^T\sum_{n=1}^{\infty}\Big\{2\lambda^2K_0T^2\Big(\psi_{1n}^2+\frac{1}{\lambda_n^2}\psi_{2n}^2\Big)\Big(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\Big)^{2N}\times$$

$$\times\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2+\frac{1}{\lambda_n^2}\lambda^2T^4K_0\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right)^{2N}\times$$

$$\times\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2\int_0^T A_n^2(\eta)d\eta)\Big\}dt\leq$$

$$\leq 2T\sum_{n=1}^{\infty}\Big\{2\lambda^2K_0T^2\left(\psi_{1n}^2+\frac{1}{\lambda_n^2}\psi_{2n}^2\right)\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right)^{2N}\times$$

$$\times\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2+\lambda^2K_0T^2T_0^2\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right)^{2N}\times$$

$$\times\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2 m\sum_{k=1}^{m}g_k^2(x_k)z_n^2(x_k)\|f_k[t,u_k(t)]\|_{H(0,T)}^2\Big\},$$

which is obtained based on the following inequalities

$$[\psi_{nt}(t,\lambda)-\psi_{nt}^N(t,\lambda)]^2\leq$$

$$\leq 2T\lambda^2\left(\psi_{1n}^2+\frac{1}{\lambda_n^2}\psi_{2n}^2\right)TK_0\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right)^{2N}\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2,$$

$$\int_0^T\Big(\varepsilon_{nt}(t,\eta,\lambda)-\varepsilon_{nt}^N(t,\eta,\lambda)\Big)^2d\eta\leq$$

$$\leq\lambda^2T^3K_0\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_n}\right)^{2N}\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_n}}\right)^2.$$

Thus, the followng relation holds

$$\|V_t(t,x)-V_t^N(t,x)\|_{H(Q)}\leq C_2(\lambda)\left(|\lambda|\frac{\sqrt{K_0T^2}}{\lambda_1}\right)^N\to 0, \tag{20}$$

where

$$C_2^2(\lambda)=4T\lambda^2K_0T^2\left(1-\frac{1}{ln\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1}}\right)\Big\{\|\psi_1(x)\|_{H(0,1)}^2+$$

$$+\frac{1}{\lambda_1^2}\|\psi_2(x)\|_{H(0,1)}^2+Tm\sum_{k=1}^{m}\|f_k[t,u_k(t)]\|_{H(0,T)}^2\overline{q}_k\left(\frac{1}{\lambda_1}+\frac{1}{6}\right)\Big\}.$$

□

5. A Finite-Dimensional Approximation of the Boundary Value Problem Solution and its Convergence

In practice we usually deal with the finite-dimensional approximations of the solution. According to (16) and (18) we find the finite-dimensional approximations of the generalized solution of problem (1) and its generalized derivative by the following formulas

$$V_r^N(t,x) = \sum_{n=1}^{r} \left\{ \psi_n^N(t,\lambda) + \frac{1}{\lambda_n} \int_0^T \varepsilon_n^N(t,\eta,\lambda) - A_n(\eta) d\eta \right\} z_n(x),$$

$$V_{rt}^N(t,x) = \sum_{n=1}^{r} \left\{ \psi_{nt}^N(t,\lambda) + \frac{1}{\lambda_n} \int_0^T \varepsilon_{nt}^N(t,\eta,\lambda) - A_n(\eta) d\eta \right\} z_n(x).$$

Initially we will prove their convergence to functions (16) and (18). This follows from the following equations

$$\begin{aligned} \|V(t,x) - V_r^N(t,x)\|^2_{H(Q)} &= \int_0^T \sum_{n=r+1}^{\infty} \Big\{ \psi_n^N(t,\lambda) + \\ &+ \frac{1}{\lambda_n} \int_0^T \varepsilon_n^N(t,\eta,\lambda) A_n(\eta) d\eta \Big\}^2 dt \le \\ \le 2 \int_0^T \sum_{n=r+1}^{\infty} & \left\{ \psi_n^{N2}(t,\lambda) + \frac{1}{\lambda_n^2} \int_0^T \varepsilon_n^{N2}(t,\eta,\lambda) A_n^2(\eta) d\eta \right\} dt \le \\ \le 8T \left(1 + \frac{\lambda^2 K_0 T^2}{(\lambda_1 - |\lambda| \sqrt{K_0 T^2})^2} \right) & \Big\{ \sum_{n=r+1}^{\infty} \psi_{1n}^2 + \frac{1}{\lambda_1^2} \sum_{n=r+1}^{\infty} \psi_{2n}^2 + \\ + mT \sum_{k=1}^{m} g_k^2(x_k) \|f_k[t,u_k(t)]\|^2_{H(0,T)} & \sum_{n=r+1}^{\infty} \frac{1}{\lambda_n^2} \Big\} \to 0 \end{aligned} \tag{21}$$

and

$$\begin{aligned} \|V_t^N(t,x) - V_{rt}^N(t,x)\|^2_{H(Q)} &= \int_0^T \sum_{n=r+1}^{\infty} \Big\{ \psi_{nt}^N(t,\lambda) + \\ &+ \frac{1}{\lambda_n} \int_0^T \varepsilon_{nt}^N(t,\eta,\lambda) A_n(\eta) d\eta \Big\}^2 dt \le \\ \le 2 \int_0^T \sum_{n=r+1}^{\infty} & \left\{ \psi_{nt}^{N2}(t,\lambda) + \frac{1}{\lambda_n^2} \int_0^T \varepsilon_{nt}^{N2}(t,\eta,\lambda) A_n^2(\eta) d\eta \right\} dt \le \end{aligned}$$

$$\leq 8T\left(1+\frac{\lambda^2 K_0 T^2}{(\lambda_1-|\lambda|\sqrt{K_0T^2})^2}\right)\Big\{\sum_{n=r+1}^{\infty}\psi_{1n}^2+\frac{1}{\lambda_1^2}\sum_{n=r+1}^{\infty}\psi_{2n}^2+ \tag{22}$$
$$+Tm\sum_{k=1}^{m}\overline{g}_k^2\|f_k[t,u_k(t)]\|^2_{H(0,T)}\sum_{n=r+1}^{\infty}\frac{1}{\lambda_n^2}\Big\}\to 0$$

for any $N=1,2,3,...,$.

Theorem 3. *Finite approximations of the generalized solution of problem (1) and its generalized derivative converge to functions (13) and (16) in norm of Hilbert space $H(Q)$.*

Proof. Assertion of the theorem follows from the following relations

$$\|V(t,x)-V_r^N(t,x)\|_{H(Q)}\leq\|V(t,x)-V^N(t,x)\|_{H(Q)}+$$
$$+\|V^N(t,x)-V_r^N(t,x)\|_{H(Q)}\to 0,$$
$$\|V_t(t,x)-V_{rt}^N(t,x)\|_{H(Q)}\leq\|V_t(t,x)-V_t^N(t,x)\|_{H(Q)}+$$
$$+\|V_t^N(t,x)-V_{rt}^N(t,x)\|_{H(Q)}\to 0,$$

□

which take place according to the formulas (18) and (20) - (5).

Conclusion

A boundary value problem of the form (1) is frequently found in practice, in particular in the study of the optimal control problem for the oscillation processes described by integro-differential equations in partial derivatives. Herewith functions $u_k(t)\in H(0,T)$, $k=1,2,3,...,m$ can be consider as the control parameters, by which the controlled process can be converted from one state to another predetermined desired state. In this context, the study of the problem (1) can be continued as the problem of optimal control.

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