International Journal of Pure and Applied Mathematics
Volume 113 No. 5 2017, 627-641
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i5.10

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# CONVERGENCE AND STABILITY OF THE IMPROVED PREDICTOR CORRECTOR METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS DRIVEN BY LIU'S PROCESS

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**Abstract:** Fuzzy differential equation (FDE) is a type of differential equation driven by Liu's process. In this paper, a concept of $\alpha$-path to FDE is presented, which is a type of certain function that solves an associate ordinary differential equation. Then, an improved predictor corrector (IPC) method is introduced to solve FDEs, which essentially solves each $\alpha$-path and produces an inverse credibility distribution of the solution. Moreover, the convergence and stability of the IPC method is presented in details. We explain that our estimation has a good degree of accuracy.

**Key Words:** Fuzzy Liu's process, Improved predictor corrector (IPC) method, Convergence, Stability

## 1. Introduction

Most of phenomena and events in the real world occur unexpectedly among which are the changes in economic and political systems, collapse of govern-

Received: —???—
Revised: —???—
Published: April 1, 2017



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ments, conflicts between tribes, wars, terrorist attacks. Thus, it is not possible to anticipate or estimate, the price of stocks, valuable papers, monetary units and precious metals accurately. Therefore, the only way find out how this factor can affect the growth or drop in the value of companies is focusing on the price of stocks.

Investigation on effects of the factors along with uncertainty theory can help better understanding and more exact modeling of these phenomena. The credibility theory was first introduced by Liu who then presented the concept of credibility measure which is powerful tool for dealing with fuzzy phenomena, to facilitate measuring of fuzzy events that are based on normality, monotonicity, self-duality, and maximality axioms.

Then the concept of fuzzy process was proposed by that introduces a particular fuzzy process with stationary and independent increment named Liu's process which is just like a stochastic process described by Brownian motion.

Since then some literatures has been published on the Liu's process and its applications in other sciences, such as economics and optimal control has been published [25]. Then Liu was inspired by stochastic notions and ito process to introduce fuzzy differential equations [10] which were driven by Liu's process for better understanding the fuzzy phenomena.

Regarding to the importance of existence and uniqueness of a solution to fuzzy differential equations driven by Liu's process, Liu investigated the existence and uniqueness of solution to the fuzzy differential equations by employing Lipschitz and Linear growth conditions [23]. Afterward, Fei studied the uniqueness of solution to the fuzzy differential equations driven by Liu's process with non-Lipschitz coefficients [4].

In this paper, we aim at obtaining numerical solutions for FDEs. First, we introduce concept of $\alpha$-path to FDE, which is a type of certain function that solves an associate ordinary differential equation. Then, an improved predictor corrector (IPC) method is designed to solve FDEs, which essentially solves each $\alpha$-path and produces an inverse credibility distribution of the solution. The IPC method is generated by combining an explicit three-step method and an implicit two-step method.

The remainder of this paper is organized as follows: Section 2 is intended to introduce some basic concepts and theorems in credibility theory and fuzzy differential equation. An explicit three-step method, an implicit two-step method for solving FDEs and, the IPC three-step algorithm are proposed. The convergence and stability of the mentioned methods are proved in Section 4. we introduced a brief survey of fundamental concepts from financial mathematics and some numerical examples are presented in Section 5. The last section contains a brief summary.

## 2. Preliminaries

In this section, we introduce some basic concepts and theorems about credibility theory and fuzzy differential equation driven by Liu's process, which are used throughout this paper.

### 2.1. Credibility theory

Suppose that $\Theta$ is a non-empty set and $\mathcal{P}$ is the power set of $\Theta$. Each element of $\mathbf{A}$ in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign a number $\mathbf{Cr}\{\mathbf{A}\}$ to each event $\mathbf{A}$ which indicates the credibility that $\mathbf{A}$ will occur. To ensure that the number $\mathbf{Cr}\{\mathbf{A}\}$ has certain mathematical properties which we intuitively expect to have a credibility, we accept the following four axioms [8]:

1. Axiom (Normality) $\mathbf{Cr}\{\Theta\} = 1$.
2. Axiom (Monotonicity) $\mathbf{Cr}\{\mathbf{A}\} \leq \mathbf{Cr}\{\mathbf{B}\}$ whenever $\mathbf{A} \subset \mathbf{B}$.
3. Axiom (Self-Duality) $\mathbf{Cr}\{\mathbf{A}\} + \mathbf{Cr}\{\mathbf{A}^c\} = 1$ for any event $\mathbf{A}$.
4. Axiom (Maximality) $\mathbf{Cr}\{\mathbf{U}_i\mathbf{A}_i\} = \sup_i \mathbf{Cr}\{\mathbf{A}_i\}$ for any events $\{\mathbf{A}_i\}$ with

$$\sup_i \mathbf{Cr}\{\mathbf{A}_i\} < 0.5.$$

**Definition 2.1** [9]. The set function $\mathbf{Cr}$ is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.

A family $\mathcal{P}$ with these four properties is called a $\sigma$-algebra. The pair $(\Theta, \mathcal{P})$ is called a measurable space, and the elements of $\mathcal{P}$ is afterwards called $\mathcal{P}$-measurable sets instead of events.

**Definition 2.2** [9]. Let $\Theta$ be a nonempty set, $\mathcal{P}$ the power set of $\Theta$, and $\mathbf{Cr}$ a credibility measure. The triple $(\Theta, \mathcal{P}, \mathbf{Cr})$ is called a credibility space.

Let $(\Theta, \mathcal{P}, \mathbf{Cr})$ be a credibility space. A filtration is a family $\{\mathcal{P}_t\}_{t\geq 0}$ of increasing sub-$\sigma$-algebras of $\mathcal{P}$ (i.e. $\mathcal{P}_t \subset \mathcal{P}_s \subset \mathcal{P}$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $\mathcal{P}_t = \bigcap_{s>t} \mathcal{P}_s$ for all $t \leq 0$. When the credibility space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{P}_0$ contains all $\mathbf{Cr}$-null sets.

We also define $\mathcal{P}_\infty = \sigma(U_{t\geq 0}\mathcal{P}_t)$ (i.e. $\sigma$-algebra generated by $U_{t\geq 0}\mathcal{P}_t$.) $\mathcal{P}$-measurable fuzzy variable is denoted by $\mathbf{L}^p(\Theta, \mathbf{R^d})$ that will be defined later.

A process is called $\mathcal{P}$-adapted, if for all $t \in [0, t]$ the fuzzy variable $x(t)$ is $\mathcal{P}$-measurable.

**Definition 2.3** [9]. A fuzzy variable is defined as a (measurable) function $\xi : (\Theta, \mathcal{P}, \mathbf{Cr}) \longrightarrow \mathbf{R}$.

**Definition 2.4** [9]. Let $\xi$ be a fuzzy variable. Then the expected value of $\xi$ is defined by

$$\mathbf{E}[\xi] = \int_0^{+\infty} \mathbf{Cr}\{\xi \geq r\}\mathrm{dr} - \int_{-\infty}^{0} \mathbf{Cr}\{\xi \leq r\}\mathrm{dr}$$

provided that at least one of the two integrals is finite. Furthermore, the variance is defined by $\mathbf{E}[(\xi - e)^2]$.

Let $\xi$ and $\eta$ be independent fuzzy variables with finite expected values. Then for any numbers $a$ and $b$, we have

$$\mathbf{E}[a\xi + b\eta] = a\mathbf{E}[\xi] + b\mathbf{E}[\eta].$$

**Definition2.5** [13]. The credibility distribution $\mu(x)$ of a fuzzy variable $\xi$ is defined by

$$\mu(x) = \max\{1, 2\mathbf{Cr}(\xi = x)\}, \quad x \in \mathbf{R}.$$

**Definition2.6** [11]. A credibility distribution $\mu(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to $x$ at which $0 < \mu(x) < 1$, and

$$\lim_{x\to -\infty} \mu(x) = 0, \quad \lim_{x\to +\infty} \mu(x) = 1.$$

In addition, the inverse function $\mu^{-1}(\alpha)$ is called the inverse credibility distribution of $\xi$.

**Definition 2.7** [9]. Let $\mathbf{T}$ be an index set and $(\Theta, \mathcal{P}, \mathbf{Cr})$ be a credibility space. A fuzzy process is a function from $\mathbf{T} \times (\Theta, \mathcal{P}, \mathbf{Cr})$ to the set of real numbers.

That is, a fuzzy process $X_t(\theta)$ is a function of two variables such that the function $X_{t^*}(\theta)$ is a fuzzy variable for each $t^*$. For each fixed $\theta^*$, the function $X_t(\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $X_t(\theta)$ is said to be sample-continuous if the sample path is continuous for almost all $\theta$.

In the following expression, we use the notation $\mathbf{x}(t)$ instead of $\mathbf{x}_t(\theta)$.

A fuzzy process is essentially a sequence of fuzzy variables indexed by time or space. As one of the most important types of fuzzy processes, the Liu's process is defined as follows.

**Definition 2.8** [13]. A fuzzy process $\mathbf{C}_t$ is said to be a Liu's process if

1. $\mathbf{C}_0 = 0$,
2. $\mathbf{C}_t$ has stationary and independent increments,
3. every increment $\mathbf{C}_{t+s} - \mathbf{C}_s$ is a normally distributed fuzzy variable with expected value **et** and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2(1 + \exp(\tfrac{\pi|x-et|}{\sqrt{6}\sigma t}))^{-1}, \qquad -\infty < x < +\infty.$$

   The parameters $e$ and $\sigma$ are called the drift and diffusion coefficients, respectively, in addition, Liu's process is said to be standard if $e = 0$ and $\sigma = 1$.

Based on Liu's process, Liu integral is defined as a fuzzy counterpart of Ito integral as follows.

**Definition 2.9** [9]. Suppose that $x(t)$ is a fuzzy process and $\mathbf{C}_t$ is a standard Liu's process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \ldots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then the Liu integral of $x(t)$ with respect to $\mathbf{C}_t$ is

$$\int_a^b x(t) d\mathbf{C}_t = \lim_{\Delta \longrightarrow 0} \sum_{i=1}^k x(t_i).(\mathbf{C}_{t_{i+1}} - \mathbf{C}_{t_i})$$

provided that the limit exists almost surely and is a fuzzy variable.

### 2.2. Fuzzy Differential Equation

In 2008, Liu [13] presented a type of fuzzy differential equations driven by the Liu's process.

**Definition2.10** [11]. Suppose $\mathbf{C}_t$ is a Liu's process and $f, g$ are some given functions,

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{C}_t \tag{1}$$

is called a fuzzy differential equation with an initial value $x(0)$. The solution is fuzzy process $x(t)$ that satisfies (1) identically in t.

Chen and Liu [4] proved the existence and uniqueness theorem of solution of fuzzy differential equation under linear growth condition and Lipschitz continuous condition.

**Theorem 2.1** [9]. The fuzzy differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{C}_t$$

has a unique solution if the coefficients $f(t,x)$ and $g(t,x)$ satisfy the linear growth condition

$$|f(t,x)|+|g(t,x)|\le \mathbf{L}(1+|x|)\quad \forall x\in\mathbf{R},\quad t\ge 0$$

and Lipschitz condition

$$|f(t,x)-f(t,y)|+|g(t,x)-g(t,y)|\le \mathbf{L}|x-y|,\forall x,y\in\mathbf{R},\quad t\ge 0$$

for some constant $\mathbf{L}$. Moreover, the solutionis sample continuous.

We now introduce some definition and theorems that form the foundation of this paper for fuzzy differential equation similar to [21] for uncertain differential equation.

**Definition 2.11** The $\alpha$-path $(0<\alpha<1)$ of a fuzzy differential equation

$$dx(t)=f(x(t),t)dt+g(x(t),t)d\mathbf{C}_t$$

with initial value $x(0)$ is a deterministic function $x^{\alpha}(0)$ with respect to t that solves the corresponding ordinary differential equation

$$dx^{\alpha}(t)=f(x^{\alpha}(t),t)dt+|g(x^{\alpha}(t),t)|\mu^{-1}(\alpha)dt \tag{2}$$

where

$$\mu^{-1}(\alpha)=\tfrac{\sqrt{6}}{\pi}\ln(\tfrac{2-\alpha}{\alpha}).$$

**Theorem 2.2** Let $x(t)$ and $x^{\alpha}(t)$ be the solution and $\alpha$-path of fuzzy differential Eq.(1), respectively. Then

$$\mathbf{Cr}\{x(t)\le x^{\alpha}(t),\quad \forall t\}=\alpha,$$

$$\mathbf{Cr}\{x(t)> x^{\alpha}(t),\quad \forall t\}=1-\alpha.$$

## 3. The Improved Predictor Corrector Method for Fuzzy Differential Equation

Theorem 2.2 can be used to numerically solve the FDEs Eq.(1) by applying any suitable numerical method for ODEs to Eq.(2). The Improved Predictor-Corrector (IPC) method is generated by combining an explicit three-step method and an implicit two-step method. We shall first briefly introduce IPC method for ordinary differential equation, then IPC method to solve FDEs. So, for a start, we focus our attention on the well-known initial value problem for ordinary differential equation

$$dx(t)=f(t,x(t))dt, \tag{3}$$

$$x(0) = x_0$$

we assume the function $f$ to be such that a unique solution exists on some interval $[0, T]$, also we are working on a uniform grid $t_n = t_0 + nh : \ 1 \leq n \leq N$ with some integer $N$ and $h = (T - t_0)/N$. By linear spline interpolation for $f(t_{i-1}, x(t_{i-1})), f(t_i, x(t_i)), f(t_{i+1}, x(t_{i+1}))$ we have

$$f_1(t, x(t)) = \frac{t_i - t}{t_i - t_{i-1}} f(t_{i-1}, x(t_{i-1})) + \frac{t - t_{i-1}}{t_i - t_{i-1}} f(t_i, x(t_i)) \quad t \in [t_{i-1}, t_i] \quad (4)$$

$$f_2(t, x(t)) = \frac{t_{i+1} - t}{t_{i+1} - t_i} f(t_i, x(t_i)) + \frac{t - t_i}{t_{i+1} - t_i} f(t_{i+1}, x(t_{i+1})) \quad t \in [t_i, t_{i+1}]. \quad (5)$$

We have already calculated approximations $x_h(t_i) \approx x(t_i) \quad i = 1, 2, \ldots, n$, that we try to obtain the approximation $x_h(t_{n+2})$ by means of the equation

$$x(t_{i+2}) = x(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{\{\frac{t_i - t}{t_i - t_{i-1}} f(t_{i-1}, x(t_{i-1})) + \frac{t - t_{i-1}}{t_i - t_{i-1}} f(t_i, x(t_i))\}dt \quad (6)$$

$$+ \textstyle\int_{t_i}^{t_{i+2}} \{\{\frac{t_{i+1}-t}{t_{i+1}-t_i} f(t_i, y(t_i)) + \frac{t-t_i}{t_{i+1}-t_i} f(t_{i+1}, y(t_{i+1}))\}dt.$$

This equation follows upon integration of (3) on the interval $[t_n, t_{n+1}]$. By integration, the following results will be obtained

$$x(t_{i+2}) = x(t_{i-1}) + \frac{h}{2}[f(t_{i-1}, x(t_{i-1})) + f(t_i, x(t_i)) + 4f(t_{i+1}, x(t_{i+1}))]. \quad (7)$$

We know neither of the expressions on the right-hand side of (7) exactly, but we do have an approximation for $x(t_i)$, namely $x_h(t_i)$, that we can use instead. The integral is then replaced by the two-point trapezoidal quadrature formula, thus given an equation for the unknown approximation $x_h(t_{i+1})$, it being

$$\begin{cases} x(t_{i+1}) = x(t_{i-1}) + \frac{h}{2}[f(t_{i-1}, x(t_{i-1})) + 2f(t_i, x(t_i)) + f(t_{i+1}, x(t_{i+1}))], \\ x(t_{i-1}) = \alpha_0, x(t_i) = \alpha_1. \end{cases}$$

where again we have to replace $x(t_i)$ and $x(t_{i+1})$ by their approximations $x_h(t_i)$ and $x_h(t_{i+1})$ , respectively. This yields the equation for the implicit two-step method

$$\begin{cases} x_h(t_{i+1}) = x_h(t_{i-1}) + \frac{h}{2}[f(t_{i-1}, x_h(t_{i-1})) + 2f(t_i, x(t_i)) + f(t_{i+1}, x_h(t_{i+1}))], \\ x_h(t_{i-1}) = \alpha_0, x_h(t_i) = \alpha_1. \end{cases} \quad (8)$$

The problem with this equation is that the unknown quantity $x_h(t_{i+1})$ appears on both sides, and due to the nonlinear nature of the function $f$, we cannot solve for $x_h(t_{i+1})$ directly in general. Therefore, we may use (8) in an iterative process, inserting a preliminary approximation for $x_h(t_{i+1})$ in the right-hand side in order to determine a better approximation that can then use. The required preliminary approximation $x_h^0(t_{i+1})$, the so called predictor, is obtain in a very similar way, only replacing the trapezoidal quadrature formula by the Simpson's rule giving the explicit three-step method is obtained as follows

$$\begin{cases} x_h^0(t_{i+1}) = x_h(t_{i-1}) + \frac{h}{2}[f(t_{i-1}, x_h(t_{i-1})) + f(t_i, x_h(t_i)) + 4f(t_{i+1}, x_h(t_{i+1}))] \\ x_h(t_{i-1}) = \alpha_0, \ x_h(t_i) = \alpha_1, \ x_h(t_{i+1}) = \alpha_2. \end{cases} \tag{9}$$

Moreover, this method said to be of the IPC (Improved Predictor Corrector) type because, in a concrete implementation, we would star by calculating the predictor in Eq. (9), use this to calculate the corrector in Eq. (8).

Now, we consider the fuzzy differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{C}_t$$

with initial value $x(0)$. According to Theorem 2.1, the fuzzy differential equation Eq. (1) has a unique solution, and its $\alpha$-path $(0 < \alpha < 1)$ is

$$\begin{cases} dx^\alpha(t) = f(x^\alpha(t), t)dt + |g(x^\alpha(t), t)|\mu^{-1}(\alpha)dt, \\ x^\alpha(0) = x(0). \end{cases} \tag{10}$$

For the sake of simplicity, we write

$$F(t, x^\alpha(t)) = f(t, x^\alpha(t)) + |g(t, x^\alpha(t))|\mu^{-1}(\alpha).$$

Then the ordinary differential Eq.(10) is reduced to

$$dx^\alpha(t) = F(t, x^\alpha(t))dt$$

$$x^\alpha(0) = x(0)$$

On the basis of IPC method, we will present IPC method for solving fuzzy differential Eq.(1) as follows.

**Algorithm** (Improved Predictor Corrector (IPC) three-step method for fuzzy differential equations) To approximate the solution of the following initial value problem

$$\begin{cases} x'(t) = F(t, x(t)), & t_0 \le t \le T, \\ x^\alpha(t_0) = \alpha_0, x^\alpha(t_1) = \alpha_1, x^\alpha(t_2) = \alpha_2, \end{cases}$$

an arbitrary positive integer $N$ is chosen.

Step 1. Let $h = \frac{T - t_0}{N}$ ,

$$w^\alpha(t_0) = \alpha_0, w^\alpha(t_1) = \alpha_1, w^\alpha(t_2) = \alpha_2,$$

Step 2. Let $i = 1$.
Step 3. Let

$$w^{(0)\alpha}(t_{i+2}) = w^\alpha(t_{i-1}) + \frac{h}{2}[F^\alpha(t_{i-1}, w(t_{i-1})) + F^\alpha(t_i, w(t_i)) + 4F^\alpha(t_{i+1}, w(t_{i+1}))],$$

Step 4. Let $t_{i+2} = t_0 + (i+2)h$,
Step 5. Let

$$w^\alpha(t_{i+2}) = w^\alpha(t_i) + (\frac{h}{2})F^\alpha(t_i, w(t_i)) + hF^\alpha(t_{i+1}, w(t_{i+1})) + (\frac{h}{2})F^\alpha(t_{i+2}, w^{(0)}(t_{i+2})),$$

Step 6. $i = i + 1$.
Step 7. If $i \leq N - 2$, go to step 3.
Step 8. The algorithm will end and $w^\alpha(T)$ approximates the real value of $X^\alpha(T)$.

## 4. Convergence

To integrate the system given in Eq.(1) from $t_0$ to a prefixed $T > t_0$, the interval $[t_0, T]$ will be replaced by a set of discrete equally spaced grid points $t_0 < t_1 < t_2 < \ldots < t_N = T$, and the exact solution $X(t, \alpha)$ is approximated by some $x(t, \alpha)$. The exact and approximate solutions at $t_n$, $0 \leq n \leq N$ are denoted by $X_n(\alpha)$, and $x_n(\alpha)$, respectively. The grid points at which the solution is calculated are $t_n = t_0 + nh$, $h = (T - t_0)/N$, $1 \leq n \leq N$.

From Eq. (8), the polygon curves

$$x(t, h, \alpha) = \{[t_0, x_0(\alpha)], [t_1, x_1(\alpha)], \ldots, [t_N, x_N(\alpha)]\},$$

are the implicit three-step approximates to $X(t, \alpha)$ over the interval $t_0 \leq t \leq t_N$. The following lemma will be applied to show the convergence of these approximates, i.e.,

$$lim_{h \to 0} x(t, h, \alpha) = X(t, \alpha)$$

**Lemma 4.1** Let a sequence of numbers $\{w_n\}_{n=0}^N$ satisfy:

$$|w_{n+1}| \leq \mathbf{A}|w_n| + \mathbf{B}|w_{n-1}| + \mathbf{C}, \qquad 0 \leq n \leq N - 1$$

for some given positive constants $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$. Then

$$|w_n| \leq (\mathbf{A}^{n-1} + \beta_1 \mathbf{A}^{n-3} B + \beta_2 \mathbf{A}^{n-5} \mathbf{B}^2 + \ldots + \beta_s \mathbf{B}^{[\frac{n}{2}]})|w_1| +$$

$$(\mathbf{A}^{n-2}\mathbf{B} + \gamma_1 \mathbf{A}^{n-4}\mathbf{B}^2 + \ldots + \gamma_t A B^{[\frac{n}{2}]})|w_0| + (\mathbf{A}^{n-2} + \mathbf{A}^{n-3} + \ldots + 1)\mathbf{C} +$$

$$(\delta_1 \mathbf{A}^{n-4} + \delta_2 \mathbf{A}^{n-5} + \ldots + \delta_m \mathbf{A} + 1) BC + (\zeta_1 \mathbf{A}^{n-6} + \zeta_2 \mathbf{A}^{n-7} + \ldots + \zeta_l \mathbf{A} + 1)\mathbf{B}^2\mathbf{C} +$$

$$(\lambda_1 \mathbf{A}^{n-8} + \lambda_2 \mathbf{A}^{n-9} + \ldots + \lambda_p \mathbf{A} + 1)\mathbf{B}^3\mathbf{C} + \ldots, \qquad when \;\; n \;\; is \;\; odd$$

and

$$|w_n| \leq (\mathbf{A}^{n-1} + \beta_1 \mathbf{A}^{n-3}\mathbf{B}\beta_2 \mathbf{A}^{n-5}\mathbf{B}^2 + \ldots + \beta_s A B^{[\frac{n}{2}]-1})|w_1|$$

$$+(\mathbf{A}^{n-2}\mathbf{B} + \gamma_1 \mathbf{A}^{n-4}\mathbf{B}^2 + \ldots + \gamma_t \mathbf{B}^{[\frac{n}{2}]})|w_0| + (\mathbf{A}^{n-2} + \mathbf{A}^{n-3} + \ldots + 1)\mathbf{C} +$$

$$(\delta_1 \mathbf{A}^{n-4} + \delta_2 \mathbf{A}^{n-5} + \ldots + \delta_m \mathbf{A} + 1) BC + (\zeta_1 \mathbf{A}^{n-6} + \zeta_2 A^{n-7} + \ldots + \zeta_l \mathbf{A} + 1)\mathbf{B}^2\mathbf{C} +$$

$$(\lambda_1 \mathbf{A}^{n-8} + \lambda_2 \mathbf{A}^{n-9} + \ldots + \lambda_p \mathbf{A} + 1)\mathbf{B}^3\mathbf{C} + \ldots, \qquad when \;\; n \;\; is \;\; even$$

where $\beta_s, \gamma_t, \delta_m, \zeta_l, \lambda_p$, are constants for all $s, t, m, l$ and $p$.

**Theorem 4.1** For any arbitrary fixed $\alpha : \;\; 0 \leq \alpha \leq 1$, the implicit two-step approximates of Eq. (8) converge to the exact solutions $X(t, \alpha)$ for $X \in \mathbf{C}^3[t_0, T]$.

*Proof.* It is sufficient to show

$$lim_{h \to 0} x_N(\alpha) = X(T, \alpha).$$

By using Taylor's theorem we have:

$$X_{n+1}(\alpha) =$$
$$X_n(\alpha) + \tfrac{h}{2} f(t_{n-1}, X_{n-1}(\alpha)) + h f(t_n, X_n(\alpha)) + \tfrac{h}{2} f(t_{n+1}, X_{n+1}(\alpha)) - \tfrac{1}{6} h^3 X'''(\xi_n),$$

where $t_n < \underline{\xi}_n$. Consequently

$$X_{n+1}(\alpha) - x_{n+1}(\alpha) =$$
$$X_{n-1}(\alpha) - x_{n-1}(\alpha) + \tfrac{h}{2}\{f(t_{n-1}, X_{n-1}(\alpha)) - f(t_{n-1}, x_{n-1}(\alpha))\} +$$

$$h\{f(t_n, X_n(\alpha)) - f(t_n, x_n(\alpha))\} + \tfrac{h}{2}\{f(t_{n+1}, X_{n+1}(\alpha)) - f(t_{n+1}, x_{n+1}(\alpha))\} -$$
$$\tfrac{1}{6} h^3 X'''(\xi_n).$$

Denote $w_n = X_n(\alpha) - x_n(\alpha)$, Then

$$|w_{n+1}| \leq h L_1 |w_n| + (1 + \tfrac{h L_2}{2})|w_{n-1}| + (\tfrac{h L_3}{2})|w_{n+1}| + \tfrac{1}{6} h^3 M,$$

where $M = \max_{t_0 \le t \le T} |X'''(t,\alpha)|$.

Set

$$L = \max\{L_1, L_2, L_3\} < \frac{1}{h},$$

then

$$|w_{n+1}| \le (\frac{2hL}{1-hL})|w_n| + (\frac{1+hL}{1-hL})|w_{n-1}| + (\frac{1}{3-3hL})h^3 M,$$

are obtained, if $h \to 0$ then $w_n \to 0$, which concludes the proof.

**Theorem 4.2** For any arbitrary fixed $\alpha : \ 0 \le \alpha \le 1$, the explicit three-step approximates of Eq. (9) converge to the exact solution $X(t,\alpha)$ for $X \in \mathbf{C}^3[t_0, \mathbf{T}]$.

*Proof.* Similar to Theorem 4.1.

**Theorem 4.3** The explicit three-step method is stable.

*Proof.* For the explicit three-step method, there exists only one characteristic polynomial $p(\lambda) = \lambda^3 - \lambda$, then it satisfies the root condition and, therefore, it is a stable method.

**Theorem 4.4** The implicit two-step method is stable.

*Proof.* Similar to Theorem 4.3.

Regarding the above-mentioned theorems, it is obvious that the IPC Three-step method is convergent and stable.

## 5. Numerical experiments

Fuzzy differential equations (FDEs) have become standard models for financial quantities such as asset prices, stock prices, interest rates, and their derivatives.

**Example 5.1** Let $a, b$ be real numbers. Consider a linear fuzzy differential equation

$$dx(t) = ax(t)dt + bx(t)d\mathbf{C}_t \tag{11}$$

with given initial value $x(0) = 1$.The explicit solution of Eq.(11) is

$$x(t) = x(0)\exp(at + b\mathbf{C}(t))$$

and its $\alpha$-path is

$$dx^\alpha(t) = (ax^\alpha(t) + |bx^\alpha(t)|\mu^{-1}(\alpha))dt.$$

The inverse credibility distribution of $x(t)$ is

$$\psi_t^{-1}(\alpha) = x(0)\exp(at + bt\mu^{-1}(\alpha)).$$

Table 1: Comparison of errors in the numerical solution using the Rung-Kutta, Adams and IPC methods for (11)

| $\alpha$ | Error(EULER) | Error(RUNG) | Error(IPM) |
|---|---|---|---|
| 0.1 | 1.808042193580918‘ | 0.9281769985563846‘ | 0.019549658444490064‘ |
| 0.2 | 1.587959722898813‘ | 0.6608581705328824‘ | 0.016895197550903962‘ |
| 0.3 | 1.4582751984957965‘ | 0.5178605149950599‘ | 0.015306752965624337‘ |
| 0.4 | 1.3637393146983499‘ | 0.4200838786965364‘ | 0.014136782818382265‘ |
| 0.5 | 1.2875751687724049‘ | 0.3451302959188256‘ | 0.013186493458830162‘ |
| 0.6 | 1.2224058081972575‘ | 0.28362043368009693‘ | 0.012367768392147704‘ |
| 0.7 | 1.1642836026596803‘ | 0.23074613350490614‘ | 0.011633093392665605‘ |
| 0.8 | 1.110794640360493‘ | 0.18369618391995957‘ | 0.010953162665008342‘ |
| 0.9 | 1.0602941039998126‘ | 0.14065458879409065‘ | 0.01030778902043017‘ |
| 1 | 1.0115406146754944‘ | 0.10034118393041669‘ | 0.00968151972950304‘ |

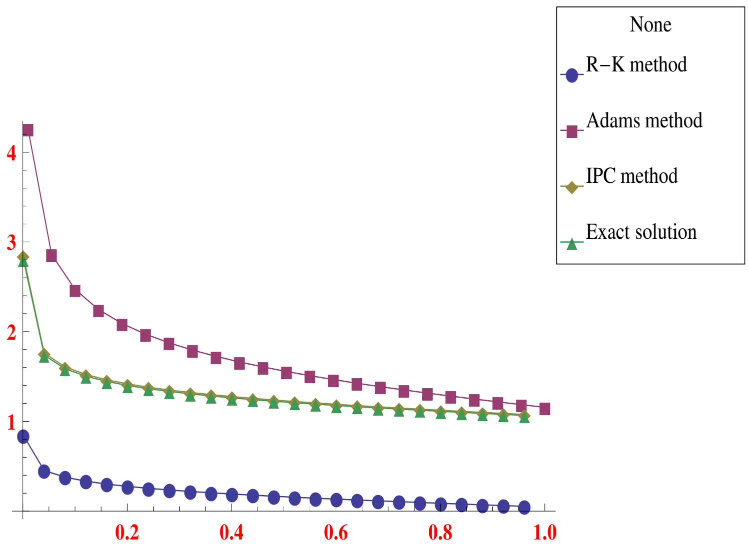


Figure 1: The errors for for $r = 0.5$ in Example 1

We choose the parameters as follows, $a = 0.5$, $b = 1.5$, $t = 10$, $N = 1000$, $h = \frac{t}{N} = 0.01$. The numerical solution obtained by Rung-Kutta method, Adams method and present solution using IPM are tabulated in Table 1 for h = 0.01. Also, the obtained errors are incorporated in this table. By looking at the results, one may conclude that the solution obtained by IPC method is the same as that of the exact solution. The solution plot is given in Figure 1.

Table 2: Comparison of errors in the numerical solution using the Rung-Kutta, Adams and IPC methods for (12)

| $\alpha$ | Error(EULER) | Error(RUNG) | Error(IPM) |
|---|---|---|---|
| 0.1 | 1.7392127789290381‘ | 4.6266621769041905‘ | 1.058798692518907‘ |
| 0.2 | 1.5545503950968755‘ | 3.592506507578089‘ | 0.7560191659930554‘ |
| 0.3 | 1.440220207977195‘ | 2.9522287816477544‘ | 0.5685590065099428‘ |
| 0.4 | 1.3541416416236476‘ | 2.470167218514831‘ | 0.4274212879140462‘ |
| 0.5 | 1.283045504886185‘ | 2.0720110193559584‘ | 0.3108493355084534‘ |
| 0.6 | 1.2209370641172477‘ | 1.724188172053291‘ | 0.20901395410035062‘ |
| 0.7 | 1.1645264988204869‘ | 1.4082748566237797‘ | 0.1165210278358837‘ |
| 0.8 | 1.1117449914121083‘ | 1.1126851756654565‘ | 0.029978444063355925‘ |
| 0.9 | 1.061133246855138‘ | 0.8292467109283896‘ | 0.05300651332533879‘ |
| 1 | 1.0115406146754944‘ | 0.5515155311338277‘ | 0.13432049497614873‘ |

By using the proposed method in Section 3, we can present the approximate solution for this example. A comparison of the numerical results with the exact solution is shown in Figure 1.

**Example 5.2** Let $a, b$ be real numbers. Consider a linear fuzzy differential equation

$$dx(t) = ax(t)dt + b\exp(at)d\mathbf{C}_t \tag{12}$$

with given initial value $x(0) = 1$.The explicit solution of Eq.(12) is

$$x(t) = \exp(at)(1 + b\mathbf{C}(t))$$

and its $\alpha$-path is

$$dx^\alpha(t) = (ax^\alpha(t) + b\exp(at)\mu^{-1}(\alpha))dt.$$

The inverse credibility distribution of $x(t)$ is

$$\psi_t^{-1}(\alpha) = (1 + t|b|\mu^{-1}(\alpha))\exp(at).$$

We choose the parameters as follows, $a = 0.5, \quad b = 1.5, \quad t = 10, \quad N = 1000, \quad h = \frac{t}{N} = 0.01$. By using the proposed method in Section 3, we can present the approximate solution for this example. To compare the numerical results with the exact solution in Figure 2.

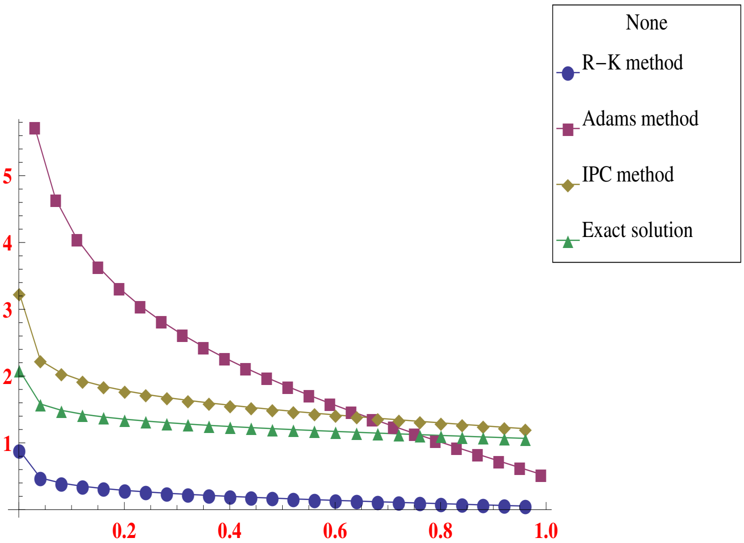


Figure 2: The errors for for $r = 0.5$ in Example 2

## 6. Conclusion

In this paper, three numerical methods based on spline interpolation, i.e. Explicit three- step method, Implicit two-step method and IPC three-step method, were discussed. We showed that our proposed IPC three-step method has more accuracy.

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