International Journal of Pure and Applied Mathematics
Volume 113 No. 5 2017, 665-678
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i5.13

ijpam.eu

L-FUZZY (K, E)-SOFT PRE-PROXIMITIES AND L-FUZZY (K, E)-SOFT CLOSURE OPERATORS

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Abstract: In this paper, we investigate the relations between L-fuzzy (K, E)-soft pre-proximities and L-fuzzy (K, E)-soft closure spaces in stsc-quantales. We give their examples.

AMS Subject Classification: 03E72, 06A15, 06F07, 54A05
Key Words: Stsc-quantales, L-fuzzy (K, E)-soft pre-proximities, L-fuzzy (K, E)-soft closure spaces, L-fuzzy soft proximity maps, L-fuzzy soft closed maps

1. Introduction

Molodtsov [14] introduced soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. Maji et al. [11,12] gave the first practical application of soft sets in decision making problems. Many

Received: February 12, 2017
Revised: March 16, 2017
Published: April 3, 2017

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researchers have contributed towards the algebraic structure of soft set theory [1-5,7]. Shabir and Naz [20] initiated the study of soft topological spaces. They defined soft topology on the collection of soft sets over X and established their several properties. Aygünoglu et.al [2] introduced the concept of (K,E)-soft topology in the sense of Šostak [9]. Cetkin et.al [3] studied (K,E)-soft proximities and discuss their properties.

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices. Höhle [9] introduced L-fuzzy topologies with algebraic structure L(cqm, quantales, MV-algebra). Kim et al. [10,14] define the the concept of L-fuzzy (K,E)-soft topologies, L- fuzzy (K,E)-soft closure operators, L-fuzzy (K,E)-soft pre-proximities in strictly two sided commutative quantales and investigated the relation between them.

In this paper, we investigate the relations between L-fuzzy (K,E)-soft pre-proximities and L-fuzzy (K,E)-soft closure spaces in stsc-quantales. We give their examples.

2. Preliminaries

Let $L = (L,\leq,\vee,\wedge,0,1)$ be a completely distributive lattice with the least element 0 and the greatest element 1 in L.

Definition 1. [8,9,18] A complete lattice $(L,\leq,\odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L,\odot)$ is a commutative semigroup,

(L2) $x = x\odot 1$, for each $x\in L$ and 1 is the universal upper bound,

(L3) $\odot$ is distributive over arbitrary joins, i.e. $(\bigvee_i x_i)\odot y = \bigvee_i (x_i\odot y)$.

There exists a further binary operation $\rightarrow$ (called the implication operator or residuated) satisfying the following condition

$$x\rightarrow y = \bigvee\{z\in L | x\odot z\leq y\}.$$

Then it satisfies Galois correspondence; i.e, $(x\odot z)\leq y$ iff $z\leq (x\rightarrow y)$.

In this paper, we always assume that $(L,\leq,\odot,\rightarrow,\oplus,^*)$ is a stsc-quantales with an order reversing involution * which is defined by

$$x\oplus y = (x^*\odot y^*)^*,\quad x^* = x\rightarrow 0$$

unless otherwise specified.

Remark 2. Every completely distributive lattice $(L, \leq, \wedge, \vee, ^*)$ with order reversing involution * is a stsc-quantale $(L, \leq, \odot, \oplus, ^*)$ with a strong negation * where $\odot = \wedge$ and $\oplus = \vee$.

Lemma 3. [8,9,18] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \leq y$ iff $x \to y = 1$.

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(7) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(8) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(10) $x \odot y = (x \to y^*)^*$ and $x \oplus y = x^* \to y$,

(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(12) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$.

(14) $x \to y = y^* \to x^*$.

(15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$.

(16) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma} (x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma} (x_i \to y_i)$,

(17) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$.

Throughout this paper, X refers to an initial universe, E and K are the sets of all parameters for X, and L^X is the set of all L-fuzzy sets on X.

Definition 4. [4] A map f is called an L- fuzzy soft set on X, where f is a mapping from E into L^X, i.e., $f_e := f(e)$ is an L- fuzzy set on X, for each $e \in E$. The family of all L- fuzzy soft sets on X is denoted by $(L^X)^E$. Let f and g be two L- fuzzy soft sets on X.

(1) f is an L-fuzzy soft subset of g and we write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) The intersection of f and g is an L- fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of f and g is an L- fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An L- fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) An L- fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.

(6) The complement of an L- fuzzy soft sets on X is denoted by f^*, where $f^* : E \to L^X$ is a mapping given by $f^*_e = (f_e)^*$, for each $e \in E$.

(7) f is called a null L- fuzzy soft set and is denoted by 0_X , if $f_e(x) = 0$, for each $e \in E$, $x \in X$.

(8) f is called an absolute L- fuzzy soft set and is denoted by 1_X , if $f_e(x) = 1$, for each $e \in E$, $x \in X$ and $(1_x)_e(x) = 1$.

Definition 5. [4] Let $\varphi : X \to Y$ and $\psi : E_1 \to E_2$ be two mappings, where E_1 and E_2 are parameters sets for the crisp sets X and Y, respectively. Then $\varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2}$ is called a fuzzy soft mapping.

(1) For $f \in (L^X)^{E_1}$, the image of f under the fuzzy soft mapping φ_ψ defined by, $\forall k \in K, \forall y \in Y$,

$$\varphi(f)_{e_2}(y) = \bigvee_{x\in\varphi^{-1}(\{y\})} (\bigvee_{e_1\in\psi^{-1}(\{e_2\})} f_{e_1}(x)).$$

(2) For $f \in (L^X)^{E_1}$, the pre-image of g defined by

$$\varphi_\psi^{-1}(g)_e(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.$$

(3) The soft mapping $\varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2}$ is called injective (resp. surjective, bijective) if f and ϕ are both injective (resp. surjective, bijective).

Definition 6. [10] A map $\mathcal{C} : K \times (L^X)^E \times L_0 \to (L^X)^E$ is called an L-fuzzy (K, E)-soft closure operator if it satisfies the following conditions;

(C1) $\mathcal{C}(k, 0_X, r) = 0_X$,

(C2) $\mathcal{C}(k, f, r) \sqsupseteq f$,

(C3) If $f_1 \sqsubseteq f_2$, then $\mathcal{C}(k, f_1, r) \sqsubseteq \mathcal{C}(k, f_2, r))$,

(C4) If $r_1 \leq r_2$, then $\mathcal{C}(k, f, r_1) \sqsubseteq \mathcal{C}(k, f, r_2)$,

(C5) $\mathcal{C}(k, f_1 \oplus f_2, r \odot s) \sqsubseteq \mathcal{C}(k, f_1, r) \oplus \mathcal{C}(k, f_2, s)$.

The pair $(X, \mathcal{C})$ is called an L-fuzzy (K, E)-soft closure space.

An L- fuzzy (K, E)-soft closure operator is called topological if

(T) $\mathcal{C}(k, \mathcal{C}(k, f, r), r) \sqsubseteq \mathcal{C}(k, f, r)$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be L-fuzzy (K, E)-soft closure operators on X. Then $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ if $\mathcal{C}_1(k, f, r) \sqsubseteq \mathcal{C}_2(k, f, r)$, for all $f \in (L^X)^E, r \in L_0, k \in K$.

Let $(X, \mathcal{C}_X)$ be L-fuzzy (K_1, E_1)-soft closure spaces and $(Y, \mathcal{C}_Y)$ be L-fuzzy (K_2, E_2)-soft closure spaces. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be maps. Then $\varphi_{\psi,\eta}$ is called an L-fuzzy soft closed map if, for each $k \in K_1, f \in (L^X)^{E_1}, r \in L_0$,

$$\varphi_{\psi,\eta}(\mathcal{C}_X(k, f, r)) \sqsubseteq \mathcal{C}_Y(\eta(k), \varphi_\psi(f), r).$$

Definition 7. [17] A mapping $\delta : K \to L^{(L^X)^E \times (L^X)^E}$ ($\delta_k = \delta(k) : (L^X)^E \times (L^X)^E \to L$) is called an L-fuzzy (K, E)-soft pre-proximity on X if it satisfies the following axioms.

(SP1) $\delta_k(0_X, 1_X) = \delta_k(1_X, 0_X) = 0$.

(SP2) If $\delta_k(f, g) \neq 1$, then $f \sqsubseteq g^*$.

(SP3) If $f_1 \sqsubseteq f_2$ and $g_1 \sqsubseteq g_2$, then $\delta_k(f_1, g_1) \leq \delta_k(f_2, g_2)$.

(SP4) $\delta_k(f_1 \odot f_2, g_1 \oplus g_2) \leq \delta_k(f_1, g_1) \oplus \delta_k(f_2, g_2)$.

The pair (X, δ) is called an L-fuzzy (K, E)-soft pre-proximity space.

An L-fuzzy (K, E)-soft pre-proximity is called an L-fuzzy (K, E)-soft quasi-proximity on X if

(PQ) $\delta_k(f, g) \geq \bigwedge_h \{\delta_k(f, h) \oplus \delta_k(h^*, g)\}$.

An L-fuzzy (K, E)-soft quasi-proximity is called an L-fuzzy (K, E)-soft proximity on X if

(SP) $\delta_k(f, g) = \delta_k(g, f)$.

Let (X, δ^1) be an L-fuzzy (K_1, E_1)-soft quasi proximity space and (Y, δ^2) be an L-fuzzy (K_2, E_2)-soft pre-proximity space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi,\eta}$ from (X, δ^1) into (Y, δ^2) is called L-fuzzy soft proximity map if

$$\delta^1_k(\varphi_\psi^{-1}(f), \varphi_\psi^{-1}(g)) \leq \delta^2_{\eta(k)}(f, g) \ \ \forall\ f, g \in (L^Y)^{E_2}, k \in K_1.$$

Lemma 8. [10] Let $\varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2}$ be a soft mapping. Then we have the following properties. For $f, f_i \in (L^X)^{E_1}$ and $g, g_i \in (L^Y)^{E_2}$,

(1) $g \sqsupseteq \varphi_\psi(\varphi_\psi^{-1}(g))$ with equality if φ_ψ is surjective,

(2) $f \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(f))$ with equality if φ_ψ is injective,

(3) if φ_ψ is injective,

$$\varphi(f)_{e_2}(y) = \begin{cases} f_{e_1}(x), & \text{if } x \in \varphi^{-1}(\{y\}), e_1 \in \psi^{-1}(\{e_2\}) \\ 0, & \text{otherwise,} \end{cases}$$

(4) $\varphi_\psi^{-1}(g^*) = (\varphi_\psi^{-1}(g))^*$,

(5) $\varphi_\psi^{-1}(\sqcup_{i \in I} g_i) = \sqcup_{i \in I} \varphi_\psi^{-1}(g_i)$,

(6) $\varphi_\psi^{-1}(\sqcap_{i \in I} g_i) = \sqcap_{i \in I} \varphi_\psi^{-1}(g_i)$,

(7) $\varphi_\psi(\sqcup_{i \in I} f_i) = \sqcup_{i \in I} \varphi_\psi(f_i)$,

(8) $\varphi_\psi(\sqcap_{i \in I} f_i) \sqsubseteq \sqcap_{i \in I} \varphi_\psi(f_i)$ with equality if φ_ψ is injective,

(9) $\varphi_\psi^{-1}(g_1 \odot g_2) = \varphi_\psi^{-1}(g_1) \odot \varphi_\psi^{-1}(g_2)$,

(10) $\varphi_\psi^{-1}(g_1 \oplus g_2) = \varphi_\psi^{-1}(g_1) \oplus \varphi_\psi^{-1}(g_2)$,

(11) $\varphi_\psi(f_1 \odot f_2) \sqsubseteq \varphi_\psi(f_1) \odot \varphi_\psi(f_2)$ with equality if φ_ψ is injective,

(12) $\varphi_\psi(f_1 \oplus f_2) \sqsubseteq \varphi_\psi(f_1) \oplus \varphi_\psi(f_2)$ with equality if φ_ψ is injective.

3. L-fuzzy (K,E)-soft pre-proximities and L-fuzzy (K,E)-soft closure operators

Theorem 9. Let (X,δ) be an L-fuzzy (K,E)-soft pre-proximity space. We define a mappings $\mathcal{C}_\delta : K \times (L^X)^E \times L_0 \to (L^X)^E$ as

$$\mathcal{C}_\delta(k,f,s) = \sqcap\{g^* \in (L^X)^E \mid \delta_k(g,f) \le r^* \}.$$

Then we have the following properties

(1) $\mathcal{C}_\delta$ is an L-fuzzy (K,E)-soft closure operator on X.

(2) If δ be an L-fuzzy (K,E)-soft quasi-proximity on X, for each $s < r$, there exist s_1, s_2 with $s_1 \odot s_2 \ge s$ such that

$$\mathcal{C}_\delta(\mathcal{C}_\delta(f,s_1),s_2) \sqsubseteq \mathcal{C}_\delta(f,r).$$

Proof. (1) (C1) Since $\delta_k(1_X, 0_X) = 0$ for each $k \in K$, we have $\mathcal{C}_\delta(k,0_X,r) = 0_X$.

(C2) Since $s \ne 0$, then $s^* \ne 1$. Since $\delta_k(g,f) \ne 1$, by (SP2), $g^* \sqsupseteq f$. Hence $\mathcal{C}_\delta(k,f,r) \sqsupseteq f$,

(C3) and (C4) are easily proved.

(C5)

$$\begin{aligned}
&\mathcal{C}_\delta(k,f_1,r_1) \oplus \mathcal{C}_\delta(f_2,r_2)\\
&= \bigwedge\{g_1^* \in (L^X)^E \mid \delta_k(g_1,f_1) \le r_1^* \}\\
&\oplus \bigwedge\{g_2^* \in (L^X)^E \mid \delta_k(g_2,f_2) \le r_2^* \}\\
&\sqsupseteq \bigwedge\{g_1^* \oplus g_2^* \mid \delta_k(g_1,f_1) \oplus \delta_k(g_2,f_2) \le r_1^* \oplus r_2^*\}\\
&\sqsupseteq \bigwedge\{(g_1 \odot g_2)^* \mid \delta_k(g_1 \odot g_2, f_1 \oplus f_2) \le (r_1 \odot r_2)^*\}\\
&\sqsupseteq \mathcal{C}_\delta(f_1 \oplus f_2, r \odot s).
\end{aligned}$$

Hence $\mathcal{C}_\delta$ is an L-fuzzy (K,E)-soft closure operator on X.

(2) Suppose that there exists $f \in (L^X)^E$ and $r, s \in L_0$ with $r > s$ and $s_1 \odot s_2 \ge s$ such that

$$\mathcal{C}_\delta(k,\mathcal{C}_\delta(k,f,s_1),s_2) \not\sqsubseteq \mathcal{C}_\delta(k,f,r).$$

From the definition of $\mathcal{C}_\delta(k,f,r)$, there exists $g \in (L^X)^E$ with $\delta_k(g,f) \le r^*$ such that

$$\mathcal{C}_\delta(\mathcal{C}_\delta(f,s_1),s_2) \not\sqsubseteq g^*.$$

On the other hand, since δ is an L-fuzzy (K,E)-soft quasi-proximity on X, $\bigwedge_{h\in(L^X)^E}(\delta_k(g,h) \oplus \delta_k(h^*,f)) \le \delta_k(f,g) \le r^*$, for each $s < r$, there exists $h \in (L^X)^E$ such that

$$\delta_k(g,h) \oplus \delta_k(h^*,f) \le s^*.$$

Then there exist $\delta_k(g,h) = s_2, \delta_k(h^*, f) = s_1$ with $s_1 \odot s_2 \geq s$ such that

$$\mathcal{C}_\delta(f, s_1) \sqsubseteq h,\ \mathcal{C}_\delta(h, s_2) \sqsubseteq g^*.$$

Then

$$\mathcal{C}_\delta(\mathcal{C}_\delta(f, s_1), s_2) \sqsubseteq g^*.$$

It is a contradiction. Thus,

$$\mathcal{C}_\delta(k, \mathcal{C}_\delta(k, f, s_1), s_2) \sqsubseteq \mathcal{C}_\delta(k, f, r).$$

□

Theorem 10. Let $(X, \mathcal{C})$ be an L-fuzzy (K,E)-soft closure space. Define a mapping $(\delta_{\mathcal{C}})_k : (L^X)^E \times (L^X)^E \to L$ by

$$(\delta_{\mathcal{C}})_k(f,g) = \bigwedge\{r^* \in L \mid \mathcal{C}(k, g, r) \sqsubseteq f^*\}.$$

Then we have the following properties.

(1) $\delta_{\mathcal{C}}$ is an L-fuzzy (K,E)-soft pre-proximity.

(2) If, for each k, r, f, there exist $s, t \in L_0$ with $s \odot t \geq r$ such that

$$\mathcal{C}(k, \mathcal{C}(k, f, s), t) \sqsubseteq \mathcal{C}(k, f, r),$$

then $\delta_{\mathcal{C}}$ is an L-fuzzy (K,E)-soft quasi-proximity on X.

(3) $\mathcal{C}_{\delta_{\mathcal{C}}} \leq \mathcal{C}$. If $\mathcal{C}(k, f, \bigvee r_i) = \bigsqcup \mathcal{C}(k, f, r_i)$ for all $k \in K$, then $\mathcal{C}_{\delta_{\mathcal{C}}}(f, r) \sqsupseteq \mathcal{C}(f, r)$.

(4) If δ is an L-fuzzy (K,E)-soft pre-proximity on X, then $\delta_{\mathcal{C}_\delta} \leq \delta$. If $\delta_k(\sqcup f_i, g) = \bigvee \delta_k(f_i, g)$ for all $k \in K$, then $\delta_{\mathcal{C}_\delta} \geq \delta$.

Proof. (1) (SP1) Since $\mathcal{C}(k, 0_X, r) = 0_X$ and $\mathcal{C}(k, 1_X, r) = 1_X$ for $r \in L_0$, we have $(\delta_{\mathcal{C}})_k(1_X, 0_X) = (\delta_{\mathcal{C}})_k(0_X, 1_X) = 0$.

(SP2) If $(\delta_{\mathcal{C}})_k(f,g) \neq 1$, for all $r \neq 0$, we have $\mathcal{C}(k, g, r) \sqsubseteq f^*$. We have $g \sqsubseteq \mathcal{C}(k, g, r) \sqsubseteq f^*$.

(SP3) If $f \sqsubseteq f_1$, $g \sqsubseteq g_1$, then $\mathcal{C}(k, g, r) \sqsubseteq \mathcal{C}(k, g_1, r)$. Thus,

$$\begin{aligned}(\delta_{\mathcal{C}})_k(f,g) &= \bigwedge\{r^* \in L \mid \mathcal{C}(k, g, r) \sqsubseteq f^*\}\\ &\leq \bigwedge\{r^* \in L \mid \mathcal{C}(k, g_1, r) \sqsubseteq f_1^*\} = (\delta_{\mathcal{C}})_k(f_1, g_1).\end{aligned}$$

(SP4)

$$\begin{aligned}
&(\delta_{\mathcal{C}})_k(f_1,g_1)\oplus(\delta_{\mathcal{C}})_k(f_2,g_2)\\
&=\bigwedge\{s^*\in L\mid \mathcal{C}(k,g_1,s)\sqsubseteq f_1^*\}\\
&\oplus\bigwedge\{t^*\in L\mid \mathcal{C}(k,g_2,t)\sqsubseteq f_2^*\}\\
&\geq\bigwedge\{s^*\oplus t^*\mid \mathcal{C}(k,g_1\oplus g_2,s\odot t)\sqsubseteq f_1^*\oplus f_2^*\}\\
&\geq\delta_{\mathcal{C}}(f_1\odot f_2,g_1\oplus g_2).
\end{aligned}$$

Hence $\delta_{\mathcal{C}}$ is an L-fuzzy (K,E)-soft pre-proximity.

(2) If, for each k,r,f, there exist $s,t\in L_0$ with $s\odot t\geq r$ such that

$$\mathcal{C}(k,\mathcal{C}(k,f,s),t)\sqsubseteq\mathcal{C}(k,f,r),$$

$$\begin{aligned}
&(\delta_{\mathcal{C}})_k(f,g)\\
&=\bigwedge\{r^*\in L\mid \mathcal{C}(k,g,r)\sqsubseteq f^*\}\\
&\geq\bigwedge\{r^*\in L\mid \mathcal{C}(\mathcal{C}(k,g,s),t)\sqsubseteq f^*,\ s\odot t\geq r\}\\
&\geq\bigwedge\{s^*\oplus t^*\in L\mid h^*=\mathcal{C}(k,g,s),\mathcal{C}(k,h^*,t)\sqsubseteq f^*,\ s\odot t\geq r\}\\
&\geq\bigwedge\{s^*\in L\mid \mathcal{C}(k,g,s)\sqsubseteq h^*\}\oplus\bigwedge\{t^*\in L\mid \mathcal{C}(k,h^*,t)\sqsubseteq f^*\}\\
&\geq\bigwedge\nolimits_{h\in(L^X)^E}((\delta_{\mathcal{C}})_k(h,g)\oplus(\delta_{\mathcal{C}})_k(f,h^*))
\end{aligned}$$

Hence $\delta_{\mathcal{C}}$ is an L-fuzzy (K,E)-soft quasi-proximity.

(3) If $\mathcal{C}(k,f,r)\sqsubseteq g^*$, then $\delta_{\mathcal{C}}(g,f)\leq r^*$. Hence $\mathcal{C}_{\delta_{\mathcal{C}}}(k,f,r)\sqsubseteq g^*$. Hence $\mathcal{C}_{\delta_{\mathcal{C}}}(k,f,r)\sqsubseteq\mathcal{C}(k,f,r)$.

Suppose

$$\mathcal{C}_{\delta_{\mathcal{C}}}(k,f,r)\not\sqsupseteq\mathcal{C}(k,f,r).$$

By the definition of $\mathcal{C}_{\delta_{\mathcal{C}}}(k,f,r)$, there exists $g\in(L^X)^E$ with that $(\delta_{\mathcal{C}})_k(g,f)\leq r^*$ such that

$$g^*\not\sqsupseteq\mathcal{C}(k,f,r).$$

Since $(\delta_{\mathcal{C}})_k(g,f)=\bigwedge\{r_i^*\in L\mid \mathcal{C}(k,f,r_i)\leq g^*\}$ and $r\leq(\delta_{\mathcal{C}})_k^*(g,f)=\bigvee r_i$,

$$\mathcal{C}(k,f,r)\sqsubseteq\mathcal{C}(k,f,\bigvee r_i)=\bigsqcup\mathcal{C}(k,f,r_i)\sqsubseteq g^*$$

It is a contradiction. Thus, $\mathcal{C}_{\delta_{\mathcal{C}}}\geq\mathcal{C}$.

(4) Let $\delta_k(g,f)\leq r^*$ for each $k\in K$ be given. Then $\mathcal{C}_\delta(k,f,r)\sqsubseteq g^*$. Then $(\delta_{\mathcal{C}_\delta})_k(g,f)\leq r^*$. Hence $(\delta_{\mathcal{C}_\delta})_k\leq\delta_k$ for each $k\in K$.

Let $\delta_k(\bigsqcup f_i,g)=\bigvee\delta_k(f_i,g)$. Suppose there exist $k\in K, f,g\in(L^X)^E$ such that

$$(\delta_{\mathcal{C}_\delta})_k(f,g)\not\geq\delta_k(f,g).$$

By the definition of $(\delta_{\mathcal{C}_\delta})_k$, there exists $s\in L_0$ such that

$$s\not\geq\delta_k(f,g),\quad \mathcal{C}_\delta(k,g,s)\sqsubseteq f^*.$$

On the other hand, since $\delta_k(\bigsqcup f_i g) = \bigvee \delta_k(f_i, g)$, by the definition of $\mathcal{C}_\delta$,

$$\delta_k(f,g) \leq \delta_k(\mathcal{C}^*_\delta(k,g,s),g) \leq s^*.$$

It is a contradiction. Hence $(\delta_{\mathcal{C}_\delta})_k \geq \delta_k$.

□

Theorem 11. Let (X,δ_1) and (Y,δ_2) be L-fuzzy (K_1,E_1)-soft and L-fuzzy (K_2,E_2)-soft pre-proximity spaces, respectively. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Let $\varphi_{\psi,\eta} : (L^X)^{E_1} \to (L^Y)^{E_2}$ be a soft map. If $\varphi_{\psi,\eta} : (X,\delta_1) \to (Y,\delta_2)$ is an L-fuzzy soft proximity map, then $\varphi_{\psi,\eta} : (X,\mathcal{C}_{\delta_1}) \to (Y,\mathcal{C}_{\delta_2})$ is an L-fuzzy soft closure map.

Proof. $\varphi_{\psi,\eta} : (X,\mathcal{C}_{\delta_1}) \to (Y,\mathcal{C}_{\delta_2})$ is an L-fuzzy soft closed map from:

$$\begin{aligned}
&\mathcal{C}_{\delta_2}(\eta(k),\varphi_{\psi,\eta}(f),r) = \bigwedge\{g^* \mid (\delta_2)_{\eta(k)}(g,\varphi_{\psi,\eta}(f)) \leq r^*\}\\
&\sqsupseteq \sqcap\{g^* \mid (\delta_1)_k(\varphi^{-1}_{\psi,\eta}(g),\varphi^{-1}_{\psi,\eta}(\varphi_{\psi,\eta}(f))) \leq r^*\}\\
&\sqsupseteq \sqcap\{\varphi_{\psi,\eta}(\varphi^{-1}_{\psi,\eta}(g^*)) \mid (\delta_1)_k(\varphi^{-1}_{\psi,\eta}(g),f) \leq r^*\}\\
&\sqsupseteq \varphi_{\psi,\eta}\Big(\sqcap\{\varphi^{-1}_{\psi,\eta}(g)^* \mid (\delta_1)_k(\varphi^{-1}_{\psi,\eta}(g),f) \leq r^*\}\Big)\\
&\sqsupseteq \varphi_{\psi,\eta}(\mathcal{C}_{\delta_1}(k,f,r))
\end{aligned}$$

□

Theorem 12. Let $(X,\mathcal{C}_X)$ and $(Y,\mathcal{C}_Y)$ be L-fuzzy (K_1,E_1)-soft and L-fuzzy (K_2,E_2)-soft closure spaces, respectively. Let $\varphi_{\psi,\eta} : (L^X)^E \to (L^Y)^{E_2}$ be a soft map. If $\varphi_{\psi,\eta} : (X,\mathcal{C}_X) \to (Y,\mathcal{C}_Y)$ is an L-fuzzy soft closure map, then $\varphi_{\psi,\eta} : (X,\delta_{\mathcal{C}_X}) \to (Y,\delta_{\mathcal{C}_Y})$ is an L-fuzzy soft proximity map.

Proof. Since $\varphi_{\psi,\eta} : (X,\mathcal{C}_X) \to (Y,\mathcal{C}_Y)$ is an L-fuzzy soft closure map,

$$\varphi_{\psi,\eta}(\mathcal{C}_X(k,\varphi^{-1}_{\psi,\eta}(f),r)) \sqsubseteq \mathcal{C}_Y(\eta(k),\varphi_{\psi,\eta}(\varphi^{-1}_{\psi,\eta}(f)),r) \sqsubseteq \mathcal{C}_Y(\eta(k),f,r).$$

$$\begin{aligned}
&(\delta_{\mathcal{C}_Y})_{\eta(k)}(g,f) = \bigwedge\{r^* \mid \mathcal{C}_Y(\eta(k),f,r) \sqsubseteq g^*\}\\
&\geq \bigwedge\{r^* \mid \varphi_{\psi,\eta}(\mathcal{C}_X(k,\varphi^{-1}_{\psi,\eta}(f),r)) \sqsubseteq g^*\}\\
&\geq \bigwedge\{r^* \mid \varphi^{-1}_{\psi,\eta}(\varphi_{\psi,\eta}(\mathcal{C}_X(k,\varphi^{-1}_{\psi,\eta}(f),r))) \sqsubseteq \varphi^{-1}_{\psi,\eta}(g^*)\}\\
&\geq \bigwedge\{r^* \mid \mathcal{C}_X(k,\varphi^{-1}_{\psi,\eta}(f),r) \sqcap \varphi^{-1}_{\psi,\eta}(g)^*\}\\
&\geq \delta_{\mathcal{C}_1}(\varphi^{-1}_{\psi,\eta}(g),\varphi^{-1}_{\psi,\eta}(f))
\end{aligned}$$

□

Example 13. Let $X = \{h_i \mid i = \{1, ..., 5\}\}$ with h_i=house and $E_X = \{e, b, w, c\}$ with e=expensive,b= beautiful, w=wooden, c= creative. Define a binary operation $\wedge$ on $[0, 1]$ by

$$x \wedge y = \min\{x, y\}, x^* = 1 - x, x \vee y = (x^* \wedge y^*)^*$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Then $([0, 1], \wedge, \to, 0, 1)$ is a stsc-quantale (ref.[8,9,18]). Let $E = \{e, b, w\} \subset E_X$, $f_1, f_2 \in ([0, 1]^X)^E$ as follows:

$$(f_1)_e = (0.8, 0.1, 0.5, 0.9, 0.6),\ (f_1)_b = (0.7, 0.9, 0.4, 0.5, 0.7)$$
$$(f_1)_w = (0.4, 0.7, 0.5, 0.6, 0.5)$$

$$(f_2)_e = (0.5, 0.9, 0.4, 0.8, 0.4),\ (f_2)_b = (0.3, 1.0, 0.2, 0.4, 0.5)$$
$$(f_2)_w = (0.5, 0.4, 0.8, 0.5, 0.1)$$

Then we obtain:

$$\begin{array}{ll} (f_1 \sqcup f_2)_e & = (0.8, 0.9, 0.5, 0.9, 0.6) \\ (f_1 \sqcup f_2)_b & = (0.7, 1.0, 0.4, 0.5, 0.7) \\ (f_1 \sqcup f_2)_w & = (0.5, 0.7, 0.8, 0.6, 0.5) \end{array}$$

$$\begin{array}{ll} (f_1 \sqcap f_2)_e & = (0.5, 0.1, 0.4, 0.8, 0.4) \\ (f_1 \sqcap f_2)_b & = (0.3, 0.9, 0.2, 0.4, 0.5) \\ (f_1 \sqcap f_2)_w & = (0.4, 0.4, 0.5, 0.5, 0.1) \end{array}$$

For $K = \{k_1, k_2\}$, we define a $[0, 1]$-fuzzy (K, E)-soft closure operator $\mathcal{C} : K \times ([0, 1]^X)^E \times (0, 1] \to ([0, 1]^X)^E$ as follows:

$$\mathcal{C}(k_1, f, r) = \begin{cases} 1_X, & \text{if } f = 1_X, r \in (0, 1], \\ f_1^*, & \text{if } f \sqsubseteq f_1^*, f \not\sqsubseteq f_1^* \sqcap f_2^*,\ r \leq 0.7, \\ f_2^*, & \text{if } f \sqsubseteq f_2^*, f \not\sqsubseteq f_1 \sqcap f_2^*,\ r \leq 0.4, \\ f_1^* \sqcap f_2^* & \text{if } f \sqsubseteq f_1^* \sqcap f_2^*,\ r \leq 0.5 \\ f_1^* \sqcup f_2^* & \text{if } f \sqsubseteq f_1^* \sqcup f_2^*, \\ & \quad f \not\sqsubseteq f_1^*,\ f \not\sqsubseteq f_2^*, r \leq 0.6, \\ 0_X, & \quad \text{otherwise.} \end{cases}$$

$$\mathcal{C}(k_2, f, r) = \begin{cases} 1_X, & \text{if } f = 1_X, r \in (0, 1], \\ f_1^*, & \text{if } f \sqsubseteq f_1^*,\ r \leq 0.5, \\ 0_X, & \text{otherwise.} \end{cases}$$

We obtain an L-fuzzy soft (K,E)-preproximity $\delta_{\mathcal{C}} : ([0,1]^X)^E \times ([0,1]^X)^E \to [0,1]$ as follows

$$(\delta_{\mathcal{C}})_{k_1}(f,g) = \begin{cases} 0, & \text{if } f = 0_X \text{ or } g = 0_X \\ 0.3, & \text{if } f \sqsubseteq f_1 \sqsubseteq g^*, f \not\sqsubseteq f_2 \\ 0.6, & \text{if } f \sqsubseteq f_1 \sqsubseteq g^*, f \not\sqsubseteq f_1 \\ 0.5, & \text{if } f \sqsubseteq f_1 \sqcup f_2 \sqsubseteq g^*, f \not\sqsubseteq f_1, f \not\sqsubseteq f_2 \\ 0.4, & \text{if } f \sqsubseteq f_1 \sqcap f_2 \sqsubseteq g^*, \\ 1, & \text{otherwise,} \end{cases}$$

$$(\delta_{\mathcal{C}})_{k_2}(f,g) = \begin{cases} 0, & \text{if } f = 0_X \text{ or } g = 0_X \\ 0.5, & \text{if } f \sqsubseteq f_1 \sqsubseteq g^*, \\ 1, & \text{otherwise,} \end{cases}$$

(1) By Theorem 10(2), since for k, r, f, there exists r with $r \wedge r = r$ with $\mathcal{C}(k, \mathcal{C}(k,f,r), r) = \mathcal{C}(k,f,r)$, $\delta_{\mathcal{C}}$ is an L-fuzzy soft (K,E)-quasi-proximity.

(2) Since $\mathcal{C}(k, f, \bigvee_{i\in I} r_i) = \sqcup_{i\in I}\mathcal{C}(k,f,r_i)$, by Theorem 10(3), $\mathcal{C}_{\delta_{\mathcal{C}}} = \mathcal{C}$.

(3) Since $\delta_{\mathcal{C}}(\sqcup_{i\in I} f_i, g) = \bigvee_{i\in I} \delta_{\mathcal{C}}(f_i, g)$, by Theorem 10(3), $\delta_{\mathcal{C}_{\delta_{\mathcal{C}}}} = \delta_{\mathcal{C}}$.

□

Example 14. Let X and E_X be given as Example 13. Let $([0,1], \odot, \oplus, \to, ^*, 0, 1)$ be a complete residuated lattice as Example 9. Let $E = \{b, w, c\} \subset E_X$, $f \in ([0,1]^X)^E$ be a fuzzy soft set as follow:

$$\begin{aligned} f_b &= (0.5, 0.3, 0.5, 0.6, 0.2) \\ f_c &= (0.1, 0.2, 0.6, 0.5, 0.5) \\ f_w &= (0.4, 0.4, 0.5, 0.6, 0.6) \end{aligned}$$

$$\begin{aligned} (f \odot f)_b &= (0.0, 0.0, 0.0, 0.2, 0.0) \\ (f \odot f)_c &= (0.0, 0.0, 0.2, 0.0, 0.0) \\ (f \odot f)_w &= (0.0, 0.0, 0.0, 0.2, 0.2) \end{aligned}$$

Let $K = \{k_1, k_2\}$ be given. Define a $[0,1]$-fuzzy (K,E)-soft closure operator $\mathcal{C} : K \times ([0,1]^X)^E \times (0,1] \to ([0,1]^X)^E$ as follows:

$$\mathcal{C}(k_1, h, r) = \begin{cases} 1_X, & \text{if } h = 1_X, r \in (0,1], \\ f^*, & \text{if } h \sqsubseteq f^*, r < 0.6 \\ f^* \oplus f^* & \text{if } h \sqsubseteq f^* \oplus f^* \\ & \quad h \not\sqsubseteq f^*, r < 0.3 \\ 0_X, & \quad \text{otherwise.} \end{cases}$$

$$\mathcal{C}(k_2, h, r) = \begin{cases} 1_X, & \text{if } h = 1_X, r \in (0,1], \\ f^*, & \text{if } h \sqsubseteq f^*, r < 0.4 \\ 0_X, & \text{otherwise.} \end{cases}$$

We obtain an L-fuzzy soft (K, E)-preproximity $\delta_{\mathcal{C}} : ([0,1]^X)^E \times ([0,1]^X)^E \to [0,1]$ as follows

$$(\delta_{\mathcal{C}})_{k_1}(h,g) = \begin{cases} 0, & \text{if } h = 0_X \text{ or } g = 0_X \\ 0.4, & \text{if } h \sqsubseteq f \sqsubseteq g^*, h \not\sqsubseteq f \odot f \\ 0.7, & \text{if } h \sqsubseteq f \odot f \sqsubseteq g^*, \\ 1, & \text{otherwise,} \end{cases}$$

$$(\delta_{\mathcal{C}})_{k_2}(h,g) = \begin{cases} 0, & \text{if } h = 0_X \text{ or } g = 0_X \\ 0.6, & \text{if } h \sqsubseteq f \sqsubseteq g^*, \\ 1, & \text{otherwise,} \end{cases}$$

We obtain a $[0,1]$-fuzzy (K, E)-soft closure operator $\mathcal{C} : K \times ([0,1]^X)^E \times (0,1] \to ([0,1]^X)^E$ as follows:

$$\mathcal{C}_{\delta_{\mathcal{C}}}(k_1, h, r) = \begin{cases} 1_X, & \text{if } h = 1_X, r \in (0,1], \\ f^*, & \text{if } h \sqsubseteq f^*, r \leq 0.6 \\ f^* \oplus f^* & \text{if } h \sqsubseteq f^* \oplus f^* \\ & \quad h \not\sqsubseteq f^*, r \leq 0.3 \\ 0_X, & \quad \text{otherwise.} \end{cases}$$

$$\mathcal{C}_{\delta_{\mathcal{C}}}(k_2, h, r) = \begin{cases} 1_X, & \text{if } h = 1_X, r \in (0,1], \\ f^*, & \text{if } h \sqsubseteq f^*, r \leq 0.4 \\ 0_X, & \text{otherwise.} \end{cases}$$

(1) By Theorem 10(2), since for k, r, f, for all s, t with $s \wedge t \leq r$ with $\mathcal{C}(k, \mathcal{C}(k, f, s), t) \neq \mathcal{C}(k, f, r)$. Moreover, $\delta_{\mathcal{C}}$ is not an L-fuzzy soft (K, E)-quasi-proximity because

$$0.4 = (\delta_{\mathcal{C}})_{k_1}(f, f^*) \not\geq \bigwedge_h \{(\delta_{\mathcal{C}})_{k_1}(f, h) \oplus (\delta_{\mathcal{C}})_{k_1}(h^*, f^*)\} = 1.$$

(2) We have $1_X = \mathcal{C}(k_1, f^*, \bigvee_{n \in N} 0.6 - \frac{1}{n}) \neq \sqcup_{n \in N} \mathcal{C}(k_1, f^*, 0.6 - \frac{1}{n}) = f^*$, by Theorem 10(3), $\mathcal{C}_{\delta_{\mathcal{C}}} \leq \mathcal{C}$ but $\mathcal{C}_{\delta_{\mathcal{C}}} \neq \mathcal{C}$.

(3) Since $\delta_{\mathcal{C}}(\sqcup_{i \in I} f_i, g) = \bigvee_{i \in I} \delta_{\mathcal{C}}(f_i, g)$, by Theorem 10(3), $\delta_{\mathcal{C}_{\delta_{\mathcal{C}}}} = \delta_{\mathcal{C}}$. □

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