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COMMUTATIVITY OF σ-PRIME Γ-RINGS WITH SEMIDERIVATIONS

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Abstract: Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and α, $\beta \in \Gamma$, I a σ-prime ideal of M and d a semiderivation associated with a function g which is surjective on I. In the paper we show some conditions on d, such that $d = 0$ or M is commutative.

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1. Introduction

C.L. Chuang [8] studied the structure of semiderivation of prime rings. He proved a structure theorem with the help of extended centroid of the classical associative rings. M. Bresar [6] also obtained the same result. J.C. Chang [7] extended some result of derivation in prime rings into semiderivations. The commutativity property of a prime ring have been investigated by H.E. Bell and

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W.S. Martindale [2] by means of semiderivations. J. Bergen and P. Grzesczule in [4] study the commutativity properties of semiprime rings with skew (semi)-derivations.

The definition and examples of semiderivations in Γ-rings have been first given by K.K. Dey and A.C. Paul [10]. Here the authors also introduced the notion σ-primenees in Γ-rings and studied centralizing automorphisms and Jordan left derivations on σ-prime Γ-rings. In [9], the authors extended some results of centralizing and commutating mappings of prime rings with semiderivations to prime Γ-ring with semiderivations. A generalization of results of C.L. Chuang [8] for prime Γ-rings has been obtained in [11]. Commutativity conditions of σ-prime Γ-rings has been studied in [9]. A.C.Paul and S. Chakrabarty [15] obtained the result $d = 0$ or $U \subseteq \mathrm{Z}(M)$, if d is acting as a homomorphism and and an anti-homomorphism in $\Gamma\sigma$-Lie ideal U of a σ-prime Γ-ring M where d is a derivation on M. Example 2.1 of [15] has been proved the existence of a σ-prime Γ-ring and an involution of a Γ-ring. In [12], A.K. Halder and A.C. Paul studied the commutativity properties of σ-prime Γ-ring with a nonzero derivation. In this paper, we obtain some commutativity results on σ-ideals of a σ-prime Γ-ring M with the help of semiderivation on M.

2. Preliminaries and Notations

In this section, we give some definitions and preliminary results that are needed for developing our results.

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ satisfying the following conditions:

1. $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$,
2. $(a\alpha b)\beta c = a\alpha(b\beta c)$,

for all $a, b, c \in M$, α, $\beta \in \Gamma$ then M is called a Γ-ring.

The notion of a Γ-ring was first introduced by N. Nabusawa [13] and then it was generalized by Barnes [1]. Throughout the paper, M represents a Γ-ring in the sense of Barnes [1] with centre $Z(M)$. A Γ-ring M is said to be a 2-*torsion free* if $2a = 0$ with $a \in M$ implies $a = 0$. A subset I of a Γ-ring M is called *left ideal* of M if I is an additive subgroup of M and $M\Gamma I \subseteq I$. The concept of *right ideal* of $M$is defined similarly. If I is both left and right ideal, then I is said to be a *two sided ideal* or simply an *ideal* of M. For elements a and b of a Γ-ring $M$and for $\alpha \in \Gamma$ we introduce the notation $[a,b]_\alpha = a\alpha b - b\alpha a$ (commutator of a and b with respect α). A Γ-ring M is called *commutative* if

$[a,b]_\alpha = 0$ for all $a,b \in M$ and $\alpha \in \Gamma$. For elements α, $\beta \in \Gamma$ and for an element a of a Γ-ring Mwe also introduce formal notation $[\alpha,\beta]_a = \alpha a\beta - \beta a\alpha$, which means

$$a[\alpha, \beta]_b c = a(\alpha b\beta - \beta b\alpha)c = (a\alpha b)\beta c - -(a\beta b)\alpha c.$$

Note that the following basic commutator identities hold

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a[\alpha, \beta]_c b + a\alpha[b, c]_\beta$$

and

$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b[\alpha, \beta]_a c + b\alpha[a, c]_\beta,$$

for all a, b, $c \in M$ and α, $\beta \in \Gamma$.

If we assume that

$$a\alpha b\beta c = a\beta b\alpha c, \tag{1.1}$$

for all a, b, $c \in M$ and α, $\beta \in \Gamma$, then we get

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta, \tag{1.2}$$

and

$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta, \tag{1.3}$$

and for all a, b, $c \in M$ and α, $\beta \in \Gamma$.

It is observed that the identities (1.2) and (1.3) mean that if by $d^r_\beta(\cdot) =[\cdot,c]_\beta$ and $d^l_\beta(\cdot) =[c, \cdot]_\beta$we denote the right and left multiplication operators, respectively, on M with respect the bracket operation then we have

$$d^r_\beta(a\alpha b) = d^r_\beta(a)\alpha c + a\alpha d^r_\beta(b)$$

and

$$d^l_\beta(a\alpha b) = d^l_\beta(a)\alpha c + a\alpha d^l_\beta(b),$$

that is the right and the left multiplication operators are derivations on M.

An additive mapping $\sigma : M \to M$ is called an involution on M if $\sigma(a+b) = \sigma(a) + \sigma(b)$, $\sigma(a\alpha b) = \sigma(b)\alpha\sigma(a)$, and $\sigma(\sigma(a)) = a$,for all a, $b \in M$ and $\alpha \in \Gamma$.

The following example of an involution of a Γ-ring has been given in [15].

Example 2.1. Let M be a Γ-ring. Define $M_1 = \{(a, b) : a, b \in M\}$and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Addition and multiplication on M_1 are defined as:

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } (a, b)(\alpha, \alpha)(c, d) = (a\alpha c, d\alpha b).$$

Under these addition and multiplication the set M_1 is a Γ_1-ring. Define a mapping

$$\sigma : M_1 \to M_1$$

by $\sigma((a, b)) = (b, a)$.

Then σ is an involution on M_1.

Given a Γ-ring M with an involution σ, we define

$$Sa_\sigma(M) = fm \in M : \sigma(m) = \pm mg,$$

which is known as the set of all symmetric and skew symmetric elements of M. Recall that M is σ-prime if

$$a\Gamma M\Gamma b = a\Gamma M\Gamma\sigma(b) = 0 \quad (a\Gamma M\Gamma b = \sigma(a)\Gamma M\Gamma b = 0) \tag{1.4}$$

implies that $a = 0$ or $b = 0$. It is clear that every prime Γ-ring having an involution σ is a σ-prime Γ-ring. The following example ensures that the converse of the above conclusion, in general, is not true (see [15]).

Example 2.2. Let M be a prime Γ-ring. Then we claim that the Γ_1-ring M_1equipped with the involution σ given in Example 2.1 is a σ-prime Γ_1-ring. Because, for any (a, b), (c, d), $(x, y) \in M_1$ and $(\alpha,\ \alpha)$, $(\beta,\ \beta) \in \Gamma_1$, we assume that

$$(a, b)(\alpha, \alpha)(x, y)(\beta, \beta)(c, d) = 0 = (a, b)(\alpha, \alpha)(x, y)(\beta, \beta)(d, c)$$

[since $\sigma((c, d)) = (d, c)$]. This gives

$$a\alpha x\beta c = 0, d\beta y\alpha b = 0, a\alpha x\beta d = 0 and c\beta y\alpha b = 0 \text{ for all } x, y \in M and \alpha, \beta \in \Gamma.$$

Therefore,

$$a\Gamma M\Gamma c = 0,\ d\Gamma M\Gamma b = 0,\ a\Gamma M\Gamma d = 0,\ \text{and } c\Gamma M\Gamma b = 0.$$

In view of the primeness of M, these yield $a = 0$ or $c = 0$, $d = 0$ or $b = 0$, $a = 0$ or $d = 0$ and $c = 0$ or $b = 0$. In all the cases, we obtain $(a, b) = 0$ or $(c, d) = 0$, which establishes our claim. But, $M_1$is not a prime Γ_1-ring. For, $(a,\ 0)(\alpha,\ \alpha)(x, y)(\beta,\ \beta)(0,\ b) = (0,\ 0)$ but $(a,\ 0)$ or $(0,\ b)$ are need not be zero, unless a =0 or b =0.

An ideal I of M is called a σ-ideal, if $\sigma(I) = I$. Through the paper the center of I is denoted by $\mathrm{Z}(I)$. An additive mapping $d : M \to M$ is said to be a derivation if $d(a\alpha b) = a\alpha d(b) + a\alpha d(b)$ holds for all a,$b \in M$, $\alpha \in \Gamma$. An additive mapping $d : M \to M$ is said to be a *semiderivation* associated with a function g: $M \to M$ if for all $a,b \in M$, $\alpha \in \Gamma$, the conditions

$$d(a\alpha b) = d(a)\alpha g(b) + a\alpha d(b) \text{ and } d(g(a)) = g(d(a))$$

hold. If g is an identity mapping of M, then all semi derivations associated with g become an ordinary derivations. If g is any endomorphism of M, then $d(x) = x- g(x)$ gives an example of a semiderivation. Here is another example of a semiderivation from [11].

Example 2.3. Let M_1 be a Γ_1-ring and M_2 be a Γ_2-ring. Consider $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$. Define addition and multiplication on M and Γ by

$$(m_1, m_2) + (m_3, m_4) = (m_1 + m_3, m_2 + m_4)(\alpha_1, \alpha_2) + (\alpha_3, \alpha_4) \\ = (\alpha_1 + \alpha_3, \alpha_2 + \alpha_4),$$

$$(m_1, m_2)(\alpha_1, \alpha_2)(m_3, m_4) = (m_1\alpha_1 m_3, m_2\alpha_2 m_4),$$

for (m_1, m_2), $(m_3, m_4) \in M$ and (α_1, α_2), $(\alpha_3, \alpha_4) \in \Gamma$.

Under these addition and multiplication operations the set M is a Γ-ring. Let δ:$M_1 \to M_1$ be an additive map and τ: $M_2 \to M_2$ be a left and right $M_2\Gamma$-module which is not a derivation. Define $d : M \to M$ such that $d((m_1, m_2)) = (0, \tau(m_2))$ and g: $M \to M$ such that $g((m_1, m_2)) = (\delta(m_1), 0)$, for $m_1 \in M_1$, $m_2 \in M_2$. Then it is clear that d is a semi-derivation of M (with associated map g) which is not a derivation.

3. Semiderivations of σ-Prime Γ-Rings

In this section, we study the commutativity property of σ-prime Γ-rings by mean of semiderivations. Throughout the section, M represents a 2-torsion free σ-prime Γ-rings satisfying the condition (1.1). Let d be a semiderivation of M such that $d\sigma = \sigma d$ associated with a function g: $M \to M$.

We begin with the following lemma from [9].

Lemma 3.1. *Let $I \neq 0$ be a σ-ideal of* M. *If $a,b \in$* M *are such that*

$$a\Gamma I\Gamma b = 0 = a\Gamma I\Gamma\sigma(b) \textit{ or } \sigma(a)\Gamma I\Gamma b = 0 = a\Gamma I\Gamma b,$$

then $a = 0$ or $b = 0$.

The following lemma is a generalization of the lemma above.

Lemma 3.2. *Let $I \neq 0$ be a σ-ideal of* M *and $a \in$ M. If $a\alpha d(x) = 0$ (or $d(x)\alpha a = 0$) for all $x \in I$ and $\alpha \in \Gamma$, then $a = 0$ or $d = 0$.*

Proof. Let $x \in I$, $m \in M$ and $\beta \in \Gamma$ then $a\alpha d(x\beta m) = 0$, and therefore

$$a\alpha d(x)\beta g(m) + a\alpha x\beta d(m) = 0, \tag{2.1}$$

and so we get $a\alpha x\beta d(m) = 0$, for all $x \in I$, $m \in M$ and α, $\beta \in \Gamma$. Hence, we have $a\Gamma I\Gamma d(m) = 0$ for all $m \in M$. Putting $\sigma(m)$ for m and using the fact $d\sigma = \sigma d$, we obtain $a\Gamma \mathrm{I}\Gamma\sigma(d(m)) = 0$ for all $m \in M$. That is, we have $a\Gamma I\Gamma d(m) = 0 = a\Gamma I\Gamma\sigma(d(m)) = 0$ for all $m \in M$. That is, we have $a\Gamma I\Gamma d(m) = 0 = a\Gamma I\Gamma\sigma(d(m))$ for all $m \in M$. Due to Lemma 3.1, we conclude that $a = 0$ or $d = 0$.

If $d(x)\alpha a = 0$, then for all $m \in M$, $x \in I$ and $\beta \in \Gamma$, we get $d(m\beta x)\alpha a = 0$ so that $d(m)\beta x\alpha a+$ $g(m)\beta d(x)\alpha a = 0$. Therefore $d(m)\Gamma I\Gamma a = 0$ for all $m \in M$. If we replace m by $\sigma(m)$ and if we can use the fact $d\sigma = \sigma d$, then we obtain $d(m)\Gamma I\Gamma a = 0 = \sigma(d(m))\Gamma I\Gamma a$. By using Lemma 2.1, we obtain $d(m) = 0$ for all $m \in M$ or $a = 0$.

Theorem 3.3. *Let $I \neq 0$ be a σ-ideal of* M. *If $d(x) \neq 0$ for all $x \in I$, then g is a homomorphism of M.*

Proof. For all $x \in I$, and m, $n \in M$, we have $x\alpha(m+ n) \in I$ for all $\alpha \in \Gamma$. So, we have

$$\begin{aligned} d(x\alpha(m+n)) &= d(x)\alpha g(m+n) + x\alpha d(m+n) \\ &= d(x)\alpha g(m+n) + x\alpha d(m) + x\alpha d(n). \end{aligned} \tag{2.2}$$

On the other hand

$$\begin{aligned} d(x\alpha(m+n)) &= d(x\alpha m + x\alpha n) = d(x\alpha m) + d(x\alpha n) \\ &= d(x)\alpha g(m) + x\alpha d(m) + d(x)\alpha g(n) + x\alpha d(n). \end{aligned} \tag{2.3}$$

Comparing (2.2) and (2.3), we obtain

$$d(x)\alpha g(m+n) - d(x)\alpha(g(m) + g(n)) = 0,$$

and so

$$d(x)\alpha(g(m+n) - g(m) - g(n)) = 0,$$

for all $x \in I$, $m, n \in M$ and $\alpha \in \Gamma$.

By Lemma 3.2, we conclude that $g(m+n) = g(m) + g(n)$, for all $x \in I$, $m, n \in M$ and $\alpha \in \Gamma$, since $d(x) \neq 0$ for $x \in I$.

Now, let $x \in I$, $m, n \in M$ and α, $\beta \in \Gamma$. Then

$$\begin{aligned} d(x\alpha(m\beta n)) &= d(x)\alpha g(m\beta n) + x\alpha d(m\beta n) \\ &= d(x)\alpha g(m\beta n) + x\alpha d(m)\beta n + x\alpha m\beta d(n). \end{aligned}$$

On the other hand

$$\begin{aligned} d((x\alpha m)\beta n) &= d(x\alpha m)\beta g(n) + x\alpha m\beta d(n) \\ &= d(x)\alpha g(m)\beta g(n) + x\alpha d(m)\beta g(n) + x\alpha m\beta d(n). \end{aligned}$$

Comparing these two equations, we find

$$d(x)\alpha(g(m\alpha n) - g(m) - g(n)) = 0$$

for all $x \in I$, $m, n \in M$ and α, $\beta \in \Gamma$.

Since $d(x) \neq 0$ for all $x \in I$, by Lemma 3.2, we have $g(m\beta n) = g(m)\beta g(n)$ for all $m, n \in M$ and $\beta \in \Gamma$.

Theorem 3.4. *Let $I \neq 0$ be a σ-ideal of* M *such that $g(I) = I$. If $d(I) \subseteq Z(I)$ then $d = 0$ or M is commutative.*

Proof. For x, $y \in I$ and $\alpha \in \Gamma$, $x\alpha y \in I$. Therefore, $d(x\alpha y) \in$Z(I). So, we have

$$d(x)\alpha g(y) + x\alpha d(y) \in Z(I),$$

for all x, $y \in I$ and $\alpha \in \Gamma$.

Commuting this term with x and using the hypothesis, we find

$$\begin{aligned} 0 =& [d(x)\alpha g(y) + x\alpha d(y), x]_\beta \\ =& [d(x)\alpha g(y), x]_\beta + [x\alpha d(y), x]_\beta \\ =& d(x)\alpha[g(y), x]_\beta + [d(x), x]_\beta \alpha g(y) + x\alpha[d(y), x]_\beta + [x, x]_\beta \alpha d(y) \\ =& d(x)\alpha[g(y), x]_\beta. \end{aligned}$$

Since $g(I) = I$, we get $[g(y),\, x]_\beta = [y,\, x]_\beta$ for all $x, y \in I, \beta \in \Gamma$. Therefore $d(x)\alpha[y,\, x]_\beta = 0$ for all x, $y \in I$ and α, $\beta \in \Gamma$. Replacing y by $y\gamma z$, for $z \in I$, $\gamma \in \Gamma$, we obtain

$$\begin{aligned} 0 =& d(x)\alpha[y\gamma z, x]_\beta \\ =& d(x)\alpha y\gamma[z, x]_\beta + d(x)\alpha[y, x]_\beta \gamma z \end{aligned}$$

$$=d(x)\alpha y\gamma[z,x]_\beta. \tag{3.1}$$

Since M is σ-prime and $\sigma d = d\sigma$, for any $x \in Sa_\sigma$(M), we have $d(x)\Gamma I\Gamma[z, x]_\beta = 0 = \sigma(d(x))\Gamma I\Gamma[z, x]_\beta$. By using Lemma 3.1, we conclude that $d(x) = 0$ or $[z, x]_\beta = 0$ for all $z \in I$and $\beta \in \Gamma$. Now let $x \in I$. As $x \pm \sigma(x) \in I\cap\ Sa_\sigma(x)$, then $d(x \pm \sigma(x)) = 0$, then $d(x \pm \sigma(x)) = 0$ or $[z, x \pm \sigma(x)]_\beta = 0$ for all $x \in I$ and $\beta \in \Gamma$. If $d(x \pm \sigma(x)) = 0$, then

$$d(x) = \pm d(\sigma(x)) = \pm\sigma(d(x)).$$

Therefore equation (3.1) becomes $d(x)\Gamma I\Gamma[z, x]_\beta = 0 = \sigma(d(x))\Gamma I\Gamma[z, x]_\beta$. In view of Lemma 3.1, $d(x) = 0$ or $[z, x]_\beta$ for all $x \in I$ and $\beta \in \Gamma$. If $[z, x \pm \sigma(x)]_\beta = 0$, then

$$[z, x]_\beta = \pm[z, \sigma(x)]_\beta = \pm\sigma([z, x]_\beta),$$

since $\sigma(I) = I$. Therefore from the equation (3.1), we have $d(x)\Gamma I\Gamma[z, x]_\beta = 0 = d(x)\Gamma I\Gamma\sigma([z, x]_\beta)$ for all $z \in I$ and $\beta \in \Gamma$. This yields that $d(x) = 0$ or $[z, x]_\beta = 0$ for all $z \in I$ and $\beta \in \Gamma$, hence we obtain that I is the union of its two additive subgroups such that $A = \{x \in I : d(x) = 0\}$and $B = \{x \in I : x \in \mathrm{Z}(I)\}$. But a group cannot be the set theoretic union of two proper subgroups. Hence $A = I$ or $B = I$. If $A = I$, then $d(x) = 0$ for all $x \in I$. If $B = I$, Then I is commutative. If $d(x) = 0$ for all $x \in I$, then for all $x \in I$, $m \in M$ and $\alpha \in \Gamma$, we have $0 = d(x\alpha m) = d(x)\alpha g(m) + x\alpha d(m) = x\alpha d(m)$. This implies that $I\Gamma d(m) = 0$. Since $I\Gamma\mathrm{M} \subseteq I$, $I\Gamma M\Gamma d(m) \subseteq I\Gamma d(m) = 0$ we have $I\Gamma M\Gamma d(m) = 0 = I\Gamma M\Gamma\sigma(d(m))$. Since M is σ-prime and $I \neq 0$, we conclude that $d(m) = 0$ for all $m \in M$. If I is commutative, then for every $x, y \in I$, $[x, y]_\alpha = 0$ for all $\alpha \in \Gamma$. Replace y by $y\beta n$ for $n \in M$ and $\beta \in \Gamma$, we obtain $0 = [x, y\beta n]_\alpha = y\beta[x, n]_\alpha +\ [x, y]_\alpha\beta n = y\beta[x, n]_\alpha$. This implies that $I\Gamma[x,n]_\alpha = 0$ and hence $I\Gamma M\Gamma[x, n]_\alpha = 0 = \sigma(I)\Gamma M\Gamma[x, n]_\alpha$. Since $I \neq 0$ and M is σ-prime, we get $[x, n]_\alpha = 0$ for all $x \in I, n \in M$ and $\alpha \in \Gamma$. Replacing x by $x\gamma m$ for $m \in M$ and $\gamma \in \Gamma$, we find that $x\gamma[m, n]_\alpha = 0$ and hence $I\Gamma[m, n]_\alpha = 0$ which gives that $I\Gamma M\Gamma[m, n]_\alpha = 0$, so that $[m, n]_\alpha = 0$ for all $m, n \in M$ and $\alpha \in \Gamma$. Therefore M is commutative.

Theorem 3.5. *Let $I \neq 0$ be a σ-ideal of M such that $g(I) = I$. If $d^2(I) = 0$, then $d = 0$.*

Proof. Putting $x = x\alpha y$ for $y \in I$ and $\alpha \in \Gamma$ in$d^2(x) = 0$ we get

$$\begin{aligned}0 =& d^2(x\alpha y) = d(d(x\alpha y)) = d(d(x)\alpha g(y) + x\alpha d(y))\\ =& d^2(x)\alpha g^2(y) + d(x)\alpha d(g(y)) + d(x)\alpha g(d(y)) + x\alpha d^2(y)\end{aligned}$$

$$=d(x)\alpha d(g(y)) + d(x)\alpha g(d(y)) = 2d(x)\alpha d(g(y)).$$

Since M is 2-torsion free and g is surjective function of M, we find $d(x)\alpha d(y) = 0$ for all x, $y \in I$ and $\alpha \in \Gamma$. In view of Lemma 3.2, we get the required result.

Theorem 3.6. *Let $I \neq 0$ be a σ-ideal of M such that $g(I) = I$ and $a \in M$. If $[d(x),\, a]_\alpha = 0$ for all $x \in I$ and $\alpha \in \Gamma$, then $d = 0$ or $a \in Z(I)$. Hence we get $a \in Z(M)$.*

Proof. We have $[d(x),\, a]_\alpha = 0$ for all $x \in I$ and $\alpha \in \Gamma$. Replacing x by $x\beta y$ for $y \in I$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} 0 =&[d(x\beta y), a]_\alpha = [d(x)\beta g(y) + x\beta d(y), a]_\alpha \\ =&[d(x)\beta g(y), a]_\alpha + [x\beta d(y), a]_\alpha \\ =&d(x)\beta[g(y), a]_\alpha + [d(x), a]_\alpha\beta g(y) + x\beta[d(y), a]_\alpha + [x, a]_\alpha\beta d(y). \end{aligned}$$

By using the hypothesis, we find

$$d(x)\beta[g(y), a]_\alpha + [x, a]_\alpha\beta d(y) = 0,$$

for all $x, y \in I$ and α, $\beta \in \Gamma$.

Replacing x by $d(x)$ in the above equation and using the hypothesis, we obtain

$$d^2(x)\beta[g(y), a]_\alpha = 0,$$

for all $x, y \in I$ and α, $\beta \in \Gamma$.

In view of the surjectiveness of g, we get $d^2(x)\beta[y,\, a]_\alpha = 0$ for all x, $y \in I$ and α, $\beta \in \Gamma$. Putting $y\gamma z$ for y where $z \in I$ and $\gamma \in \Gamma$, we have

$$0 = d^2(x)\beta[y\gamma z,\, a]_\alpha = d^2(x)\beta y\gamma[z,\, a]_\alpha + d^2(x)\beta[y,\, a]_\alpha\gamma z = d^2(x)\beta y\gamma[z,\, a]_\alpha$$

for all $x, y \in I$ and α, $\beta \in \Gamma$. This implies that $d^2(x)\Gamma I\Gamma[z,\, a]_\alpha = 0$ for all $x, y \in I$ and α,$\beta \in \Gamma$. Replacing x by $\sigma(x)$ and using $\sigma d = d\sigma$, we find $\sigma(d^2(x))\Gamma I\Gamma[z,\, a]_\alpha = 0$ for all x,$y \in I$ and α, $\beta \in \Gamma$. In view of Lemma 3.1, we obtain $d^2(x) = 0$ or $[z,\, a]_\alpha = 0$ for all $z \in I$ and $\alpha \in \Gamma$. If $d^2(x) = 0$ for all $x \in I$, then $d = 0$ by Theorem 3.5. If $[z,\, a]_\alpha = 0$ for all $z \in I$ and $\alpha \in \Gamma$, $a \in$Z(I). Now replacing z by $z\beta m$ for $m \in I$ and $\beta \in \Gamma$, we have

$$0 = [z\beta m,\, a]_\alpha = z\beta[m,\, a]_\alpha + [z,\, a]_\alpha\beta m = z\beta[m,\, a]_\alpha = 0 \text{ for all } z \in I.$$

Replacing z by $z\gamma x$ for $x \in I$ and $\gamma \in \Gamma$, we obtain $z\gamma x\beta[m, a]_\alpha = 0$. By Theorem 3.1, we conclude that $a \in$Z(M).

Theorem 3.7. *If $I \neq 0$ is a σ-ideal of M such that $g(I) = I$. If $[d(I), d(I)]_\alpha = 0$ for all $\alpha \in \Gamma$, then $d = 0$ or M is commutative.*

Proof. Since $[d(I), d(I)]_\alpha = 0$ for all $\alpha \in \Gamma$ in view of Theorem 2.6, we obtain d =0 or $d(I) \subseteq$ Z(I), therefore by Theorem 3.4. we have $d = 0$ or M is commutative.

Theorem 3.8. *Let $I \neq 0$ be a σ-ideal of M. If $[d(x), x]_\alpha = 0$ for all $x \in I$ and $\alpha \in \Gamma$, then $d = 0$ or M is commutative.*

Proof. According to the hypothesis of the theorem we have $[d(x), x]_\alpha = 0$ for all $x \in I$ and $\alpha \in \Gamma$. Putting $x + y$ for x and using (2.1), we find

$$[d(x), y]_\alpha + [d(y), x]_\alpha = 0 \text{ for all } x, y \in I \text{ and } \alpha \in \Gamma.$$

Replacing y by $y\beta x$, and using (2.1) in the above equation, we obtain

$$\begin{aligned}
0 =&[d(x), y\beta x]_\alpha + [d(y\beta x), x]_\alpha \\
=&y\beta[d(x), x]_\alpha + [d(x), y]_\alpha\beta x + [d(y)\beta g(x) + y\beta d(x), x]_\alpha \\
=&[d(x), y]_\alpha\beta x + d(y)\beta[g(x), x]_\alpha + [d(y), x]_\alpha\beta g(x) + y\beta[d(x), x]_\alpha \\
&+ [y, x]_\alpha\beta d(x) \\
=&[d(x), y]_\alpha\beta x + d(y)\beta[g(x), x]_\alpha + [d(y), x]_\alpha\beta g(x) + [y, x]_\alpha\beta d(x).
\end{aligned}$$

Since g$(I) = I$, we have

$$([d(x), y]_\alpha + [d(y), x]_\alpha)\beta x + [y, x]_\alpha\beta d(x) = [y, x]_\alpha\beta d(x).$$

Hence we have, $[y, x]_\alpha\beta d(x) = 0$ for all $x \in I$ and α, $\beta \in \Gamma$. Replacing y by $z\gamma y$ for $z \in I$ and $\gamma \in \Gamma$, we obtain $[z, x]_\alpha\gamma y\beta d(x) = 0$. That is $[z, x]_\alpha\Gamma I\Gamma d(x) = 0$ for all $x, z \in I$ and $\alpha \in \Gamma$. By using the same arguments which are used in the last part of the proof of the Theorem 3.4, we obtain the required result.

Theorem 3.9. *Let $I \neq 0$ be a σ-ideal of M such that $g(I) = I$. If $d([x, y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.*

Proof. Writing $x\beta y$ for y in the hypothesis, we find
$0 = d([x, x\beta y]_\alpha) = d(x\beta[x, y]_\alpha + [x, x]_\alpha\beta y)$

$= d(x)\beta g([x,y]_\alpha) + x\beta d([x,y]_\alpha)$
$= d(x)\beta g([x,y]_\alpha)$.
In view of Theorem 3.3, g is a homomorphism of M. Hence we have $d(x)\beta([g(x),\ g(y)]_\alpha = 0$ for all $x, y \in I$ and $\alpha,\ \beta \in \Gamma$.
Since g(I) = I, we get

$$[g(x),\ g(y)]_\alpha = [x,y]_\alpha \text{ for all } x, y \in \text{I and } \alpha \in \Gamma.$$

Hence, we obtain $d(x)\beta([x,y]_\alpha = 0$ for all $x, y \in I$ and $\alpha,\ \beta \in \Gamma$. Replacing y by $y\gamma z$ for $z \in I$ and $\gamma \in \Gamma$, we get
$d(x)\beta([x,y\gamma z]_\alpha = d(x)\beta y\gamma[x,z]_\alpha + d(x)\beta[x,y]_\alpha\gamma z = d(x)\beta y\gamma[x,z]_\alpha = 0$.
That is $d(x)\Gamma I\Gamma[x,z]_\alpha = 0$ for all $x, z \in I$ and $\alpha,\ \beta \in \Gamma$. If $x \in Sa_\sigma(M)$, then $x = pm\sigma(x)$ so the above equation becomes $d(\sigma(x))\Gamma I\Gamma[x,z]_\alpha = 0$ for all $x \in Sa_\sigma(M)$, $z \in I$ and $\alpha \in \Gamma$. Since $\sigma d = d\sigma$, we obtain $d(\sigma(x))\Gamma I\Gamma[x,z]_\alpha = \sigma(d(x))\ \Gamma I\Gamma[x,z]_\alpha = 0$ for all $x \in Sa_\sigma(M)$, $z \in I$ and $\alpha \in \Gamma$. Therefore, $d(x) = 0$ or $[x,z]_\alpha = 0$ for all $x \in Sa_\sigma(M)$, $z \in I$ and $\alpha \in \Gamma$. If $x \in I \cap Sa_\sigma(M)$, then $x \pm \sigma(x) \in Sa_\sigma(M)$. As a result we have $d(x \pm \sigma(x)) = 0$ or $[x \pm \sigma(x),\ z]_\alpha = 0$ for all $z \in I$. Using the same arguments used in the last part of the Theorem 3.4, we have the required result.

Theorem 3.10. *Let $I \neq 0$ be a σ-ideal of M such that $g(I) = I$. If $d([x,y]_\alpha) = \pm[x,y]_\alpha$, for all $x, y \in I$ and $\alpha \in \Gamma$, then $d = 0$ or M is commutative.*

Proof. For x,$y \in I$ and $\beta \in \Gamma$, we have $x\beta y \in$I. Therefore putting $x\beta y$ for y in the hypothesis, we have

$$d([x,x\beta y]_\alpha) = \pm[x,x\beta y]_\alpha,$$

for all $x, y \in I$ and $\alpha,\ \beta \in \Gamma$. This implies that $d(x\beta[x,y]_\alpha) = \pm x\beta[x,y]_\alpha$ for all $x, y \in I$ and $\alpha \in \Gamma$, so that $d(x)\beta g([x,y]_\alpha)$ =0 for all $x, y \in I$ and $\alpha \in \Gamma$. In view of the hypothesis, the last relation yields $d(x)\beta g([x,y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$. Using the similar arguments used in the last part of the proof of Theorem 3.9, we have the result.

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