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AN INFINITE SERVER QUEUEING MODEL WITH VARYING ARRIVAL AND DEPARTURES RATES FOR HEALTHCARE SYSTEM

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Abstract: In this paper, we consider the infinite server queues with time-varying arrival and departure pattern when the parameters are varying with time. Here we give an extension of our previous work on infinite server queueing model considering inpatient department to study the improvement in the varying service rate with the help of probability generating function techniques which results in difference differential equations. With an infinite number of servers providing service to the system, aim is to find an optimal solution to the distribution of service time over a given period. Simulation techniques are used to demonstrate the effectiveness of this model.

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Key Words: transient solution, infinite server queue, $M/M/\infty$, generating functions, inpatient department

1. Introduction

Inpatient Department is one of the most important component of hospital man-

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agement and is visited by a large number of patients. Once the patients writes an admission note he/she becomes an 'inpatient'formally. Here we consider the city Guwahati, which is the rapidly developing city in the North-eastern region of India. There is heavy inflow of patients in this region from neighboring rural and semi urban areas or from towns because of the availability of advanced health facilities especially in Public hospitals.

The public hospitals in any major city in a developing country like India faces heavy rush in the inpatient department constantly because of several reasons. It not only has to cater to the moderately and low socio-economic population of the city who prefer public hospitals, coming to its own outpatient department, but also patients who are referred by family doctors or through emergency medicine department, by public hospitals of neighboring rural and semi-urban areas, Thus it requires an efficient patient management system.

A queueing system consists of one or more servers that provide service of some sort to the arrivals at the queue. The arriving rate and sometime the rate of service vary with time. In queueing theory the $M/M/\infty$ queue is a multi-server queueing model where every patient does not have to wait because they experience immediate service from the server. Queueing system is primarily concerned with congestion (e.g. waiting and blocking) related to inadequate resources. The infinite server is thus of great importance for multi-server systems with varying arrivals as it serves as the basis for important over-load analysis, which characterizes the total load faced by the system, and serves as the basis for much useful analysis. Hence it is often seen that such a service is of utmost importance in the health service sector. Here the demand is random and service time distribution is exponentially distributed.

The transient solution of queueing problems is associated with varying arrival and departure routes. Saaty [19] has considered the transient single server model and derived the queue size of the distribution. Collings and Stoneman [6] have also considered the transient $M/M/\infty$ model and derived the queue size distribution for a time dependent arrival and departure rates in the form of probability generating function. Mahanta et al.[13] proposed a single server queueing model for severe diseases especially in outpatient department.

Parthasarathy and Sharafali [18] have also derived a simple method for the time dependent solution for a number of patients in the system in the multi server model. Abol'nikov [1] derives the generating function of the queue size distribution at any time. Shanbhag [22]considers the general service time distribution and shows that if the departure rate is one at a point of time and the initial queue size is empty, then the resulting queue size distribution is always Poisson with a time dependent parameter. Clarke [5] studied queues resulting from non homogeneous Poisson processes and provided a complete theoretical solution to the single server model.

Leese and Boyd [12] provides a useful discussion of the mathematical method that have been suggested. Bagchi and Templeton [2] also applied the queues resulting from non homogeneous Poisson processes and their method is not applicable to the more general sorts of time depending. Mohit et al. [16] too considered the $M/M/\infty$ model and obtained the probabilities and expected queue length of the system for a time dependent values of arrival and service rates. Feldman [9], Green [10], Hampshire [11], Mandelbaum [14], Massey [15], Newell [17] have worked on the performance of queues with time-varying arrival rates and their application. Also, Collings and Stoneman [6] has done a study on a $M/M/\infty$ queue with deterministic time varying arrivals and service rates.

Zhou and Gans [23], state that time-varying service rate problems have not been studied in the literature. The key difference between Zhou and Gans [23] and ours is that, they consider service speeds that change only when a customer completes service. Unlike so in our paper, the server speed cannot change during the middle of a service. In addition, the service rates mainly take only two values in Zhou and Gans [23]. Boxma and Kurkova [4] considers an M/G/1 queue where the speed of the server alternates between two values with high speed periods having exponential distribution and low speed periods having a general distribution. Motivated by the transportation system where if an incident occurs on a road segment all the vehicles on the road have to lower their speed until that incident is cleared, Baykal-Gursoy and Xiao [3] considers an $M/M/\infty$ queueing system subject to random interruptions of exponential distributed durations. El-Sherbiny et al. [8] proposed a transient solution for infinite server queues by assuming Poisson arrivals and exponential service times but their equations were extremely abstract with respect to managerial use in operations planning. Satoh and Yanagihara [21] consider a new approach of modelling growth curves is developed which uses time-varying coefficients. Satoh et al.[20] proposed a logistic regression model for the survival time, using time-varying coefficients and the NewtonRaphson method. Dhar et al. [7] compared the single server as well as multiple servers queuing model in an outpatient department.

Our work is founded on the findings from an inpatient department (IPD) in a public hospital where the daily fluctuation of the number of patients is large and that has limited beds to serve patients admitted from the emergency department and other sources.

Here we offer a more realistic example of the hospital admission by developing a queueing system that takes into account the transfer of patients from the emergency department to other units in the hospital. The main issue that arises in this setting is the possibility of the beds being full in those units, which causes patients, whose treatment is waiting in the hospital, to be blocked from being transferred and hence to continue waiting for beds until it becomes available in hospital units. This effectively lengthens the duration of their stay and reduces the hospital efficiency, where some patients may be waiting to get a bed into an inpatient department. Here we study $M/M/\infty$ queue model numerically. Moreover, this infinite server queueing model is used to understand how the number of beds affects the inpatient department for patients.

The main purpose of this paper is to study a queue whose service capacity varies over time. That is, the rate of the server with which it serves a patient is determined by an external environment process. In particular, we assume that the server rate changes according to a Continuous Time Markov Chain that is independent of the arrival process and service requirements of the patient. Each patient brings a certain random amount of work, however, the rate at which this work is completed is time-varying. For example, the server serves at different rates (per hours) over time to serve a request that needs a certain number patients.

2. Problem Formulation for Transient Infinite Server Queueing Model ($M/M/\infty$)

The birth and death equations for an infinite server queue are:

$$\begin{aligned} \frac{dP_n(t)}{dt} &= P_{n-1}(t)\lambda + P_{n+1}(t)\ \mu + P_n(t)(\lambda+\mu), n \geq 1 \\ \frac{dP_0(t)}{dt} &= -\lambda P_0(t) + \mu P_1(t) \end{aligned} \tag{2.1}$$

We consider the following initial conditions for the system:
$\lambda = \lambda \sum_{i=0}^{1} P_{i1}$ *and* $\mu = \mu \sum_{j=0}^{1} P_{j1}$
Let us consider a modified infinite server queueing system with patients arriving at random and with an exponential distribution of service. The chance of a new arrival in time t is $\lambda \sum_{i=0}^{1} P_{i1}$ and the chance of a departure is $\mu \sum_{j=0}^{1} P_{j1}$, so long as there is at least one patient present. Otherwise, it is zero.

Let $P_n(t)$ be the probability that there are n patients in the system at time and let $G(z,t)$ be the probability generating function as defined below

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n \tag{2.2}$$

The proposed the birth-death equations are obtained by a applying the initial conditions defined above on the equations (2.1) and (2.2). Hence,

$$\begin{aligned}\frac{dP_n(t)}{dt} &= P_{n-1}(t)\lambda\sum_{i=0}^{1}P_{i1} + P_{n+1}(t)\mu\sum_{j=0}^{1}P_{j1}\\ &\quad +P_n(t)(\lambda+\mu), n\geq 1n\geq \qquad (2.3)\end{aligned}$$

where $P_{i1} = P(S = i|A = 1)$ and $P_{j1} = P(S = j|SE = 1)$ for all $i, j = 0, 1$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)\ \sum_{i=0}^{1}\ P_{i1} + \mu P_1(t)\ \sum_{j=0}^{1}\ P_{j1} \qquad (2.4)$$

Now differentiating (2.3) and (2.4) with respect to t and using (2.1) and (2.2), we derive the following equation as expressed in terms of probability generating function:

$$\frac{dG(z,t)}{dt} - (1-z)\mu\ \sum_{j=0}^{1}P_{j1}\frac{dG(z,t)}{dz} = -\lambda(1-z)\sum_{i=0}^{1}\ P_{i1}G(z,t) \qquad (2.5)$$

Solving the equation (2.6) by the usual method , we get

$$\frac{dt}{1} = \frac{dz}{-(1-z)\mu\ \sum_{j=0}^{1}\ P_{j1}} = \frac{dG}{-\lambda(1-z)G\ \sum_{i=0}^{1}\ P_{i1}} \qquad (2.6)$$

Now, the first two expression in (2.7) can be written in the following form

$$\begin{aligned} c_1 &= (1-z)e^{-\mu\ t\sum_{j=0}^{1}\ P_{j1}}\\ &= (1-z)e^{-\sum_{j=0}^{1}\ P_{j1}\int_0^t\mu(t)dt}\\ &= (1-z)I(t) \qquad (2.7)\end{aligned}$$

where c_1 is a constant and $I(t) = e^{-\sum_{j=0}^{1}\ P_{j1}\int_0^t\mu\ tdt}$
Again from the last two terms of (2.7),we get

$$\begin{aligned} c_2 &= Ge^{-\frac{\lambda\ \sum_{i=0}^{1}\ P_{i1}}{\mu\ \sum_{j=0}^{1}\ P_{j1}}}\\ &= Ge^{-\frac{\sum_{i=0}^{1}\ P_{i1}}{\sum_{j=0}^{1}\ P_{j1}}\int_0^t\frac{\lambda(t)}{\mu(t)}dt}\\ &= GI^{\square}(t) \qquad (2.8)\end{aligned}$$

where c_2 is a constant and $I^{\square}(t) = e^{-\frac{\sum_{i=0}^{1} P_{i1}}{\sum_{j=0}^{1} P_{j1}} \int_0^t \frac{\lambda(t)}{\mu(t)} dt}$.

The general solution is given by

$$c_2 = f(c_1)$$

3. Results and Discussion

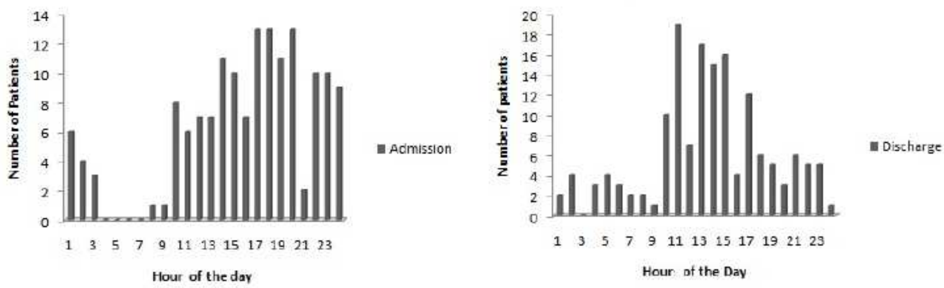

Figure 3.1: The admission and discharge pattern of patients of an in-patient department.

Figure 3.1 shows that the admission and discharge pattern of incoming and out-coming patients in an inpatient department. It also depicts that the service systems typically have time-varying arrival and departure rates, usually with significant variation over the day. For instance, the arrival rate of patients in an inpatient department might vary from 0 (during the late night) to 24 hours of a whole day. Because of such time varying arrivals and departures, it is difficult to analyze the system performance. It is no longer possible to apply the steady-state analysis associated with queueing models having constant arrival rates, commonly found in many books. Table 3.1 summarizes the number of those patients who spent on average time (in terms of days). For better graphical representations of the below summarized table as shown in Figure 3.1. It is observed that for four diseases out of thirteen, patients spent on average longer time in inpatient department. For example in the first disease in Table 3.1, the average time spent by patients in a hospital IPD were seven days. Figure 3.2 [(a)-(d)] contains arrival pattern of those patients who wait for bed with arrival rate varying with day, weeks, months and years respectively. It also depicts the

Table 3.1: The number of those patients who spent on bed in terms of days during the period

Disease	Number of Patients	Average number of days (in IPD)
Fever	150	7
Pain Abdomen	46	5
Gastric	190	6
Anaemia	90	2
Diabetic	101	1
Appendicitis	17	8
Dengu	19	3
Typhoid	26	2.5
Asthma	36	1.5
vomiting	61	0.5
Cholelithiasis	42	2
Cholecystitis	23	1
Choleccystectomy	6	1

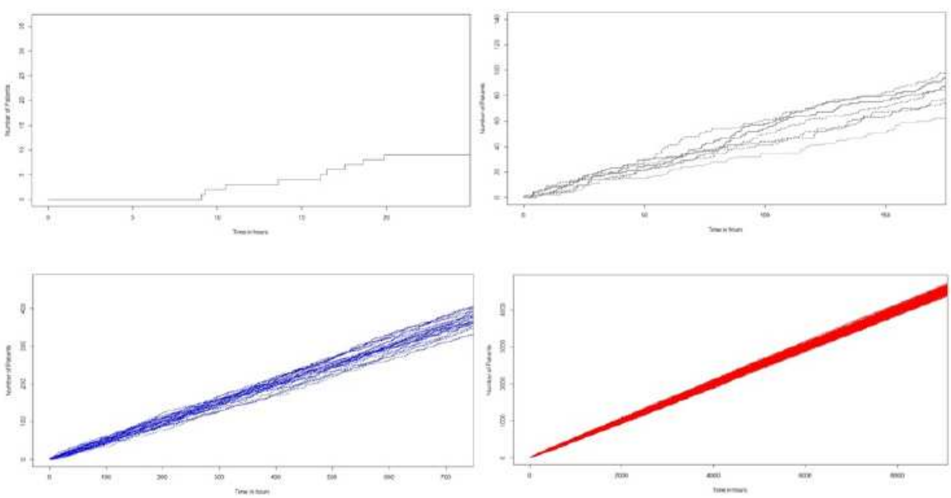

Figure 3.2: Average queue size variation when arrival rate is 0.5.

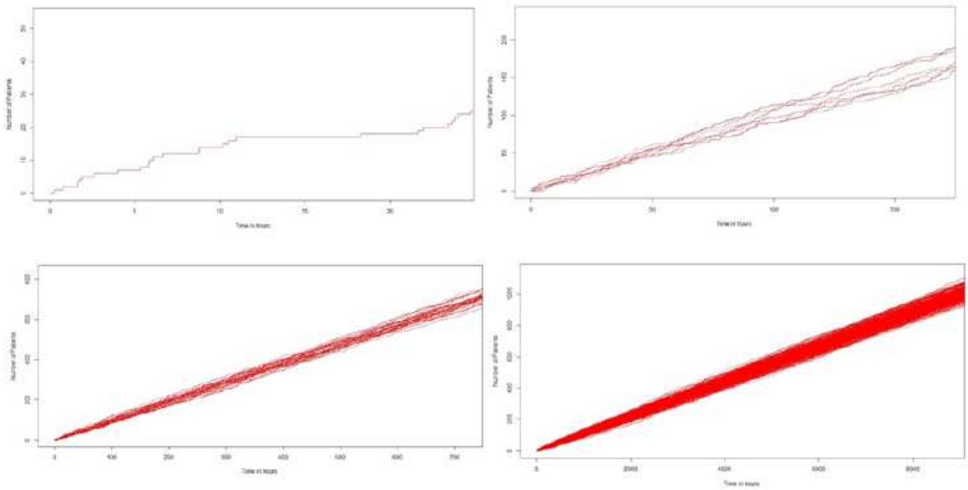

Figure 3.3: Average queue size variation when arrival rate is $\lambda = 0.5 * (1 + 0.9 * \cos \frac{\pi t}{24})$.

average queue size when arrival rate is 0.5. As observed the average queue size increases with time in terms of days, weeks,months and years.

Figure 3.3 [(a)-(d)] shows that the average queue size when arrival rate is varies with time t i.e. $\lambda = 0.5 * (1 + 0.9 * \cos \frac{\pi t}{24})$ in terms of the days, weeks, months and years respectively. For instance, considering arrival in terms of days, vomiting records the lowest time spent on bed over the period of half days as compared to fever, pain abdomen, gastric and appendicitis that records seven, five, six and eight respectively. It seems that fever is not a severe disease as compared to other disease and which create large variation for bed those patients who need immediate service and also compared to the other diseases. Thus one would not expect the variation to have a large effect on the queue size.

Figure 3.4 shows that the average queue size when arrival rate is varying with time t i.e. $\lambda = 0.5 * (1 + 0.9 * \cos \frac{\pi t - 24}{24})$. It also illustrates the variation in the average queue size at the end of each day assuming the arrival and departure rates are vary i.e the arrival rate is increasing while the departure rate is decreasing. The arrival rate is vary with time and that experienced in most inpatient department and it will effect the day variation of the patients who wait for bed.

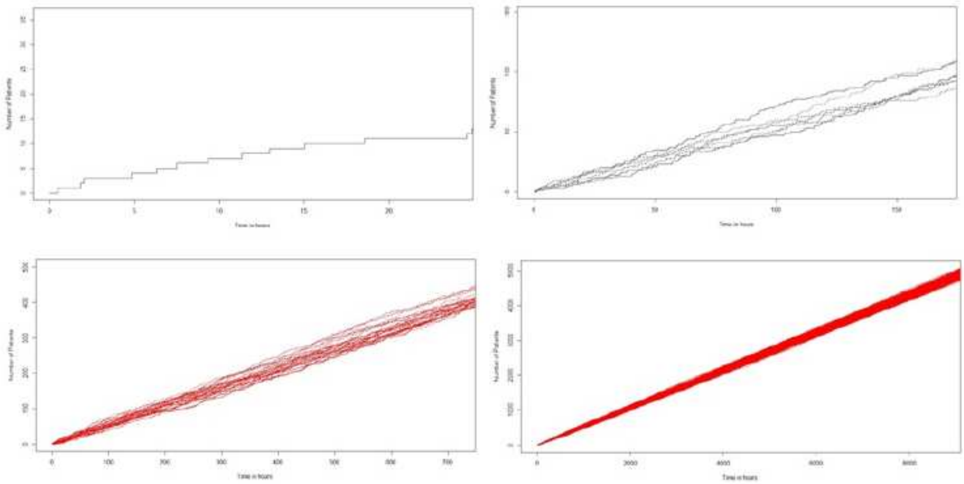

Figure 3.4: Average queue size variation when arrival rate is $\lambda = 0.5 * (1 + 0.9 * \cos \frac{\pi t - 24}{24})$

4. Conclusion

This study obtains the average queue size of an infinite server queues with Poisson arrivals and exponential service times when the parameters of both distributions are allowed to vary with time. The transient probabilities of the system are given by using the generating function and difference differential equations.

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References

[1] L. M. Abol'nikov. A nonstationary queueing problem for a system with an infinite number of channels for a group arrival of requests. *Problemy Peredachi Informatsii*, **4(3)** (1968) 99-102 .

[2] T. P. Bagchi and James G. C. Templeton.*Numerical Methods in Markov Chains and Bulk Queues*. Springer Berlin Heidelberg, (1972) **doi:** 10.1007/978-3-642-80712-1.

[3] M. Baykal-Gursoy and W. Xiao. Stochastic decomposition in $M/M/\infty$ queues with markov modulated service rates. *Queueing Systems*, **48(1-2)** (2004) 75-88.

[4] O. J. Boxma and I.A. Kurkova. The M / G /1 queue with two service speeds. **Adv. in Appl. Probab. 33**, 2 (2001) 520–540, **doi:** 10.1239/aap/999188327.

[5] A. B. Clarke. A Waiting Line Process of Markov Type. Ann. Math. Statist. **27**, 2 (1956) 452–459. **doi:** 10.1214/aoms/1177728268.

[6] T. Collings and C. Stoneman. The $M/M/\infty$ Queue with Varying Arrival and Departure Rates, *Oper. Res.* **24**, 4 (1976) 760-773, **doi:** 10.1287/opre.24.4.760.

[7] K. K. Das, S. Dhar and L. B Mahanta. Comparative study of waiting and service costs of single and multiple server system: A case study on an outpatient department.*International Journal of Scientific Footprints*,**3(2)** (2014) 18-30, **doi:** 10.22576/ijsf.

[8] A. El-Sherbiny, H. Zamani, N. Ismail, G. Mokaddis, M. Ghazal, A. El-Desokey, M. S. El-Sherbeny, E. Al-Esayeh, and O. Olotu. Transient solution to an infinite server queue with varying arrival and departure rate. *Journal of Mathematics and Statistics*, **6(1)** (2010) 1-3, **doi:** 10.3844/jmssp.2010.1.3.

[9] Z. Feldman, A. Mandelbaum, W. A. Massey, and W. Whitt. Staffing of time-varying queues to achieve time-stable performance. *Management Science*,**54(2)** (2008) 324-338, **doi:** 10.1287/mnsc.1070.0821.

[10] L. V. Green, P. J Kolesar and W. Whitt. Coping with Time-Varying Demand When Setting Staffing Requirements for a Service System. *Production and Operations Management*, 16 (2007) 1339, **doi:** 10.1111/j.1937-5956.2007.tb00164.x.

[11] R. C. Hampshire and W. A. Massey. Dynamic optimization with applications to dynamic rate queues. *TUTORIALS in Operations Research, INFORMS Society*, (2010) 210-247. **doi:** 10.1287/educ.1100.0077

[12] E.L. Leese and D.W. Boyd. Numerical methods of determining the transient behaviour of queues with variable arrival rates.*J. Can. Oper. Res. Soc*, 4 (1966) 1-13.

[13] L. B. Mahanta, K. K. Das, and S. Dhar. A queuing model for dealing with patients with severe disease. *Electronic Journal of Applied Statistical Analysis*, **9(2)** (2016) 362-370, **doi:** 10.1285/i20705948v9n2p362.

[14] A. Mandelbaum, W. A. Massey, and M. I. Reiman. Strong approximations for markovian service networks. *Queueing Systems*, **30(1-2)** (1998) 149-201, **doi:** 10.1023/A:1019112920622

[15] W. A. Massey. The analysis of queues with time-varying rates for telecommunication models. *Telecommunication Systems*, 21 (2002), 173-204, **doi:** 10.1023/A:1020990313587.

[16] D.V. Gupta Mohit and P.K Sharma. Transient solution of$M/M/\infty$ queue with varying arrival and service rate. *VSRD Technical & Non-Technical Journal* (2010) 119-128.

[17] C. Newell. *Applications of queueing theory*, **4**. *Springer Science & Business Media*, (2013), **doi:** 10.1007/978-94-009-5970-5.

[18] P.R. Parthasarathy and M. Sharafali. Transient solution to the many-server poisson queue: a simple approach. *Journal of Applied Probability* (1989) 58-594, **doi:** 10.2307/3214415.

[19] T. L. Saaty. *Elements of queueing theory*, **423**, *McGraw-Hill* New York, (1961).

[20] K. Satoh, T. Tonda, and S. Izumi. Logistic regression model for survival time analysis using time-varying coefficients. *American Journal of Mathematical and Management Sciences*, **35(4)** (2016) 353-360, **doi:** 10.1080/01966324.2016.1215945.

[21] K. Satoh and H. Yanagihara. Estimation of varying coefficients for a growth curve model. *American Journal of Mathematical and Management Sciences*, **30 (3-4)** (2010) 243-256. **doi:** 10.1080/01966324.2010.10737787.

[22] D. N. Shanbhag. On infinite server queues with batch arrivals. *Journal of Applied Probability*, **3(1)** (1966) 274-279, **doi:** 10.2307/3212053 .

[23] Y. P. Zhou, N. Gans, et al. A single-server queue with markov modulated service times. submitted for publication, (1999).