International Journal of Pure and Applied Mathematics
Volume 113 No. 5 2017, 595-607
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i5.7

ijpam.eu

FIXED POINT THEOREMS FOR GENERALIZED (α-ψ)-CONTRACTIVE MAPPINGS IN MULTIPLICATIVE METRIC SPACES

Parveen Kumar[1], Shin Min Kang[2 §], Sanjay Kumar[3], Chahn Yong Jung[4]

[1,3]Departement of Mathematics
Deenbandhu Chhotu Ram University
of Science and Technology
Murthal, Sonepat, 131039, Haryana, INDIA
[2]Department of Mathematics and RINS
Gyeongsang National University
Jinju, 52828, KOREA
[4]Department of Business Administration
Gyeongsang National University
Jinju 52828, KOREA

Abstract: In this paper, we discuss generalized (α-ψ)-contractive mappings in setting of multiplicative metric spaces and obtain the existence and the uniqueness of fixed points for these mappings.

AMS Subject Classification: 47H10, 54H25
Key Words: multiplicative metric spaces, generalized (α-ψ)-contractive mappings, α-admissible mappings, fixed points

1. Introduction and Preliminaries

It is well know that the set of positive real numbers $\mathbb{R}^n_+$ is not complete according

Received: November 30, 2016
Revised: February 1, 2017
Published: March 31, 2017

§Correspondence author

to the usual metric. To overcome this problem, in 2008, Bashirov et al. [2] introduced the concept of multiplicative metric spaces as follows:

Definition 1.1. Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x,y) \geq 1$ for all $x,y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,y) \leq d(x,z) \cdot d(z,y)$ for all $x,y,z \in X$ (multiplicative triangle inequality).

Then the mapping d together with X, that is, (X,d) is a multiplicative metric space.

Example 1.2. ([3]) Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d^* : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n_+$ and $|\cdot|^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore $(\mathbb{R}^n_+, d^*)$ is a multiplicative metric space.

One can refer to [3] for detailed multiplicative metric topology.

Definition 1.3. Let (X,d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X is said to be

(1) a *multiplicative convergent* to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x,y) < \epsilon\}$, $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, that is, $d(x_n, x) \to 1$ as $n \to \infty$.

(2) a *multiplicative Cauchy sequence* if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m,n \geq N$, that is, $d(x_n, x_m) \to 1$ as $n,m \to \infty$.

(3) We call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

In 2012, Özavsar and Çevikel [3] gave the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings in a multiplicative metric space.

Definition 1.4. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a *multiplicative contraction* if there exists a real number $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^{\lambda}(x, y) \quad \text{for all } x, y \in X.$$

In 2012, Samet et al. [4] introduced the concept of α-admissible mappings and established fixed point theorems for these mappings in complete metric spaces. In fact, these results extend and generalize many existing fixed point results present in the literature.

Definition 1.5. Let X be a nonempty set, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is *α-admissible* if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.

Example 1.6. Let $X = [0, \infty)$. Define mappings $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \sqrt{x} \text{ for all } x \in X \quad \text{and} \quad \alpha(x, y) = \begin{cases} 2 & \text{if } x \geq y, \\ 0 & \text{if } x < y, \end{cases}$$

respectively. Then T is an α-admissible mapping.

Example 1.7. Let $X = \mathbb{R}$. Define mappings $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \ln |x| & \text{if } x \neq 0, \\ 7 & \text{if } x = 0 \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} e^{x-y} & \text{if } 0 < y \leq x, \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Then T is an α-admissible mapping.

Denote with Ψ_1 the family of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Also, Samet et al. [4] introduced the notion of α-ψ-contractive mappings in a metric space as follows:

Definition 1.8. Let T be a mapping of a metric space (X, d) into itself. Then T is said to be an *α-ψ-contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_1$ such that

$$\alpha(x, y) \cdot d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

The α-ψ-contraction was further generalized by Alsulami et al. [1] in the setting of a generalized metric space to rational contraction known as (α, ψ)-rational type-I as follows:

Let Ψ_2 be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(1) ψ is upper semi-continuous and non-decreasing;
(2) $\{\psi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t > 0$;
(3) $\psi(t) < t$ for every $t > 0$.

Definition 1.9. Let T be a mapping of a generalized metric space (X, d) into itself. Then T is said to be an (α, ψ)*-rational type-I contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_2$ such that

$$\alpha(x, y) \cdot d(Tx, Ty) \leq \psi(M(x, y))$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{d(Tx, x), d(Ty, y), d(x, y), \frac{d(Tx, x) \cdot d(Ty, y)}{1 + d(x, y)}, \frac{d(Tx, x) \cdot d(Ty, y)}{1 + d(Tx, Ty)}\right\}.$$

2. Main Results

Now we formulate the notion of α-ψ-contractive mappings in the context of a multiplicative metric space as follows:

Let Ψ_3 be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(i) ψ is upper semi-continuous and non-decreasing;
(ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ;
(iii) $\psi(t) < t$ for every $t > 0$.

Definition 2.1. Let T be a mappings of a multiplicative metric space (X, d) into itself. Then T is said to be a *generalized* $(\alpha$-$\psi)$*-contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_3$ such that

$$\alpha(x, y) \cdot d(Tx, Ty) \leq \psi(M(x, y)) \tag{2.1}$$

for all $x, y \in X$, where

$$M(x,y) = \max\Big\{d(Tx,x), d(Ty,y), d(x,y), (d(Tx,y)\cdot d(Ty,x))^{1/2}, \frac{d(Tx,x)\cdot d(Ty,y)}{1+d(x,y)}, \frac{d(Tx,y)\cdot d(Ty,x)}{1+d(x,y)}, \frac{d(Tx,y)\cdot d(Ty,x)}{1+d(Tx,Ty)}\Big\}.$$

Now, we establish our main results for α-ψ-contractive mappings in multiplicative metric spaces.

Theorem 2.2. *Let T be a mapping of a complete multiplicative metric space (X, d) into itself and $\alpha : X \times X \to [0, \infty)$ be a given function satisfying the following conditions:*
$(C1)$ T is an α-admissible mapping;
$(C2)$ T is a generalized $(\alpha,\, \psi)$-contractive mapping;
$(C3)$ there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;
$(C4)$ T is continuous.
Then T has a fixed point.

Proof. From $(C3)$, there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. We construct a sequence $\{x_n\}$ in X by

$$x_n = T^n x_0 = Tx_{n-1}$$

for all $n \in \mathbb{N}$. It is obvious that if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T.

Consequently, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible, we become

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1.$$

This implies

$$\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1 \quad \text{and} \quad \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0. \tag{2.2}$$

By similar arguments, we have $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1$ and hence

$$\alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+2}) \geq 1 \quad \text{for all } n \geq 0. \tag{2.3}$$

From $(C2)$ and (2.2), putting $x = x_n$ and $y = x_{n-1}$ in (2.1), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1}) \cdot d(Tx_n, Tx_{n-1}) \\ &\leq \psi(M(x_n, x_{n-1})), \end{aligned}$$

where

$$\begin{aligned} &M(x_n, x_{n-1}) \\ &= \max\Big\{ d(Tx_n, x_n), d(Tx_{n-1}, x_{n-1}), d(x_n, x_{n-1}, \\ &\quad (d(Tx_n, x_{n-1}) \cdot d(Tx_{n-1}, x_n))^{1/2}, \frac{d(Tx_n, x_n) \cdot d(Tx_{n-1}, x_{n-1})}{1 + d(x_n, x_{n-1})}, \\ &\quad \frac{d(Tx_n, x_{n-1}) \cdot d(Tx_{n-1}, x_n)}{1 + d(x_n, x_{n-1})}, \frac{d(Tx_n, x_{n-1}) \cdot d(Tx_{n-1}, x_n)}{1 + d(Tx_n, Tx_{n-1})} \Big\} \\ &= \max\Big\{ d(x_{n+1}, x_n), d(x_n, x_{n-1}), d(x_n, x_{n-1}), \\ &\quad (d(x_{n+1}, x_{n-1}) \cdot d(x_n, x_n))^{1/2}, \frac{d(x_{n+1}, x_n) \cdot d(x_n, x_{n-1})}{1 + d(x_n, x_{n-1})}, \\ &\quad \frac{d(x_{n+1}, x_{n-1}) \cdot d(x_n, x_n)}{1 + d(x_n, x_{n-1})}, \frac{d(x_{n+1}, x_{n-1}) \cdot d(x_n, x_n)}{1 + d(x_{n+1}, x_n)} \Big\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \end{aligned} \tag{2.4}$$

since

$$\frac{d(x_{n+1}, x_n) \cdot d(x_n, x_{n-1})}{1 + d(x_n, x_{n-1})} \leq d(x_n, x_{n+1}),$$

$$\frac{d(x_{n+1}, x_{n-1}) \cdot d(x_n, x_n)}{1 + d(x_n, x_{n-1})} \leq d(x_n, x_{n+1})$$

and

$$\frac{d(x_{n+1}, x_{n-1}) \cdot d(x_n, x_n)}{1 + d(x_{n+1}, x_n)} \leq d(x_{n-1}, x_n).$$

Case 1. If for some n, we have $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$, then

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \psi(M(x_n, x_{n-1})) \\ &= \psi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}), \end{aligned} \tag{2.5}$$

which is impossible.

Case 2. If $M(x_n, x_{n-1}) = d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \psi(d(x_{n-1}, x_n)). \tag{2.6}$$

Repeating the above arguments, we have

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_0, x_1)) \quad \text{for all } n \in \mathbb{N}.$$

From (ii), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 1. \tag{2.7}$$

Putting $x = x_{n+1}$ and $y = x_{n-1}$ in (2.1) and using (2.3) we have

$$\begin{aligned} d(x_{n+2}, x_n) &= d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \alpha(x_{n+1}, x_{n-1}) \cdot d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \psi(M(x_{n+1}, x_{n-1})), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &M(x_{n+1}, x_{n-1}) \\ &= \max\Big\{ d(Tx_{n+1}, x_{n+1}), d(Tx_{n-1}, x_{n-1}), d(x_{n+1}, x_{n-1}), \\ &\quad (d(Tx_{n+1}, x_{n-1}) \cdot d(Tx_{n-1}, x_{n+1}))^{1/2}, \frac{d(Tx_{n+1}, x_{n+1}) \cdot d(Tx_{n-1}, x_{n-1})}{1 + d(x_{n+1}, x_{n-1})}, \\ &\quad \frac{d(Tx_{n+1}, x_{n-1}) \cdot d(Tx_{n-1}, x_{n+1})}{1 + d(x_{n+1}, x_{n-1})}, \frac{d(Tx_{n+1}, x_{n-1}) \cdot d(Tx_{n-1}, x_{n+1})}{1 + d(Tx_{n+1}, Tx_{n-1})} \Big\} \\ &= \max\Big\{ d(x_{n+2}, x_{n-1}), d(x_n, x_{n-1}), d(x_{n-1}, x_{n+1}), \\ &\quad (d(x_{n+2}, x_{n-1}) \cdot d(x_n, x_{n-1}))^{1/2}, \frac{d(x_{n+2}, x_{n+1}) \cdot d(x_n, x_{n-1})}{1 + d(x_{n-1}, x_{n+1})}, \\ &\quad \frac{d(x_{n+2}, x_{n-1}) \cdot d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n+2}, x_{n-1}) \cdot d(x_n, x_{n+1})}{1 + d(x_n, x_{n+2})} \Big\}. \end{aligned}$$

Define $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Then

$$\begin{aligned} &M(x_{n+1}, x_{n-1}) \\ &= \max\left\{ b_{n-1}, b_{n-1}, a_{n-1}, b_{n-1}, \frac{b_{n+1} \cdot b_{n-1}}{1 + a_{n-1}}, \frac{b_{n-1} \cdot b_n}{1 + a_{n-1}}, \frac{b_{n-1} \cdot b_n}{1 + a_n} \right\}. \end{aligned}$$

If $M(x_{n+1}, x_{n-1}) = b_{n-1}$, then taking lim sup as $n \to \infty$ in (2.8) and using (2.7) and upper semi-continuity of ψ we get

$$\begin{aligned} 1 \le \limsup_{n\to\infty} a_n &\le \limsup_{n\to\infty} (M(x_{n+1}, x_{n-1})) \\ &= \psi\Big(\limsup_{n\to\infty} M(x_{n+1}, x_{n-1})\Big) = \psi(1) = 1. \end{aligned}$$

Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_{n+2}, x_n) = 1.$$

If $M(x_{n+1}, x_{n-1}) = a_{n-1}$ or $\frac{b_{n+1}\cdot b_{n-1}}{1+a_{n-1}}$, or $\frac{b_{n-1}\cdot b_n}{1+a_{n-1}}$, or $\frac{b_{n-1}\cdot b_n}{1+a_n}$, then (2.8) implies

$$a_n \le \psi(a_{n-1}) < a_{n-1},$$

due to the property (iii) of ψ, that is, the sequence $\{a_n\}$ is positive monotone decreasing, and hence, it converges to some $t \ge 1$.

Assume that $t > 1$. Now, by (2.8), we get

$$t = \limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \psi(a_{n-1}) = \psi\Big(\limsup_{n\to\infty} a_{n-1}\Big) = \psi(t) < t,$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_{n+2}, x_n) = 1. \tag{2.9}$$

Let $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \ne n$. Without loss of generality, we assume that $m > n + 1$. Since $\alpha(x_m, x_{m-1}) \ge 1$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) = d(x_m, Tx_m) \\ &= d(Tx_{m-1}, Tx_m) \\ &\le \alpha(x_{m-1}, x_m) \cdot d(Tx_{m-1}, Tx_m) \\ &\le \psi(M(x_{m-1}, x_m)), \end{aligned} \tag{2.10}$$

where

$$
\begin{aligned}
&M(x_{m-1}, x_m) \\
&= \max\Big\{(d(Tx_{m-1}, x_{m-1}), d(Tx_m, x_m), d(x_{m-1}, x_m), \\
&\quad (d(Tx_{m-1}, x_m) \cdot d(Tx_m, x_{m-1}))^{1/2}, \frac{d(Tx_{m-1}, x_{m-1}) \cdot d(Tx_m, x_m)}{1 + d(x_{m-1}, x_m)}, \\
&\quad \frac{d(Tx_{m-1}, x_m) \cdot d(Tx_m, x_{m-1})}{1 + d(x_{m-1}, x_m)}, \frac{d(Tx_{m-1}, x_m) \cdot d(Tx_m, x_{m-1})}{1 + d(Tx_{m-1}, Tx_m)}\Big\} \\
&= \max\Big\{d(x_m, x_{m-1}), d(x_{m+1}, x_m), d(x_{m-1}, x_m), \\
&\quad d(x_m, x_m) \cdot d(x_{m+1}, x_{m-1}))^{1/2}, \frac{d(x_m, x_{m-1}) \cdot d(x_{m+1}, x_m)}{1 + d(x_{m-1}, x_m)}, \\
&\quad \frac{d(x_m, x_m) \cdot d(x_{m+1}, x_{m-1})}{1 + d(x_{m-1}, x_m)}, \frac{d(x_m, x_m) \cdot d(x_{m+1}, x_{m-1})}{1 + d(x_m, x_{m+1})}\Big\} \\
&= \max\{d(x_m, x_{m-1}), d(x_{m+1}, x_m)\}.
\end{aligned}
\tag{2.11}
$$

If $M(x_{m-1}, x_m) = d(x_m, x_{m-1})$, then (2.10) implies

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \psi(d(x_{m-1}, x_m)) \\
&\leq \psi^{m-n}(d(x_n, x_{n+1})) \quad \text{for all } m, n \in \mathbb{N}.
\end{aligned}
\tag{2.12}
$$

If on the other hand, $M(x_{m-1}, x_m) = d(x_{m+1}, x_m)$, then from (2.10) we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \psi(d(x_{m+1}, x_m)) \\
&\leq \psi^{m-n+1}(d(x_n, x_{n+1})) \quad \text{for all } m, n \in \mathbb{N}.
\end{aligned}
\tag{2.13}
$$

Using the property (iii) of ψ, the two inequalities (2.12)and (2.13) imply

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

which is impossible. Thus, $x_n \neq x_m$ for all $m, n \in \mathbb{N}$ with $m \neq n$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence, that is, $\lim_{n\to\infty} d(x_n, x_{n+k}) = 1$ for all $k \in \mathbb{N}$. We have already proved the cases for $k = 1$ and $k = 2$ in (2.7) and (2.9), respectively.

Take arbitrary $k \geq 3$. We discuss two cases.

Case 1. Suppose that $k = 2m+1$, where $m \geq 1$. Using the triangle inequality, we have

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \\ &\le d(x_n, x_{n+1}) \cdot d(x_{n+1}, x_{n+2}) \cdots d(x_{n+2m}, x_{n+2m+1}) \\ &\le \prod_{p=n}^{n+2m} \psi^p(d(x_0, x_1)) \qquad (2.14) \\ &\le \prod_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 1 \quad \text{as } n \to \infty. \end{aligned}$$

Case 2. Suppose that $k = 2m$, where $m \ge 2$. Using the triangle inequality, we have

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \\ &\le d(x_n, x_{n+2}) \cdot d(x_{n+2}, x_{n+3}) \cdots d(x_{n+2m-1}, x_{n+2m}) \\ &\le d(x_n, x_{n+2}) \cdot \prod_{p=n}^{n+2m-1} \psi^p(d(x_0, x_1)) \\ &\le d(x_n, x_{n+2}) \cdot \prod_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 1 \quad \text{as } n \to \infty. \end{aligned}$$

By (2.9), we have $\lim_{n\to\infty} d(x_{n+2}, x_n) = 1$, therefore $\lim_{n\to\infty} d(x_n, x_{n+k}) = 1$ for all $k \ge 3$. Hence we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 1. \qquad (2.15)$$

Finally, we show that the limit x^* of the sequence $\{x_n\}$ is a fixed point of T. Since T is continuous. Then from (2.15) we have

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 1.$$

By uniqueness of limit, x^* is a fixed point of T. This completes the proof. □

Theorem 2.3. *Let T be a mapping of a complete multiplicative metric space (X, d) and $\alpha : X \times X \to [0, \infty)$ be a given function satisfying the conditions $(C1)$-$(C3)$ and the following conditions:*

$(C5)$ if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x, x_{n(k)}) \ge 1$ for all k.

Then T has a fixed point.

Proof. Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \geq 0$. By Theorem 2.2, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ and from (2.2) there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x^*, x_{n(k)}) \geq 1$ for all $k \in \mathbb{N}$.

Now from inequality (2.1) with $x = x_{n(k)}$ and $y = x^*$, we have

$$\begin{aligned} d(x_{n(k)+1}, Tx^*) &= d(Tx_{n(k)}, Tx^*) \\ &\leq \alpha(x_{n(k)}, x^*) \cdot d(Tx_{n(k)}, Tx^*) \\ &\leq \psi(M(x_{n(k)}, x^*)), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} &M(x_{n(k)}, x^*) \\ &= \max\Big\{ d(Tx_{n(k)}, x_{n(k)}), d(Tx^*, x^*), d(x_{n(k)}, x^*), \\ &\quad (d(Tx_{n(k)}, x^*) \cdot d(Tx^*, x_{n(k)}))^{1/2}, \frac{d(Tx_{n(k)}, x_{n(k)}) \cdot d(Tx^*, x^*)}{1 + d(x_{n(k)}, x^*)}, \\ &\quad \frac{d(Tx_{n(k)}, x^*) \cdot d(Tx^*, x_{n(k)})}{1 + d(x_{n(k)}, x^*)}, \frac{d(Tx_{n(k)}, x^*) \cdot d(Tx^*, x_{n(k)})}{1 + d(Tx_{n(k)}, Tx^*)} \Big\} \\ &= \max\Big\{ d(x_{n(k)+1}, x_{n(k)}), d(Tx^*, x^*), d(x_{n(k)}, x^*), \\ &\quad (d(x_{n(k)+1}, x^*) \cdot d(Tx^*, x_{n(k)}))^{1/2}, \frac{d(x_{n(k)+1}, x_{n(k)}) \cdot d(Tx^*, x^*)}{1 + d(x_{n(k)}, x^*)}, \\ &\quad \frac{d(x_{n(k)+1}, x^*) \cdot d(Tx^*, x_{n(k)})}{1 + d(x_{n(k)}, x^*)}, \frac{d(x_{n(k)+1}, x^*) \cdot d(Tx^*, x_{n(k)})}{1 + d(x_{n(k)+1}, Tx^*)} \Big\}. \end{aligned} \tag{2.17}$$

Letting $k \to \infty$ in (2.17), we obtain $M(x_{n(k)}, x^*) = d(x^*, Tx^*)$. Therefore, taking the limit as $k \to \infty$ in inequality (2.16), we have

$$d(x^*, Tx^*) \leq \psi(d(x^*, Tx^*)) < d(x^*, Tx^*),$$

which implies $x^* = Tx^*$, that is, x^* is a fixed point of T. This completes the proof. □

To assure the uniqueness of fixed points, we consider the following hypothesis:

(H) If for all $x, y \in F(T)$, where $F(T)$ is a set of fixed points of T, we have $\alpha(x, y) \geq 1$.

Theorem 2.4. *Adding condition* (H) *to the hypothesis of Theorem 2.2* (*resp. Theorem 2.3*), *we obtain the uniqueness of fixed points of* T.

Proof. Suppose that x^* and y^* ($x^* \neq y^*$) are two fixed points of T. Then by the hypothesis (H), $\alpha(x^*, y^*) \geq 1$. Hence, from (2.1) with $x = x^*$ and $y = y^*$ we have

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \alpha(x^*, y^*) \cdot d(Tx^*, Ty^*) \\ &\leq \psi(M(x^*, y^*)), \end{aligned}$$

where

$$\begin{aligned} M(x^*, y^*) = \max\Bigg\{ & d(Tx^*, x^*), d(Ty^*, y^*), d(x^*, y^*), \\ & (d(Tx^*, y^*) \cdot d(Ty^*, x^*))^{1/2}, \frac{d(Tx^*, x^*) \cdot d(Ty^*, y^*)}{1 + d(x^*, y^*)}, \\ & \frac{d(Tx^*, y^*) \cdot d(Ty^*, x^*)}{1 + d(x^*, y^*)}, \frac{d(Tx^*, y^*) \cdot d(Ty^*, x^*)}{1 + d(Tx^*, Ty^*)} \Bigg\} \\ = \max\Bigg\{ & d(x^*, x^*), d(y^*, y^*), d(x^*, y^*), \\ & (d(x^*, y^*) \cdot d(y^*, x^*))^{1/2}, \frac{d(x^*, x^*) \cdot d(y^*, y^*)}{1 + d(x^*, y^*)}, \\ & \frac{d(x^*, y^*) \cdot d(y^*, x^*)}{1 + d(x^*, y^*)}, \frac{d(x^*, y^*) \cdot d(y^*, x^*)}{1 + d(x^*, y^*)} \Bigg\} \\ = d(x^*, y^*). & \end{aligned}$$

Hence, we get

$$d(x^*, y^*) \leq \psi(d(x^*, y^*)) < d(x^*, y^*),$$

which is impossible, that is, $x^* = y^*$. Hence T has a unique fixed point. This completes the proof. □

References

[1] H.H. Alsulami, S. Chandok, M.A. Taoudi, I.M. Erhan, Some fixed point theorems for (α, ψ)-rational type contractive mappings, *Fixed Point Theory Appl.*, **97** (2015) 12 pages, **doi:** 10.1186/s13663-015-0332-3

[2] A.E. Bashirov, E.M. Kurplnara, A. Ozyapici, Multiplicative calculus and its applications, *J. Math. Anal. Appl.*, **337** (2008) 36-48. **doi:** 10.1016/j.jmaa.2007.03.081

[3] M. Özavsar, A.C. Çevikel, Fixed points of multiplicative contraction mappings on multiplicative metric spaces, *arXiv:* 1205.5131v1 [*math.GM*], 2012, 14 pages.

[4] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.*, **75** (2012) 2154-2165, **doi:** 10.1016/jj.na.2011.10.014