LEFT FILTERS IN TERNARY SEMIGROUPS

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Abstract: In this paper we consider the left ternary filters in a ternary semigroup. We analyze some relations between the left ternary filters and completely prime ideals of a ternary semigroup $T$.

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1. Introduction

Lee S.K. and Lee S.S. in [7], introduced the notion of a left(right) filters in a po-semigroup and gave a characterization of the left(right) filters of $T$ in terms of the right(left) prime ideals. Kwon Y.I. [4] and Kostaq H. in [5], characterized filters in ordered semigroups. In [10] Subramanyeswara Rao etc defined some relations between the filters of partially ordered $\Gamma$-semigroups $S$. In this paper, we analyze some relations between the left ternary filters and completely prime ideals of a ternary semigroup.
Let \(x, y, z\) \(\in F\). It is a contradiction. Thus \(F\) is completely prime. Therefore \(xyz\) is a contradiction. Thus possible \(\iff\) ternaryfilter completely primeideal provided \(x, y, z\) \(\in a\ \alpha\). Let \(a, b, c\ \in \alpha\). Define a relation on \(T\) as \(I_T \cup \{(x, y, z); (x, z, y)\}\). Then \(T\) is a ternarysemigroup and \(\{x, y, z\}; \{y\}\) are all ternaryfilters of \(T\).

**Theorem 2.3.** The nonempty intersection of two left ternaryfilters of a ternarysemigroup \(T\) is also a left ternaryfilter of \(T\).

**Proof.** Let \(A, B\) be two left ternaryfilters of \(T\). Let \(a, b, c \in T, \ ABC \in A \cap B.\) \(a, b, c \in T; \ ABC \in A; \) \(A\) is a left ternaryfilter of \(T \Rightarrow a \in A.\) \(a, b, c \in T; \ ABC \in B; \) \(B\) is a left ternaryfilter of \(T \Rightarrow a \in B.\) \(a \in A; \ a \in B \Rightarrow a \in A \cap B.\) \(a, b, c \in T, \ ABC \in A \cap B \Rightarrow a \in A \cap B.\) Therefore \(A \cap B\) is a left ternaryfilter of \(T\).

**Theorem 2.4.** The nonempty intersection of a family of left ternaryfilters of a ternarysemigroup \(T\) is also a left ternaryfilter.

**Proof.** Let \(\{F_{\alpha}\}_{\alpha \in \Delta}\) be a family of left ternaryfilters of \(T\) and let \(F = \bigcap_{\alpha \in \Delta} F_{\alpha}.\) Let \(a, b, c \in T, \ ABC \in F.\) Now \(ABC \in F \Rightarrow \ ABC \in \bigcap_{\alpha \in \Delta} F_{\alpha} \Rightarrow ABC \in F_{\alpha}\) for each \(\alpha \in \Delta.\) \(\ ABC \in F_{\alpha}; \ F_{\alpha}\) is a left ternaryfilter of \(T \Rightarrow a \in F_{\alpha}\) for each \(\alpha \in \Delta \Rightarrow a \in \bigcap_{\alpha \in \Delta} F_{\alpha} \Rightarrow a \in F.\) Therefore \(F\) is a left ternaryfilter of \(T.\)

**Definition 2.5.** [9] An ideal \(A\) of a ternarysemigroup \(T\) is known as a completely primeideal provided \(x, y, z \in T; \ xyz \in A \Rightarrow\) either \(x \in A\) or \(y \in A\) or \(z \in A.\)

**Theorem 2.6.** A nonempty subset \(F\) of a ternary subsemigroup \(T\) is a left ternaryfilter \(\Leftrightarrow T \setminus F\) is a completely prime rightidealeal of \(T\) or empty.

**Proof.** Suppose that \(T \setminus F \neq \emptyset.\) Let \(x \in T \setminus F;\) \(y, z \in T.\) Suppose if possible \(xyz \notin T \setminus F.\) Then \(xyz \in F.\) Since \(F\) is a left ternaryfilter, \(x \in F.\) It is a contradiction. Thus \(xyz \in T \setminus F\) and so \((T \setminus F)TT \subseteq T \setminus F.\) Therefore \(T \setminus F\) is a right ternaryideal. Now shall we prove that \(T \setminus F\) is a completelyprime. Let \(x, y, z \in T\) and \(xyz \in T \setminus F.\) Suppose if possible \(x \notin T \setminus F;\) \(y \notin T \setminus F\) and \(z \notin T \setminus F.\) Then \(x, y, z \in F.\) Since \(F\) is a ternary subsemigroup of \(T, \ xyz \in F.\) It is a contradiction. Thus \(x \in T \setminus F\) or \(y \in T \setminus F\) or \(z \in T \setminus F.\) Hence \(T \setminus F\) is completely prime. Therefore \(T \setminus F\) is a completely prime rightidealeal of \(T.\)
Contrary assume that $T \setminus F$ is a completely prime right ternary ideal of $T$ or empty. If $T \setminus F = \emptyset$ then $F = T$. Thus $F$ is a left ternary filter of $T$. Suppose that $T \setminus F$ is a completely prime right ternary ideal of $T$. Let $x, y, z \in F$. Suppose if possible $xyz \notin F$. Then $xyz \in T \setminus F$. Since $T \setminus F$ is a completely prime, $x \in T \setminus F$ or $y \in T \setminus F$ or $z \in T \setminus F$. It is a contradiction. Thus $xyz \in F$ and hence $F$ is a ternary subsemigroup of $T$. Let $x, y, z \in T$; $xyz \in F$. If $x \notin F$ then $x \in T \setminus F$. Since $T \setminus F$ is a completely prime right ideal of $T$, $xyz \in (T \setminus F)TT \subseteq T \setminus F$. It is a contradiction. Thus $x \notin F$. Therefore $F$ is a left ternary filter of $T$.

**Definition 2.7.** [10] A ternary ideal $P$ of a ternary semigroup $T$ is known as prime ideal provided $A, B$ and $C$ are ideals of $T$ and $ABC \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

**Corollary 2.8.** Let $T$ be a ternary semigroup and $F$ is a left ternary filter of $T$. Then $T \setminus F$ is a prime right ternary ideal of $T$ or empty.

**Proof.** Since $F$ is a left ternary filter; by theorem 2.6, $T \setminus F$ is a completely prime right ternary ideal of $T$ or empty.

We now introduce the notion of a $c$-system of a ternary semigroup.

**Definition 2.9.** Let $T$ be a ternary semigroup. A non empty subset $A$ of $T$ is called a $c$-system of $T$ if for each $a, b \in A$ and $c \in T$ there exist an element $x \in A$ such that $x = abc$.

A non empty subset $A$ of a ternary semigroup $T$ is a $c$-system of $T$ if for each $a, b \in A$ there exist an element $c \in A$ such that $c \in aTb$.

**Theorem 2.10.** Every ternary subsemigroup of a ternary semigroup $T$ is a $c$-system.

**Proof.** Let $S$ be a ternary subsemigroup of $T$ and $a, b, c \in T$. Since $S$ is a ternary subsemigroup of $T$, $abc \in T$. Let $x = abc$. Therefore there exist an element $x \in S$ such that $x = abc$. Therefore $S$ is a $c$-system.

**Theorem 2.11.** A ternary ideal $P$ of a ternary semigroup $T$. If $T \setminus P$ is either a $c$-system of $T$ or empty then $P$ is completely prime.

**Proof.** Assume that $T \setminus P$ is a $c$-system of $T$ or $T \setminus P$ is empty. If $T \setminus P$ is empty then $P = T$ and hence $P$ is a completely prime. Suppose that $T \setminus P$ is a $c$-system of $T$. Let $a, b, c \in T$ and $abc \in P$. Suppose if possible $a \notin P$; $b \notin P$ and $c \notin P$. Then $a \in T \setminus P$; $b \in T \setminus P$ and $c \in T \setminus P$.

Since $T \setminus P$ is a $c$-system, there exists $x \in T \setminus P$ such that $x = abc$. $x = abc \in P$. Since $P$ is a completely prime ideal of $T$, we have $x \in P$. It is a contradiction. Hence either $a \in P$ or $b \in P$ or $c \in P$. Therefore $P$ is a completely prime ternary ideal of $T$. 

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**Definition 2.12.** A ternary ideal $A$ of a ternary semigroup $T$ is called a completely semiprime ideal provided $x^3 \in A; x \in T \Rightarrow x \in A$.

**Theorem 2.13.** Every completely prime ideal of a ternary semigroup $T$ is a completely semiprime ideal of $T$.

*Proof.* Let $A$ be a completely prime ideal of a ternary semigroup $T$. Suppose that $x \in T$ and $x^3 \in A$. Since $A$ is a completely prime ideal of $T$, $x \in A$. Therefore $T$ is a completely semiprime ideal.

**Theorem 2.14.** A ternary ideal $P$ of a ternary semigroup $T$. If $T \setminus P$ is either a $c$-system of $T$ or empty then $P$ is a completely semiprime ideal.

*Proof.* By Theorem 2.11; $P$ is completely prime. By Theorem 2.13; $P$ is a completely semiprime ideal.

We now introduce the notion of a $d$-system of a ternary semigroup.

**Definition 2.15.** Let $T$ be a ternary semigroup. A non empty subset $A$ of $T$ is called a $d$-system of $T$ if for each $a \in A$ and $b \in T$, there exist an element $x \in A$ such that $x = aba$.

A non empty subset $A$ of a ternary semigroup $T$ is a $d$-system of $T \iff$ for each $a \in A$ there exist $x \in A$ such that $x \in aTa$.

**Theorem 2.16.** An ideal $P$ of a ternary semigroup $T$ is a completely semiprime $\iff T \setminus P$ is a $d$-system of $T$ or empty.

*Proof.* Assume that $P$ is completely semiprime ideal of $T$ and $T \setminus P \neq \emptyset$. Let $a \in T \setminus P$. Then $a \notin P$. Suppose if possible $x \notin aTa$ for every $x \in T \setminus P$ and $aTa \subseteq P$. $aTa \subseteq P$, Since $P$ is a completely semiprime, $a \in P$. It is a contradiction. Therefore there exist an element $x \in T \setminus P$ such that $x = aba; b \in T$. Therefore $T \setminus P$ is a $d$-system of $T$.

Contrary suppose that $T \setminus P$ is a $d$-system of $T$ or $T \setminus P$ is empty. If $T \setminus P$ is empty then $P = T$ and hence $P$ is completely semiprime. Suppose that $T \setminus P$ is a $d$-system of $T$. Let $a \in T$ and $a^3 \in P$. Suppose if possible $a \notin P$. Then $a \in T \setminus P$. Since $T \setminus P$ is a $d$-system, there exists an element $x \in T \setminus P$ such that $x = aba$ for $b \in T$. $x = aba \in aTa \subseteq P$. Therefore $x \in P$. It is a contradiction. Hence $a \in P$. Thus $P$ is a completely semiprime ideal of $T$.

We now introduce the notion of a left ternary filter of $T$ generated by $A$.

**Definition 2.17.** Let $T$ be a ternary semigroup and $A$ be a non empty subset of $T$. The smallest leftfilter of $T \subseteq A$ is known as a left ternary filter of $T$ generated by $A$ and is symbolized by $F_l(A)$.

**Theorem 2.18.** The left ternary filter of a ternary semigroup $T$ generated by a non empty subset $A$ of $T$ is the intersection of all left ternary filters of $T \subseteq A$.

*Proof.* Let $\Delta$ be the set of all left ternary filter of $T \subseteq A$. Since $T$ itself is a
left ternary filter of \( T \subseteq A, T \in \Delta \). So \( \Delta \neq \phi \).

Let \( F^* = \bigcap_{\alpha \in \Delta} F \). Since \( A \subseteq F \) for all \( F \in \Delta \), \( A \subseteq F^* \). So \( F^* \neq \phi \). By theorem 2.3, \( F^* \) is a left ternary filter of \( T \subseteq A \). Clearly \( A \subseteq K \) and \( K \) is a left ternary filter of \( T \). Therefore \( K \in \Delta \Rightarrow F^* \subseteq K \). Therefore \( F^* \) is the smallest left ternary filter of \( T \subseteq A \) and hence \( F^* \) is the left ternary filter of \( T \) generated by \( A \).

**Definition 2.19.** A left ternary filter \( F \) of a ternary semigroup \( T \) is known as a principal left filter provided \( F \) is a left ternary filter generated by \( \{a\} \) for some \( a \in T \). It is symbolized by \( F_l(a) \).

**Example 2.20.** As in the example 2.2, \( T \) is a ternary semigroup and \( F_l(a) = \{a, b, c\} \), \( F_l(b) = \{b\} \) and \( F_l(c) = \{c\} \) are all the principal left ternary filters of the ternary semigroup \( T \).

**Corollary 2.21.** Let \( T \) be a ternary semigroup and \( a \in T \). Then \( F_l(a) \) is the least left ternary filter of \( T \) containing \( \{a\} \).

For every \( a \in T \), the intersection of all left ternary filters containing \( \{a\} \) is again a left ternary filter and thus the least left ternary filter containing \( \{a\} \).

**References**


