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SCHUR-GEOMETRIC CONVEXITY OF STOLARSKY'S EXTENDED MEAN VALUES

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Abstract: In the recent years, the Schur convexity and Schur-geometrically convexity of Stolarsky's mean values have attracted the attention of a considerable number of mathematicians and researchers. In this paper, the Schur-geometric convexity of Stolarsky's extended type mean values are discussed.

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1. Introduction

The eminent researchers and scholars have explored the prominence and applications of means and their inequalities to filed of science and technology, see [1]. In previous studies literature (see [8], [9], [15]) have revealed various results on means and inequalities including contra harmonic mean.

In [19], have studied the different properties of Stolarsky (extended) two parameter mean values, which are defined as follows;

$$E_{p,q}(a,b) = \begin{cases} \left[\frac{p}{q}\frac{(a^q-b^q)}{(a^p-b^p)}\right]^{\frac{1}{q-p}} & pq(q-p)(a-b) \neq 0; \\ e^{\left(\frac{-1}{q}+\frac{(a^q lna - b^q lnb)}{(a^q-b^q)}\right)} & p = q \neq 0; \\ \left(\frac{(a^q-b^q)}{q(lna-lnb)}\right)^{\frac{1}{q}} & p = 0, q \neq 0; \\ \sqrt{ab} & p = q = 0; \\ a & a = b > 0. \end{cases} \tag{1}$$

The essential for this paper recalls some of the known means. The Arithmetic mean in weighted form is

$$A_{r,s}(a,b) = ra + sb = A(a,b;r,s);$$

such that $a, b > 0$ and $r + s = 1$, where r and s are the weights.

The Stolarsky means $E_{p,q}(a,b)$ are C^∞ functions on the domain (p,q,a,b) : $p,\ q \in R; a, b > 0$. Obviously, Stolarsky means $E_{p,q}(a,b)$ are symmetric with respect to a, b and p, q.

Most of the classical two variable means are special cases of $E_{p,q}(a,b)$, Stolarsky mean.

For example:

$$E_{1,2}(a,b) = \frac{a+b}{2} \qquad \text{is the Arithmetic mean.}$$

$$E_{0,0}(a,b) = \sqrt{ab} \qquad \text{is the Geometric mean.}$$

$$E_{-1,-2}(a,b) = \frac{2ab}{a+b} \qquad \text{is the Harmonic mean.}$$

$$E_{0,1}(a,b) = \frac{a-b}{lna-lnb} \qquad \text{is the Logarithmic mean.}$$

$$E_{1,1}(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}} \qquad \text{is the Identric mean.}$$

$$E_{r,2r}(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} \qquad \text{is the } r^{th} \text{ Power mean.}$$

The basic properties of Stolarsky means, as well as their comparison theorems, log-convexities, and inequalities are studied in papers (see [6], [21], [22]).

In recent years, a considerable number of mathematicians attention have been attracted by the Schur convexity and Schur geometrically convexity of various means (see [2]-[4], [7], [11]-[14], [16]). Qi [17] first proved that the Stolarsky means $E_{p,q}(a,b)$ are Schur convex on $(-\infty, 0] \times (-\infty, 0]$ and Schur concave on $[0,\infty) \times [0,\infty)$ with respect to (p,q) for fixed $a, b > 0$ with $a \neq b$. Yang [25] improved Qi's result and proved that Stolarsky means $E_{p,q}(a,b)$ are Schur convex with respect to (p,q) for fixed $a, b > 0$ with $a \neq b$ if and only if $p+q < 0$ and Schur concave if and only if $p+q > 0$.

Qi et al. [16] tried to obtain the Schur convexity of $E_{p,q}(a,b)$ with respect to (a,b) for fixed (p,q). Shi et al. [18] worked on the same and obtained a sufficient condition for the Schur convexity of $E_{p,q}(a,b)$ with respect to (a,b). Chu and Zhang [3] improved Shi's results and gave a necessary and sufficient condition. This perfectly solved the Schur convexity of Stolarsky means with respect to (a,b).

For the Schur geometrically convexity, Chu and Zhang [2] proved that Stolarsky means $E_{p,q}(a,b)$ are Schur geometrically convex with respect to $(a,b) \in (0,\infty)\times(0,\infty)$ if $p+q \geq 0$ and Schur geometrically concave if $p+q \leq 0$. Li et al. [7] also investigated the Schur geometrically convexity of generalized exponent mean $I_p(a,b)$.

The purpose of this paper is to investigate Schur-geometric convexity of Stolarsky's extended type mean values $N_{p,q}(a,b;r,s)$.

In [1], the weighted contra harmonic mean is defined on the basis of proportions by,

$$C_{r,s}(a,b) = \frac{ra^2 + sb^2}{ra + sb} = C(a,b;r,s);$$

where $a, b > 0$ and r, s are weights such that $r + s = 1$.

This work has led to a new family introduction of Stolarsky's extended type mean values in weighted forms in *two* and n variables.

For two variables $a, b > 0$, $p, q \in R$ and r, s are weights such that $r + s = 1$. Consider a new mean in the following form;

$$N_{p,q}(a, b; r, s) = \left[\frac{p^2}{q^2}\frac{C(a^q, b^q; r, s) - A(a^q, b^q; r, s)}{C(a^p, b^p; r, s) - A(a^p, b^p; r, s)}\right]^{\frac{1}{q-p}}.$$

Which is equivalently,

$$N_{p,q}(a, b; r, s) = \left[\frac{p^2}{q^2}\left(\frac{ra^p + sb^p}{ra^q + sb^q}\right)\left(\frac{ra^{2q} + sb^{2q} - (ra^q + sb^q)^2}{ra^{2p} + sb^{2p} - (ra^p + sb^p)^2}\right)\right]^{\frac{1}{q-p}}.$$

Which is equivalently,

$$N_{p,q}(a, b; r, s) = \left[\frac{p^2}{q^2}\left(\frac{ra^p + sb^p}{ra^q + sb^q}\right)\left(\frac{a^q - b^q}{a^p - b^p}\right)^2\right]^{\frac{1}{q-p}}.$$

In [23], author introduced the homogeneous function with two parameters p and q by,

$$H_{f,p,q}(a, b) = \left[\frac{f(a^q, b^q)}{f(a^p, b^p))}\right]^{\frac{1}{q-p}} \qquad pq(q-p) \neq 0$$

and studied its monotonicity and deduced some inequalities involving means, where f is a homogeneous function for a and b.

In particular, $f = A$ is the Arithmetic mean of a and b.

$$H_{A,p,q}(a, b) = \begin{cases} \left[\frac{(a^q+b^q)}{(a^p+b^p)}\right]^{\frac{1}{q-p}} & pq(q-p) \neq 0; \\ G_{A,q}(a, b) = a^{\frac{a^q}{a^q+b^q}} b^{\frac{b^q}{a^q+b^q}} & p = q \neq 0; \\ \sqrt{ab} & p = q = 0; \\ a & a = b > 0. \end{cases} \qquad (2)$$

Here $G_{A,q}(a, b) = Z_q(a, b) = Z^{\frac{1}{q}}(a^q, b^q) = Z_q$, where $Z(a, b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$ is named power-exponential mean between two positive numbers a and b.

In weighted form,$f = A_{r,s}(a,b) = ra + sb.$, and

$$H_{A,p,q}(a,b) = \begin{cases} \left[\frac{(ra^q+sb^q)}{(ra^p+sb^p)}\right]^{\frac{1}{q-p}} & p \neq q; \\ G_{A,q}(a,b) = a^{\frac{ra^q}{ra^q+qb^q}} b^{\frac{sb^q}{ra^q+sb^q}} & p = q \neq 0; \\ a^r b^s & p = q = 0; \\ a & a = b > 0;. \end{cases} \tag{3}$$

In [20], author has introduced and studied the various properties and log-convexity results of the class W of weighted two parameter means which are given by;

$$W_{p,q}(a,b;r,s)$$
$$= \begin{cases} \left[\frac{p^2}{q^2}\left(\frac{ra^q+sb^q-a^{rq}b^{sq}}{ra^p+sb^p-a^{rp}b^{sp}}\right)\right]^{\frac{1}{q-p}} & pq(q-p)(a-b) \neq 0; \\ \left[\frac{2}{ln^2(a/b)}\left(\frac{ra^q+sb^q-a^{rq}b^{sq}}{rsq^2}\right)\right]^{\frac{1}{q}} & q(a-b) \neq 0, p = 0; \\ exp\left(\frac{-2}{q} + \frac{ra^q lna + sb^q lnb - (rlna+slnb)a^{rq}b^{sq}}{ra^q+sb^q-a^{rq}b^{sq}}\right) & p = q, q \neq 0; \\ a^{(r+1)/3}b^{(s+1)/3} & a \neq b, p = q = 0; \\ a & a = b > 0. \end{cases} \tag{4}$$

The above definitions lead to express the mean values $N_{p,q}(a,b;r,s)$ in the following form;

$$N_{p,q}(a,b;r,s) = \left[\frac{ra^p+sb^p}{ra^q+sb^q}\right]^{\frac{1}{q-p}} \left[\left(\frac{p}{q}\frac{a^q-b^q}{a^p-b^p}\right)^{\frac{1}{q-p}}\right]^2.$$

which is equivalently,

$$N_{p,q}(a,b;r,s) = \left[\frac{f(a^p,b^p;r,s)}{f(a^q,b^q;r,s))}\right]^{\frac{1}{q-p}} E^2_{p,q}(a,b)$$

or

$$N_{p,q}(a,b;r,s) = H_{f=A(a,b;r,s)}(a,b;p,q)E^2_{p,q}(a,b) \tag{5}$$

Here $f = A(a,b;r,s)$ is Arithmetic mean in weighted form.

The various properties and identities concerning to $N_{p,q}(a,b;r,s)$ are also studied by K. M. Nagaraja and et.al.

The laborious calculations give the following different cases of the mean value $N_{p,q}(a,b;r,s)$.

$$N_{p,q}(a,b;r,s) = \begin{cases} \left[\frac{p^2}{q^2}\left(\frac{ra^p+sb^p}{ra^q+sb^q}\right)\left(\frac{a^q-b^q}{a^p-b^p}\right)^2\right]^{\frac{1}{q-p}} & pq(q-p)(a-b)\neq 0; \\ \left[\frac{1}{ln^2(a/b)}\left(\frac{1}{ra^q+sb^q}\right)\left(\frac{a^q-b^q}{q}\right)^2\right]^{\frac{1}{q}} & s(a-b)\neq 0, r=0; \\ exp\left(\frac{-2}{q}-\frac{ra^q lna+sb^q lnb}{ra^q+sb^q}+2\frac{a^q lna-b^q lnb}{a^q-b^q}\right) & p=q, q\neq 0; \\ a^{1-r}b^{1-s} & a\neq b, p=q=0; \\ a & a=b>0. \end{cases} \tag{6}$$

2. Definition and Properties

In 1923, Schur introduced Schur convexity [10], and it has many important applications in analytic inequalities [5], linear regression, graphs and matrices, combinatorial optimization, information theoretic topics, Gamma functions, stochastic orderings, reliability, and other related fields. Some of the definitions are recall for convenience of readers.

Definition 2.1. *[10] Let $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)$* $\in R^n$

1. *x is majorized by y (in symbol $x\prec y$) If $\sum_{i=1}^k x_{[i]}\leq\sum_{i=1}^k y_{[i]}$, and $\sum_{i=1}^n x_{[i]}=\sum_{i=1}^n y_{[i]}$, where $x_{[1]}\geq,...,\geq x_{[n]}$ and $y_{[1]}\geq,...,\geq y_{[n]}$ are rearrangements of x and y in descending order.*

2. *$x\geq y$ means $x_i\geq y_i$ for all $i=1,2,....,n$. Let $\Omega\in R^n(n\geq 2)$. The function $\varphi:\Omega\to R$ is said to be increasing if $x\geq y$ implies $\varphi(x)\geq\varphi(y)$. φ is said to be decreasing if and only if $-\varphi$ is increasing.*

3. *$\Omega\subseteq R^n$ is called a convex set if $(\alpha x_1+\beta y_1,....,\alpha x_n+\beta y_n)\in\Omega$ for every x and $y\in\Omega$ where $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.*

4. *Let $\Omega\subseteq R^n$. The function $\varphi:\Omega\to R$ be said to be a Schur convex function on Ω if $x\prec y$ on Ω implies $\varphi(x)\leq\varphi(y)$. φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex.*

Definition 2.2. *[24] Let* $x = (x_1, x_2, ..., x_n)$ *and* $y = (y_1, y_2, ..., y_n) \in R^n_+$
$\Omega \subseteq R^n$ *is called geometrically convex set if* $(x_1^\alpha y_1^\beta, ..., x_1^\alpha y_1^\beta) \in \Omega$ *for all* x *and* $y \in \Omega$ *where* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta = 1$.
Let $\Omega \subseteq R^n_+$. *The function* $\varphi : \Omega \to R_+$ *is said to be Schur geometrically convex function on* Ω *if* $(lnx_1, ..., lnx_n) \prec (lny_1, ..., lny_n)$ *on* Ω *implies* $\varphi(x) \leq \varphi(y)$. φ *is said to be a Schur geometrically concave function on* Ω *if and only if* $-\varphi$ *is Schur geometrically convex.*

Definition 2.3. $\Omega \subseteq R^n$ *is called symmetric set if* $x \in \Omega$ *implies* $Px \in \Omega$ *for every* $n \times n$ *permutation matrix* P.
The function $\varphi : \Omega \to R$ *is called symmetric if for every permutation matrix* P, $\varphi(Px) = \varphi(x)$ *for all* $x \in \Omega$.

Lemma 2.1. *[25] Let* $\Omega \subseteq R^n$ *be symmetric with non empty interior geometrically convex set* Ω^0 *and let* $\varphi : \Omega \to R_+$ *be continuous on* Ω *and differentiable in* Ω^0. *Then* φ *is Schur-geometrically convex (Schur-geometrically concave) on* Ω *if and only if* φ *is symmetric on* Ω *and*

$$(lnx_1 - lnx_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0). \tag{7}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$.

3. Main Results

In this section, we shall prove some of the lemmas required for proving main theorem.

Lemma 3.1. *Stolarsky's extended family type mean* $N_{p,q}(a, b; r, s)$ *are Schur-geometric convex or Schur-geometric concave with respect to* $(a, b) \in (0, \infty) \times (0, \infty)$ *if and only if* $g(t) \geq 0$ *or* $g(t) \leq 0$ *for all* $t > 0$, *where*

$$g(t) = g_{p,q}(t)$$

$$= \begin{cases} \left[\frac{(p-q)Sinh2(p+q)t-(p+q)Sinh2(p-q)t+6pSinh2qt-6qSinh2pt}{pq(p-q)}\right] & pq(p-q) \neq 0; \\ \left[\frac{2Sinh2qt-4qtCosh2qt-12qt+6Sinh2qt}{-q^2}\right] & p = 0, q \neq 0; \\ \left[\frac{2Sinh2pt-4ptCosh2pt-12pt+6Sinh2pt}{-p^2}\right] & p \neq 0, q = 0; \\ \left[\frac{Sinh4qt-4qt+6Sinh2qt-12qtCosh2qt}{q^2}\right] & p = q \neq 0; \\ 0 & p = q = 0; \end{cases} \tag{8}$$

Proof. Let Stolarsky's extended family type means $N = N_{p,q}(a,b;r,s)$ for $pq(q-p) \neq 0$ is defined as;

$$N = N_{p,q}(a,b;r,s) = \left[\frac{p^2}{q^2}\left(\frac{ra^p+sb^p}{ra^q+sb^q}\right)\left(\frac{a^q-b^q}{a^p-b^p}\right)^2\right]^{\frac{1}{q-p}} \tag{9}$$

Let $r = s$.

Taking logarithm on both sides, differentiate partially with respect to a, multiply by a and on rearranging leads to

$$a\frac{\partial N}{\partial a} = \frac{N}{p-q}\left[\frac{qa^q}{a^q+b^q} - \frac{pa^p}{a^p+b^p} + 2\frac{pa^p}{a^p-b^p} - 2\frac{qa^q}{a^q-b^q}\right]. \tag{10}$$

Similarly,

$$b\frac{\partial N}{\partial b} = \frac{N}{p-q}\left[\frac{qb^q}{a^q+b^q} - \frac{pb^p}{a^p+b^p} + 2\frac{pb^p}{a^p-b^p} - 2\frac{qb^q}{a^q-b^q}\right] \tag{11}$$

then,

$$(lna - lnb)\left(a\frac{\partial N}{\partial a} - b\frac{\partial N}{\partial b}\right) = \frac{(lna-lnb)N}{p-q}\left[\Delta\right] \tag{12}$$

where,

$$\Delta = \left[\frac{q(a^q-b^q)}{a^q+b^q} - \frac{p(a^p-b^p)}{a^p+b^p} + 2\frac{p(a^p+b^p)}{a^p-b^p} - 2\frac{q(a^q+b^q)}{a^q-b^q}\right].$$

Substituting $ln\sqrt{a/b} = t$ and using $sinhx = \frac{1}{2}\left(e^x - e^{-x}\right)$; $Coshx = \frac{1}{2}\left(e^x + e^{-x}\right)$, we have

$$\Delta = \sqrt{ab}\left[q\frac{Sinhqt}{Coshqt} - p\frac{Sinhpt}{Coshpt} + 2p\frac{Coshpt}{Sinhpt} - 2q\frac{Coshqt}{Sinhqt}\right].$$

For $pq(p-q) \neq 0$, using the product into sum formula for hyperbolic functions leads to

$$(lna - lnb)\left(a\frac{\partial N}{\partial a} - b\frac{\partial N}{\partial b}\right) = \frac{(pq)N(a-b)\sqrt{ab}}{SinhptSinhqtCoshptCoshqt}\left[g_{p,q}(t)\right], \tag{13}$$

where

$g_{p,q}(t) =$

$$\left[\frac{(p-q)Sinhp2(p+q)t-(p+q)Sinhp2(p-q)t+6pSinh2qt-6qSinh2pt}{pq(p-q)}\right]. \tag{14}$$

In case of $p \neq q = 0$. Since $N_{p,q} \in C^{\infty}$, we have

$$\frac{\partial N_{p,0}}{\partial a} = \lim_{q\to 0}\frac{\partial N_{p,q}}{\partial a}, \qquad \frac{\partial N_{p,0}}{\partial b} = \lim_{q\to 0}\frac{\partial N_{p,q}}{\partial b},$$

$$\frac{\partial N_{p,p}}{\partial a} = \lim_{q\to p}\frac{\partial N_{p,p}}{\partial a}, \qquad \frac{\partial N_{p,p}}{\partial b} = \lim_{q\to p}\frac{\partial N_{p,p}}{\partial b},$$

$$\frac{\partial N_{0,0}}{\partial a} = \lim_{p\to 0}\frac{\partial N_{p,q}}{\partial a}, \qquad \frac{\partial N_{0,0}}{\partial b} = \lim_{p\to 0}\frac{\partial N_{p,q}}{\partial b}.$$

Thus

$$(lna-lnb)\left(a\frac{\partial N_{p,0}}{\partial a}-b\frac{\partial N_{p,0}}{\partial b}\right) = \lim_{q\to 0}\left[(lna-lnb)\left(a\frac{\partial N_{p,0}}{\partial a}-b\frac{\partial N_{p,0}}{\partial b}\right)\right]$$

$$= \lim_{q\to 0}\left(\frac{(pq)N_{p,q}(a-b)\sqrt{ab}}{SinhptSinhqtCoshptCoshqt}\left[g_{p,q}(t)\right]\right) = \frac{(p)N_{p,0}(a-b)\sqrt{ab}}{tSinhptCoshpt}\left[g_{p,0}(t)\right].$$

Likewise for $q \neq p = 0$

$$(lna-lnb)\left(a\frac{\partial N_{0,q}}{\partial a}-b\frac{\partial N_{0,q}}{\partial b}\right) = \lim_{p\to 0}\left[(lna-lnb)\left(a\frac{\partial N_{0,q}}{\partial a}-b\frac{\partial N_{0,q}}{\partial b}\right)\right]$$

$$= \lim_{p\to 0}\left(\frac{(pq)N_{p,q}(a-b)\sqrt{ab}}{SinhptSinhqtCoshptCoshqt}\left[g_{p,q}(t)\right]\right) = \frac{(q)N_{0,q}(a-b)\sqrt{ab}}{tSinhqtCoshqt}\left[g_{0,q}(t)\right].$$

For $q = p \neq 0$:

$$(lna-lnb)\left(a\frac{\partial N_{p,p}}{\partial a}-b\frac{\partial N_{p,p}}{\partial b}\right) = \lim_{q\to p}\left(\frac{(pq)N_{p,q}(a-b)\sqrt{ab}}{Sinh^2ptCosh^2pt}\left[g_{p,q}(t)\right]\right)$$

$$= \frac{p^2N_{p,p}(a-b)\sqrt{ab}}{SinhptCoshpt}\left[g_{p,p}(t)\right].$$

For $q = p = 0$:

$$(lna-lnb)\left(a\frac{\partial N_{0,0}}{\partial a}-b\frac{\partial N_{0,0}}{\partial b}\right) = \lim_{p\to 0}\left(a\frac{(pq)N_{p,p}(a-b)\sqrt{ab}}{Sinh^2ptCosh^2pt}\left[g_{p,p}(t)\right]\right)$$

$$= \frac{N_{0,0}(a-b)\sqrt{ab}}{t^2}\left[g_{0,0}(t)\right].$$

By summarizing all cases above yield;

$$(lna - lnb)\left(a\frac{\partial N}{\partial a} - b\frac{\partial N}{\partial b}\right)$$
$$= \begin{cases} \frac{pqN(a-b)\sqrt{ab}}{SinhptSinhqtCoshptCoshqt}\left[g_{p,q}(t)\right] & pq(p-q)\neq 0; \\ \frac{(q)N_{0,q}(a-b)\sqrt{ab}}{2tSinhqtCoshqt}\left[g_{0,q}(t)\right] & p=0, q\neq 0; \\ \frac{pN_{p,0}(a-b)\sqrt{ab}}{2tSinhptCoshpt}\left[g_{p,0}(t)\right] & p\neq 0, q=0; \\ \frac{2p^2N_{p,p}(a-b)\sqrt{ab}}{2SinhptCoshpt}\left[g_{p,p}(t)\right] & p=q\neq 0; \\ \frac{N_{0,0}(a-b)\sqrt{ab}}{2t^2}\left[g_{0,0}(t)\right] & p=q=0. \end{cases} \quad (15)$$

Since $(lna - lnb)\left(a\frac{\partial N}{\partial a} - b\frac{\partial N}{\partial b}\right)$ is symmetric with respect to a and b, without loss of generality we assume $a > b$, then $t = ln\sqrt{a/b} > 0$. It is easy to verify that $\frac{N(a-b)\sqrt{ab}}{2} > 0$, $\frac{p}{sinhpt}$, $\frac{q}{sinhqt} > 0$ if $pq \neq 0$ for $t > 0$. Thus by Lemma 2.1 Stolarsky means $N_{p,q}(a,b)$ are Schur-geometric convex (Schur-geometric concave) with respect to $(a,b) \in (0,\infty)\times(0,\infty)$ if and only if $(lna - lnb)\left(a\frac{\partial N}{\partial a} - b\frac{\partial N}{\partial b}\right)(\geq)(\leq)0$, if and only if $g(t) = g_{p,q}(t)(\geq)(\leq)0$ for all $t > 0$.

This completes the proof of Lemma 3.1. □

Lemma 3.2. *The function $g(t) = g_{p,q}(t)$ defined by (3.1) and $g^{\prime}(t) = \frac{\partial g_{p,q}(t)}{\partial t}$ both are symmetric with respect to p and q, and both continuous with respect to p and q on $R \times R$.*

Proof. It is easy to check that $g_{p,q}(t)$ and $\frac{\partial g_{p,q}(t)}{\partial t}$ are symmetric with respect to p and q. That is $\frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{q,p}(t)}{\partial t}$.

By Lemma 3.1, we note that $g(t) = g_{p,q}(t)$ is continuous with respect to $(p,\ q)$ on $R \times R$. Finally, we prove that $g^{\prime}(t) = \frac{\partial g_{p,q}(t)}{\partial t}$ is also continuous with respect to $(p,\ q)$ on $R \times R$.

Simple calculations yield the following cases.

Case (i) For $pq(p-q) \neq 0$

$$g^{\prime}(t) = \frac{\partial g_{p,q}(t)}{\partial t}$$

$$= \frac{2(p-q)(p+q)(Cosh2(p+q)t - Cosh2(p-q)t) + 12pq(Cosh2qt - Cosh2pt)}{pq(p-q)}.$$

Case (ii) For $p = 0, q \neq 0$

$$g^{|}(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{4qCosh4qt - 4q - 24q^2 tSinh2qt}{q^2}$$

Case (iii) For $q = 0, p \neq 0$

$$g^{|}(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{4pCosh4pt - 4p - 24p^2 tSinh2pt}{p^2}.$$

Case (iv) For $p = q \neq 0$

$$g^{|}(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{-8q^2 tSinh2qt - 12q + 12qCosh2qt}{-q^2}.$$

Case (v) For $p = q = 0$

$$g^{|}(t) = \frac{\partial g_{p,q}(t)}{\partial t} = 0.$$

It is obvious that $\frac{\partial g_{p,q}(t)}{\partial t}$ is continuous with respect to (p, q) on $R \times R$, with $pq(p-q) \neq 0$. Further, we have

$$\lim_{q \to 0} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,0}(t)}{\partial t}; \qquad \lim_{p \to 0} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{0,q}(t)}{\partial t};$$

$$\lim_{q \to p} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,p}(t)}{\partial t}; \qquad \lim_{p \to 0} \frac{\partial g_{p,p}(t)}{\partial t} = \frac{\partial g_{0,0}(t)}{\partial t}.$$

Hence, $g^{|}(t)$ is continuous for all (p, q)$\in R \times R$. □

This completes the proof of Lemma 3.2.

Lemma 3.3. $\lim\limits_{t \to 0, t > 0} \frac{g_{p,q}(t)}{t^3} = \frac{-8}{3}(p+q)$.

Proof. It is easy to check that firs1t and second derivatives of $g(t) = 0$, at $t = 0$. In the case of $pq(p-q) \neq 0$. Applying L-Hospital's rule(three times)yield;

$$\lim_{t\to 0, t>0} \frac{g_{p,q}(t)}{t^3} = \lim_{t\to 0, t>0} \frac{g_{p,q}^{'}(t)}{3t^2} = ... = \lim_{t\to 0, t>0} \frac{g_{p,q}^{'''}(t)}{6} = \frac{-8(p+q)}{3}.$$

Similarly, $p = 0, q \neq 0$

$$\lim_{t\to 0, t>0} \frac{g_{0,q}(t)}{t^3} = \lim_{t\to 0, t>0} \frac{g_{0,q}^{'}(t)}{3t^2} = ... = \lim_{t\to 0, t>0} \frac{g_{0,q}^{'''}(t)}{6} = \frac{-8q}{3}$$

for $q = 0, p \neq 0$

$$\lim_{t\to 0, t>0} \frac{g_{p,0}(t)}{t^3} = \lim_{t\to 0, t>0} \frac{g_{p,0}^{'}(t)}{3t^2} = ... = \lim_{t\to 0, t>0} \frac{g_{p,0}^{'''}(t)}{6} = \frac{-8p}{3}$$

for $p = q \neq 0$

$$\lim_{t\to 0, t>0} \frac{g_{p,p}(t)}{t^3} = \lim_{t\to 0, t>0} \frac{g_{p,p}^{'}(t)}{3t^2} = ... = \lim_{t\to 0, t>0} \frac{g_{p,p}^{'''}(t)}{6} = \frac{-16p}{3}.$$

for $p = q = 0$

$$\lim_{t\to 0, t>0} \frac{g_{0,0}(t)}{t^3} = \lim_{t\to 0, t>0} \frac{g_{0,0}^{'}(t)}{3t^2} = ... = \lim_{t\to 0, t>0} \frac{g_{0,0}^{'''}(t)}{6} = 0.$$ □

By summarizing the above cases, we conclude the proof of Lemma 3.3.

The proof of our main result stated below follows from the above lemmas:

Theorem 3.1. *For fixed* $(p, q) \in R \times R$ *and* $r = s$,

1. *Stolarsky's extended family type means* $N_{p,q}(a, b; r, s)$ *are Schur-geometric convex with respect to* (a, b) *if* $\frac{-8(p+q)}{3} \leq 0$.

2. *Stolarsky's extended family type means* $N_{p,q}(a, b; r, s)$ *are Schur-geometric concave if* $\frac{-8(p+q)}{3} \geq 0$.

4. Conclusion

In the case when $r \neq s$, the convexity and concavity of Stolarsky's extended family type means $N_{p,q}(a, b; r, s)$ is an open problem.

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