ONE-TWELVETH STEP CONTINUOUS BLOCK METHOD FOR THE SOLUTION OF $y''' = f(x, y, y', y'')$

E.A. Areo¹ §, M.T. Omojola²

^{1,2}Department of Mathematical Sciences
Federal University of Technology
Akure, NIGERIA

Abstract: We consider direct solution to third order ordinary differential equations in this paper. Collocation and interpolation approach was adopted to generate a continuous hybrid multistep method. We adopted the use of power series as a basis function for approximate solution. We evaluated at off-grid points to get a continuous hybrid multistep method. Block method was later adopted to generate the independent solution at selected off-grid points. The properties of the block viz: order, zero stability and region of absolute stability are investigated. Our method was tested on third order ordinary differential equation and found to give better result when compared with existing methods.

Key Words: block method, approximate solutions, zero-stability, region of absolute stability, continuous hybrid multistep method

1. Introduction

In this paper, we considered the method of approximate solution of the general second order initial value problem of the form

We consider the solution to general third order initial value problem of the form

$$y''' = f(x, y, y', y''), \quad y^k(x_0) = y_0^k, \quad k = 0, 1, 2, \quad y \in R, \quad (1)$$

where x_0 is the initial point and f is continuous within the interval of integration

Received: November 3, 2015

Revised: April 26, 2017

Published: April 28, 2017

© 2017 Academic Publications, Ltd.

url: www.acadpubl.eu

§Correspondence author

and satisfies the existence and uniqueness condition.

Many real life problems in sciences, engineering, biology and social sciences are model of third order ordinary differential equations. Some of these models do not always have theoretical solutions, thus numerical methods are often employed to solve them. Researchers in most cases always use method of reduction of higher order ODEs into system of first order ODEs to solve (1). This technique though quite good, is bedeviled with many problems such tediousness, complexity of the method, waste of time, and the need for large computer storage memories because of too many auxiliary functions, etc. Conventionally, higher order ordinary differential equations are solved directly by the predictor-corrector method where separate predictors are developed to implement the corrector and Taylor series expansion adopted to provide the starting values. Predictor-corrector methods are extensively studied by [1-12]. These authors proposed linear multistep methods with continuous coefficient, which have advantage of evaluation at all points within the grid over the proposed method in [4]. The major setbacks of predictor-corrector method are extensively discussed by [6]. Many research works have been grounded on Linear Multistep Method (LMM) and Linear Multistep Hybrid Method (LMHM) where Collocation and Interpolation are done at selected grid and off-grid points. There was then need to interpolate and collocate at grid and off-grid points with evaluation to be done at off-grid points only since this gives higher accuracy and large region of absolute stability. We apply the One-Twelfth Step Method introduced for the direct solution of Third Order Differential Equations with x_n being the only grid point.

2. Mathematical Formulation of the Method

We consider the simple power series as a basis function for approximation:

$$Y(x) = \sum_{j=0}^{r+s-1} a_j \phi_j(x), \quad (2)$$

where $\phi_j(x) = x^j$ and $x \in [a, b]$, a_j 's are coefficients to be determined and is a polynomial of degree $r + s - 1$. We construct a K-step Collocation Method by imposing the following conditions on (2)

$$Y(x_{n+j}) = y_{n+j}, \quad j = 0, 1, 2, \dots, r-1, \quad (3)$$

$$Y'(x_{n+j}) = f_{n+j}, \quad j = 0, 1, 2, \dots, s-1. \quad (4)$$

Substituting (1) into (5) gives

$$f(x, y, y, y) = \sum j(j-1)(j-2)a_j x^{j-3}. \quad (5)$$

We shall consider the grid point of step length $\frac{1}{12}$ with constant step-size (h), where $h = x_{n+i} - x_i, i = 0, 1, 2$ and off-grid points at $x_{n+\frac{1}{48}}, x_{n+\frac{1}{24}}, x_{n+\frac{1}{16}}$.

Interpolating (3) at $x = x_n, x_{n+\frac{1}{48}}$ and $x_{n+\frac{1}{24}}$ and collocating (6) at $x = x_n, x_{n+\frac{1}{48}}, x_{n+\frac{1}{24}}, x_{n+\frac{1}{16}}, x_{n+\frac{1}{12}}$ to give a system of non-linear equation of the form

$$AX = B. \quad (6)$$

Here

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]^T,$$

$$B = [y_n, y_{n+\frac{1}{48}}, y_{n+\frac{1}{24}}, f_n, f_{n+\frac{1}{48}}, f_{n+\frac{1}{24}}, f_{n+\frac{1}{16}}, f_{n+\frac{1}{12}}]^T,$$

and

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+\frac{1}{48}} & x_{n+\frac{1}{48}}^2 & x_{n+\frac{1}{48}}^3 & x_{n+\frac{1}{48}}^4 & x_{n+\frac{1}{48}}^5 & x_{n+\frac{1}{48}}^6 & x_{n+\frac{1}{48}}^7 \\ 1 & x_{n+\frac{1}{24}} & x_{n+\frac{1}{24}}^2 & x_{n+\frac{1}{24}}^3 & x_{n+\frac{1}{24}}^4 & x_{n+\frac{1}{24}}^5 & x_{n+\frac{1}{24}}^6 & x_{n+\frac{1}{24}}^7 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{48}} & 20x_{n+\frac{1}{48}}^2 & 120x_{n+\frac{1}{48}}^3 & 210x_{n+\frac{1}{48}}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{24}} & 60x_{n+\frac{1}{24}}^2 & 120x_{n+\frac{1}{24}}^3 & 210x_{n+\frac{1}{24}}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{16}} & 60x_{n+\frac{1}{16}}^2 & 120x_{n+\frac{1}{16}}^3 & 210x_{n+\frac{1}{16}}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{12}} & 60x_{n+\frac{1}{12}}^2 & 120x_{n+\frac{1}{12}}^3 & 210x_{n+\frac{1}{12}}^4 \end{bmatrix}.$$

Solving (6) for the a'_j s and substituting back into (3) and after much algebraic

simplification, we obtained

$$\begin{aligned}
a_0 &= y_n \\
a_1 &= \frac{793}{5806080} h^3 f_{n+\frac{1}{48}} - \frac{19}{645120} h^3 f_{n+\frac{1}{24}} + \frac{79}{5806080} h^3 f_{n+\frac{1}{16}} \\
&\quad - \frac{29}{11612160} h^3 f_{n+\frac{1}{12}} + \frac{307}{11612160} h^3 f_n + 96 y_{n+\frac{1}{48}} - 24 y_{n+\frac{1}{24}} - 72 y_n \\
a_2 &= -\frac{101}{11520} h^3 f_{n+\frac{1}{48}} + \frac{31}{11520} h^3 f_{n+\frac{1}{24}} - \frac{41}{34560} h^3 f_{n+\frac{1}{16}} + \frac{1}{4608} h^3 f_{n+\frac{1}{12}} \\
&\quad - \frac{233}{69120} h^3 f_n - 2304 y_{n+\frac{1}{48}} + 1152 y_{n+\frac{1}{24}} + 1152 y_n \\
a_3 &= \frac{1}{6} h^3 f_n \\
a_4 &= 8 h^3 f_{n+\frac{1}{48}} - 6 h^3 f_{n+\frac{1}{24}} + \frac{8}{3} h^3 f_{n+\frac{1}{16}} - \frac{1}{2} h^3 f_{n+\frac{1}{12}} - \frac{25}{6} h^3 f_n \\
a_5 &= -\frac{832}{5} h^3 f_{n+\frac{1}{48}} + \frac{912}{5} h^3 f_{n+\frac{1}{24}} - \frac{448}{5} h^3 f_{n+\frac{1}{16}} + \frac{88}{5} h^3 f_{n+\frac{1}{12}} + 56 h^3 f_n \\
a_6 &= \frac{6912}{5} h^3 f_{n+\frac{1}{48}} - \frac{9216}{5} h^3 f_{n+\frac{1}{24}} + \frac{5376}{5} h^3 f_{n+\frac{1}{16}} - \frac{1152}{5} h^3 f_{n+\frac{1}{12}} \\
&\quad - 384 h^3 f_n \\
a_7 &= -\frac{147456}{35} h^3 f_{n+\frac{1}{48}} + \frac{221184}{35} h^3 f_{n+\frac{1}{24}} - \frac{147456}{35} h^3 f_{n+\frac{1}{16}} \\
&\quad + \frac{36864}{35} h^3 f_{n+\frac{1}{12}} + \frac{36864}{35} h^3 f_n
\end{aligned} \tag{7}$$

where

$$q = \frac{x - x_n}{h}, \frac{dq}{dx} = \frac{1}{h}. \tag{8}$$

We hereby present the continuous schemes

$$\begin{aligned}
y_{n+\frac{1}{16}} &= y_n - 3 y_{n+\frac{1}{48}} + 3 y_{n+\frac{1}{24}} + \frac{1}{26542080} h^3 f_{n+\frac{1}{12}} - \frac{1}{6635520} h^3 f_{n+\frac{1}{16}} \\
&\quad + \frac{7}{1474560} h^3 f_{n+\frac{1}{24}} + \frac{29}{6635520} h^3 f_{n+\frac{1}{48}} + \frac{1}{26542080} h^3 f_n \\
y_{n+\frac{1}{12}} &= 3 y_n - 8 y_{n+\frac{1}{48}} + 6 y_{n+\frac{1}{24}} + \frac{1}{6635520} h^3 f_{n+\frac{1}{12}} + \frac{13}{3317760} h^3 f_{n+\frac{1}{16}} \\
&\quad + \frac{7}{368640} h^3 f_{n+\frac{1}{24}} + \frac{43}{3317760} h^3 f_{n+\frac{1}{48}} + \frac{1}{6635520} h^3 f_n
\end{aligned} \tag{9}$$

Evaluating y' at $x = x_n, \quad x_{n+\frac{1}{48}}, \quad x_{n+\frac{1}{24}}, \quad x_{n+\frac{1}{16}}, \quad x_{n+\frac{1}{12}},$ we obtain

$$\begin{aligned}
hy'_n &= \frac{793}{5806080} h^3 f_{n+\frac{1}{48}} - \frac{19}{645120} h^3 f_{n+\frac{1}{24}} + \frac{79}{5806080} h^3 f_{n+\frac{1}{16}} - \frac{29}{11612160} h^3 f_{n+\frac{1}{12}} \\
&+ \frac{307}{11612160} h^3 f_n + 96 y_{n+\frac{1}{48}} - 24 y_{n+\frac{1}{24}} - 72 y_n \\
hy'_{n+\frac{1}{48}} &= -24 y_n + 24 y_{n+\frac{1}{24}} + \frac{1}{4644864} h^3 f_{n+\frac{1}{12}} - \frac{1}{1161216} h^3 f_{n+\frac{1}{16}} - \frac{1}{430080} h^3 f_{n+\frac{1}{24}} \\
&- \frac{383}{5806080} h^3 f_{n+\frac{1}{48}} - \frac{79}{23224320} h^3 f_n \\
hy'_{n+\frac{1}{24}} &= 24 y_n - 96 y_{n+\frac{1}{48}} + 72 y_{n+\frac{1}{24}} + \frac{13}{11612160} h^3 f_{n+\frac{1}{12}} - \frac{47}{5806080} h^3 f_{n+\frac{1}{16}} \\
&+ \frac{97}{1935360} h^3 f_{n+\frac{1}{24}} + \frac{583}{5806080} h^3 f_{n+\frac{1}{48}} + \frac{13}{11612160} h^3 f_n \\
hy'_{n+\frac{1}{16}} &= 72 y_n - 192 y_{n+\frac{1}{48}} + 120 y_{n+\frac{1}{24}} + \frac{1}{4644864} h^3 f_{n+\frac{1}{12}} + \frac{163}{5806080} h^3 f_{n+\frac{1}{16}} \\
&+ \frac{13}{28672} h^3 f_{n+\frac{1}{24}} + \frac{1801}{5806080} h^3 f_{n+\frac{1}{48}} + \frac{89}{23224320} h^3 f_n \\
hy'_{n+\frac{1}{12}} &= 120 y_n - 288 y_{n+\frac{1}{48}} + 168 y_{n+\frac{1}{24}} + \frac{391}{11612160} h^3 f_{n+\frac{1}{12}} + \frac{503}{1161216} h^3 f_{n+\frac{1}{16}} \\
&+ \frac{569}{645120} h^3 f_{n+\frac{1}{24}} + \frac{3061}{5806080} h^3 f_{n+\frac{1}{48}} + \frac{11}{2322432} h^3 f_n
\end{aligned} \tag{10}$$

Evaluating the second derivatives at y' at $x = x_n, x_{n+\frac{1}{48}}, x_{n+\frac{1}{24}}, x_{n+\frac{1}{16}}, x_{n+\frac{1}{12}},$

$$\begin{aligned}
h^2 y''_n &= -\frac{101}{5760} h^3 f_{n+\frac{1}{48}} + \frac{31}{5760} h^3 f_{n+\frac{1}{24}} - \frac{41}{17280} h^3 f_{n+\frac{1}{16}} + \frac{1}{2304} h^3 f_{n+\frac{1}{12}} \\
&- \frac{233}{34560} h^3 f_n - 4608 y_{n+\frac{1}{48}} + 2304 y_{n+\frac{1}{24}} + 2304 y_n \\
h^2 y''_{n+\frac{1}{48}} &= 2304 y_n - 4608 y_{n+\frac{1}{48}} + 2304 y_{n+\frac{1}{24}} - \frac{1}{8640} h^3 f_{n+\frac{1}{12}} + \frac{1}{1440} h^3 f_{n+\frac{1}{16}} \\
&- \frac{13}{5760} h^3 f_{n+\frac{1}{24}} + \frac{1}{864} h^3 f_{n+\frac{1}{48}} + \frac{1}{1920} h^3 f_n \\
h^2 y''_{n+\frac{1}{24}} &= 2304 y_n - 4608 y_{n+\frac{1}{48}} + 2304 y_{n+\frac{1}{24}} + \frac{7}{34560} h^3 f_{n+\frac{1}{12}} \\
&- \frac{5}{3456} h^3 f_{n+\frac{1}{16}} + \frac{7}{640} h^3 f_{n+\frac{1}{24}} + \frac{193}{17280} h^3 f_{n+\frac{1}{48}} - \frac{1}{34560} h^3 f_n \\
h^2 y''_{n+\frac{1}{16}} &= 2304 y_n - 4608 y_{n+\frac{1}{48}} + 2304 y_{n+\frac{1}{24}} - \frac{1}{2880} h^3 f_{n+\frac{1}{12}} + \frac{37}{4320} h^3 f_{n+\frac{1}{16}} \\
&+ \frac{139}{5760} h^3 f_{n+\frac{1}{24}} + \frac{13}{1440} h^3 f_{n+\frac{1}{48}} + \frac{1}{3456} h^3 f_n \\
h^2 y''_{n+\frac{1}{12}} &= 2304 y_n - 4608 y_{n+\frac{1}{48}} + 2304 y_{n+\frac{1}{24}} + \frac{239}{34560} h^3 f_{n+\frac{1}{12}} \\
&+ \frac{157}{5760} h^3 f_{n+\frac{1}{16}} + \frac{19}{1152} h^3 f_{n+\frac{1}{24}} + \frac{209}{17280} h^3 f_{n+\frac{1}{48}} - \frac{1}{3840} h^3 f_n
\end{aligned} \tag{11}$$

2.1. Formation of the Block for One-Twelfth Step Block Method

We implement the block method proposed by Fatunla [10]

$$\begin{aligned}
 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{48}} \\ y_{n+\frac{1}{24}} \\ y_{n+\frac{1}{16}} \\ y_{n+\frac{1}{12}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{1}{48}} \\ y_{n-\frac{1}{24}} \\ y_{n-\frac{1}{16}} \\ y_n \end{bmatrix} \\
 & + \begin{bmatrix} 0 & 0 & 0 & \frac{1}{48} \\ 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{16} \\ 0 & 0 & 0 & \frac{1}{12} \end{bmatrix} \begin{bmatrix} y'_{n+\frac{1}{48}} \\ y'_{n+\frac{1}{24}} \\ y'_{n+\frac{1}{16}} \\ y'_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4608} \\ 0 & 0 & 0 & \frac{1}{1152} \\ 0 & 0 & 0 & \frac{1}{512} \\ 0 & 0 & 0 & \frac{1}{288} \end{bmatrix} \begin{bmatrix} y''_{n-\frac{1}{48}} \\ y''_{n-\frac{1}{24}} \\ y''_{n-\frac{1}{16}} \\ y''_n \end{bmatrix} \\
 & + h^2 \begin{bmatrix} \frac{107}{111476736} & -\frac{103}{185794560} & \frac{43}{185794560} & -\frac{47}{1114767360} \\ \frac{83}{8709120} & -\frac{1}{290304} & \frac{13}{8709120} & -\frac{19}{69672960} \\ \frac{69}{2293760} & -\frac{9}{2293760} & \frac{5}{1376256} & -\frac{3}{4587520} \\ \frac{17}{272160} & \frac{1}{362880} & \frac{1}{90720} & -\frac{1}{870912} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{48}} \\ f_{n+\frac{1}{24}} \\ f_{n+\frac{1}{16}} \\ f_{n+\frac{1}{12}} \end{bmatrix} \\
 & + h^2 \begin{bmatrix} 0 & 0 & 0 & \frac{113}{123863040} \\ 0 & 0 & 0 & \frac{331}{69672960} \\ 0 & 0 & 0 & \frac{53}{4587520} \\ 0 & 0 & 0 & \frac{31}{1451520} \end{bmatrix} \begin{bmatrix} f_{n-\frac{1}{48}} \\ f_{n-\frac{1}{24}} \\ f_{n-\frac{1}{16}} \\ f_n \end{bmatrix}
 \end{aligned} \tag{12}$$

Writing (12) explicitly, gives

$$\begin{aligned}
 y_{n+\frac{1}{48}} &= y_n + \frac{1}{48} h y'_n + \frac{1}{4608} h^2 y''_n + \frac{107}{111476736} h^3 f_{n+\frac{1}{48}} - \frac{103}{185794560} h^3 f_{n+\frac{1}{24}} + \frac{43}{185794560} h^3 f_{n+\frac{1}{16}} \\
 &\quad - \frac{47}{1114767360} h^3 f_{n+\frac{1}{12}} + \frac{113}{123863040} h^3 f_n \\
 y_{n+\frac{1}{24}} &= y_n + \frac{1}{24} h y'_n + \frac{1}{1152} h^2 y''_n + \frac{83}{8709120} h^3 f_{n+\frac{1}{48}} - \frac{1}{290304} h^3 f_{n+\frac{1}{24}} + \frac{13}{8709120} h^3 f_{n+\frac{1}{16}} \\
 &\quad - \frac{19}{69672960} h^3 f_{n+\frac{1}{12}} + \frac{331}{69672960} h^3 f_n \\
 y_{n+\frac{1}{16}} &= y_n + \frac{1}{16} h y'_n + \frac{1}{512} h^2 y''_n + \frac{69}{2293760} h^3 f_{n+\frac{1}{48}} - \frac{9}{2293760} h^3 f_{n+\frac{1}{24}} + \frac{5}{1376256} h^3 f_{n+\frac{1}{16}} \\
 &\quad - \frac{3}{4587520} h^3 f_{n+\frac{1}{12}} + \frac{53}{4587520} h^3 f_n \\
 y_{n+\frac{1}{12}} &= y_n + \frac{1}{12} h y'_n + \frac{1}{288} h^2 y''_n + \frac{17}{272160} h^3 f_{n+\frac{1}{48}} + \frac{1}{362880} h^3 f_{n+\frac{1}{24}} + \frac{1}{90720} h^3 f_{n+\frac{1}{16}} \\
 &\quad - \frac{1}{870912} h^3 f_{n+\frac{1}{12}} + \frac{31}{1451520} h^3 f_n
 \end{aligned} \tag{13}$$

$$\begin{aligned}
y'_{n+\frac{1}{48}} &= y'_n + \frac{1}{48} h y''_n + \frac{1}{6144} h^2 f_{n+\frac{1}{48}} - \frac{47}{552960} h^2 f_{n+\frac{1}{24}} + \frac{29}{829440} h^2 f_{n+\frac{1}{16}} - \frac{7}{1105920} h^2 f_{n+\frac{1}{12}} \\
&\quad + \frac{367}{3317760} h^2 f_n \\
y'_{n+\frac{1}{24}} &= y'_n + \frac{1}{24} h y''_n + \frac{1}{1440} h^2 f_{n+\frac{1}{48}} - \frac{1}{6912} h^2 f_{n+\frac{1}{24}} + \frac{1}{12960} h^2 f_{n+\frac{1}{16}} - \frac{1}{69120} h^2 f_{n+\frac{1}{12}} \\
&\quad + \frac{53}{207360} h^2 f_n \\
y'_{n+\frac{1}{16}} &= y'_n + \frac{1}{16} h y''_n + \frac{13}{10240} h^2 f_{n+\frac{1}{48}} + \frac{3}{20480} h^2 f_{n+\frac{1}{24}} + \frac{1}{6144} h^2 f_{n+\frac{1}{16}} - \frac{1}{40960} h^2 f_{n+\frac{1}{12}} \\
&\quad + \frac{49}{122880} h^2 f_n \\
y'_{n+\frac{1}{12}} &= y'_n + \frac{1}{12} h y''_n + \frac{1}{540} h^2 f_{n+\frac{1}{48}} + \frac{1}{2160} h^2 f_{n+\frac{1}{24}} + \frac{1}{1620} h^2 f_{n+\frac{1}{16}} + \frac{7}{12960} h^2 f_n
\end{aligned} \tag{14}$$

$$\begin{aligned}
y''_{n+\frac{1}{48}} &= y''_n + \frac{323}{17280} h f_{n+\frac{1}{48}} - \frac{11}{1440} h f_{n+\frac{1}{24}} + \frac{53}{17280} h f_{n+\frac{1}{16}} - \frac{19}{34560} h f_{n+\frac{1}{12}} + \frac{251}{34560} h f_n \\
y''_{n+\frac{1}{24}} &= y''_n + \frac{31}{1080} h f_{n+\frac{1}{48}} + \frac{1}{180} h f_{n+\frac{1}{24}} + \frac{1}{1080} h f_{n+\frac{1}{16}} - \frac{1}{4320} h f_{n+\frac{1}{12}} + \frac{29}{4320} h f_n \\
y''_{n+\frac{1}{16}} &= y''_n + \frac{17}{640} h f_{n+\frac{1}{48}} + \frac{3}{160} h f_{n+\frac{1}{24}} + \frac{7}{640} h f_{n+\frac{1}{16}} - \frac{1}{1280} h f_{n+\frac{1}{12}} + \frac{9}{1280} h f_n \\
y''_{n+\frac{1}{12}} &= y''_n + \frac{4}{135} h f_{n+\frac{1}{48}} + \frac{1}{90} h f_{n+\frac{1}{24}} + \frac{4}{135} h f_{n+\frac{1}{16}} + \frac{7}{1080} h f_{n+\frac{1}{12}} + \frac{7}{1080} h f_n
\end{aligned} \tag{15}$$

3. Basic Properties of One-Twelfth Step Method

3.1. Order and Error Constant of the Block

$$\Delta [y(x); h] = \mathbf{A}^{(0)} \mathbf{Y}_m^{(i)} - \sum_{i=0}^k \frac{(jh)}{i} y_n^{(i)} - h^{(3-i)} [\mathbf{d}_i f(y_n) + \mathbf{b}_i \mathbf{F}(\mathbf{Y}_m)], \tag{16}$$

Expanding the form Y_m and $F(Y_m)$ in Taylor series and comparing coefficients of h , we obtained

$$\begin{aligned}
\Delta [y(x); h] &= C_0 y(x) + C_1 h y'(x) + \dots + C_p h^p y^p(x) \\
&\quad + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \dots \tag{17}
\end{aligned}$$

Definition. The linear operator and the associated block method are said to be of order p if $C_0 = C_1 = \dots = C_p = C_{p+1} = 0, C_{p+2} \neq 0$. C_{p+2} is called the error constant. It implies that the local truncation error is given by $T_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O(h^{p+3})$.

Expanding the block in Taylor series expansion gives

$$\left(\begin{aligned} & \sum_{j=0}^{\infty} \frac{\left(\frac{1}{48}\right)^j}{j!} y_n^j - y_n - \frac{1}{48} h y_n' - \frac{1}{4608} h^2 y_n'' - \frac{113}{123863040} h^3 y_n''' \\ & - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} \left[-\frac{107}{111476736} \left(\frac{1}{48}\right)^j + \frac{103}{185794560} \left(\frac{1}{24}\right)^j - \frac{43}{185794560} \left(\frac{1}{16}\right)^j + \frac{47}{1114767360} \left(\frac{1}{12}\right)^j \right] \\ & \sum_{j=0}^{\infty} \frac{\left(\frac{1}{24}\right)^j}{j!} y_n^j - y_n - \frac{1}{24} h y_n' - \frac{1}{1152} h^2 y_n'' - \frac{331}{69672960} h^3 y_n''' \\ & - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} \left[-\frac{83}{8709120} \left(\frac{1}{48}\right)^j + \frac{1}{290304} \left(\frac{1}{24}\right)^j - \frac{13}{8709120} \left(\frac{1}{16}\right)^j + \frac{19}{69672960} \left(\frac{1}{12}\right)^j \right] \\ & \sum_{j=0}^{\infty} \frac{\left(\frac{1}{16}\right)^j}{j!} y_n^j - y_n - \frac{1}{16} h y_n' - \frac{1}{512} h^2 y_n'' - \frac{53}{4587520} h^3 y_n''' \\ & - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} \left[-\frac{69}{2293760} \left(\frac{1}{48}\right)^j + \frac{9}{2293760} \left(\frac{1}{24}\right)^j - \frac{5}{1376256} \left(\frac{1}{16}\right)^j + \frac{3}{4587520} \left(\frac{1}{12}\right)^j \right] \\ & \sum_{j=0}^{\infty} \frac{\left(\frac{1}{12}\right)^j}{j!} y_n^j - y_n - \frac{1}{12} h y_n' - \frac{1}{288} h^2 y_n'' - \frac{31}{1451520} h^3 y_n''' \\ & - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} \left[-\frac{17}{272160} \left(\frac{1}{48}\right)^j - \frac{1}{362880} \left(\frac{1}{24}\right)^j - \frac{1}{90720} \left(\frac{1}{16}\right)^j + \frac{1}{870912} \left(\frac{1}{12}\right)^j \right] \end{aligned} \right). \quad (18)$$

Comparing the coefficients of h , the order of the block is $p = 5$, with error constant $C_{p+3} = \left[\frac{139}{1136188586899537920}, \frac{1}{1268067619307520}, \frac{1}{519519244124160}, \frac{1}{277389791723520} \right]^T$.

3.2. Consistency

In numerical analysis, it is necessary that the method satisfies the necessary and sufficient conditions.

A numerical method is said to be consistent if the following conditions are satisfied

1. The order of the scheme must be greater than or equal to 1 i.e. $p \geq 1$.
2. $\sum_{j=0}^k \alpha_j = 0$
3. $\rho(r) = \rho'(r) = 0$
4. $\rho'''(r) = 3!\sigma(r)$

Where, $\rho(r)$ and $\sigma(r)$ are the first and second characteristics polynomials of our method. According to [3], the first condition is a sufficient condition for the associated block method to be consistent. Our method is order $p \geq 1$. Hence it is consistent

3.3. Zero Stability of the Method

The general form of block method is given as

$$\mathbf{A}^{(0)} \mathbf{Y}_m = \mathbf{A}^{(i)} \mathbf{Y}_{m-1} + h^{mu} \left[\mathbf{B}^{(i)} Y_m + \mathbf{B}^{(i)} Y_{m-1} \right] \quad (19)$$

Applying (14) to (16) gives

$$\left| \left[\lambda A^{(0)} - A^{(i)} \right] \right| = \left| \left[\lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] \right| = 0$$

$$\lambda^4 - \lambda^3 = 0, \lambda = 0, 0, 0, 1$$

Since no root has modulus greater than one and $|\lambda| = 1$ is simple, the block method is zero stable in the $h \rightarrow 0$

4. Region of Absolute Stability of the Block Method

We express the stability matrix

$$M(z) = V + zB(M - zA)^{-1}U \quad (20)$$

together with the stability function

$$p(\eta, z) = \det(\eta I - M(z)) \quad (21)$$

Hence, we express the block method (16) in form of

$$\begin{bmatrix} Y \\ \text{---} \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ \text{---} & \text{---} & \text{---} \\ B & & V \end{bmatrix} \begin{bmatrix} h^3 f(y) \\ \text{---} \\ Y_{i-1} \end{bmatrix} \quad (22)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{113}{123863040} & \frac{107}{111476736} & -\frac{103}{185794560} & \frac{43}{185794560} & -\frac{47}{1114767360} \\ \frac{331}{69672960} & \frac{83}{8709120} & -\frac{1}{290304} & \frac{13}{8709120} & -\frac{19}{69672960} \\ \frac{53}{4587520} & \frac{69}{2293760} & -\frac{9}{2293760} & \frac{5}{1376256} & -\frac{3}{4587520} \\ \frac{31}{1451520} & \frac{17}{272160} & \frac{1}{362880} & \frac{1}{90720} & -\frac{1}{870912} \end{bmatrix},$$

$$\begin{aligned}
 B &= \begin{bmatrix} \frac{113}{123863040} & \frac{107}{111476736} & -\frac{103}{185794560} & \frac{43}{185794560} & -\frac{47}{1114767360} \\ \frac{53}{4587520} & \frac{69}{2293760} & -\frac{9}{2293760} & \frac{5}{1376256} & -\frac{3}{4587520} \\ \frac{31}{1451520} & \frac{17}{272160} & \frac{1}{362880} & \frac{1}{90720} & -\frac{1}{870912} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\
 U &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} y_n \\ y_{n+\frac{1}{48}} \\ y_{n+\frac{1}{24}} \\ y_{n+\frac{1}{16}} \\ y_{n+\frac{1}{12}} \end{bmatrix}, \quad f(y) = \begin{bmatrix} f_n \\ f_{n+\frac{1}{48}} \\ f_{n+\frac{1}{24}} \\ f_{n+\frac{1}{16}} \\ f_{n+\frac{1}{12}} \end{bmatrix}, \\
 Y_{i-1} &= \begin{bmatrix} y_{n+\frac{1}{48}} \\ y_n \end{bmatrix}, \quad Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{48}} \\ y_{n+\frac{1}{12}} \end{bmatrix}.
 \end{aligned} \tag{23}$$

The elements of the matrices A, B, U and V are substituted and computing the stability function with Maple software yield, the stability polynomial of the method which is then plotted in MATLAB environment with the Newton-Raphson Method where $N = 401$ and $tol = 10^{-11}$ to produce the required absolute stability region of the method, as shown by the figure below

5. Implementation of the Method

In this section, we discuss the strategy for the implementation of the method. In addition, the performance of the method is tested on some examples of third order initial value problems in Ordinary Differential Equations. Absolute error of the approximate solution are then compared with the existing methods. In particular, the comparison are made with those proposed by Olabode [16] and Adesanya et al. [2]

Discussion of the results of the methods are also done here.

5.1. Numerical Experiments

The method is tested on some numerical problems to test the accuracy of the proposed methods and our results are compared with the results obtained using existing methods. The following problems are taken as test problems:

5.1.1. Problem 1

$$y''' = 3\sin x, y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1 \tag{24}$$

Exact solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Figure 1: Region of absolute stability of our method

5.1.2. Problem 2

$$y''' = e^x, y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1 \quad (25)$$

Exact solution: $y(x) = 2 + 2x^2 + e^x$

5.1.3. Problem 3

$$y''' = -y, y(0) = 1, y'(0) = -1, y''(0) = 1, h = 0.1 \quad (26)$$

Exact solution: $y(x) = e^{-x}$

5.1.4. Problem 4

$$y''' + y' = 0, y(0) = 0, y'(0) = 1, y''(0) = 2, h = 0.05 \quad (27)$$

Exact solution: $y(x) = 2(1 - \cos x) + \sin x$

Table 1: Result of test problem 1

X-value	Exact Result	Computed Result	Error in our Method	Error in Hybrid Gbenga [19] Hybrid	Error in Olabode [16]
0.10000000	0.990012495834	0.990012495834	8.881784e-16	3.317602e-11	1.65922e-10
0.20000000	0.960199733524	0.960199733524	4.440892e-16	2.123026e-10	4.76275e-10
0.30000000	0.911009467377	0.911009467377	6.661338e-16	5.374488e-10	6.23182e-10
0.40000000	0.843182982009	0.843182982009	1.665335e-16	9.268841e-10	19.9134e-10
0.50000000	0.757747685671	0.757747685671	1.998401e-15	1.450962e-09	3.28882e-10
0.60000000	0.656006844729	0.656006844729	3.108624e-15	2.109886e-09	1.27096e-09
0.70000000	0.539526561853	0.539526561853	3.996803e-15	2.827094e-09	4.84653e-09
0.80000000	0.410120128041	0.410120128041	4.607426e-15	3.657408e-09	1.09585e-08
0.90000000	0.269829904812	0.269829904812	5.218048e-15	4.601151e-09	2.0188e-08
1.00000000	0.120906917604	0.120906917604	5.870304e-15	5.593764e-09	3.53956e-08
1.10000000	-0.034211635723	-0.034211635723	1.304512e-15	6.669647e-09	5.66233e-08
1.20000000	-0.192926736570	-0.192926736570	8.965051e-15	7.829207e-09	8.35700e-08

Table 2: Result of test problem 2

X-value	Exact Result	Computed Result	Error in our Method	Error in Olabode [16]
0.10000000	3.125170918076	3.125170918076	2.664535e-15	9.24352e-10
0.20000000	3.301402758160	3.301402758160	4.440892e-16	18.3983e-10
0.30000000	3.529858807576	3.529858807576	3.108624e-15	24.2400e-10
0.40000000	3.811824697641	3.811824697641	6.661338e-15	53.5873e-10
0.50000000	4.148721270700	4.148721270700	9.769963e-15	7.00128e-10
0.60000000	4.542118800391	4.542118800391	2.042810e-14	3.90509e-10
0.70000000	4.993752707470	4.993752707470	2.131628e-14	6.52952e-09
0.80000000	5.505540928492	5.505540928492	1.865175e-14	2.15075e-08
0.90000000	6.079603111157	6.079603111157	2.220446e-14	3.88430e-08
1.00000000	6.718281828459	6.718281828459	2.131628e-14	6.15410e-08
1.10000000	7.424166023946	7.424166023946	7.993606e-15	9.00536e-08
1.20000000	8.200116922737	8.200116922737	4.796163e-14	1.27263e-07

Table 3: Result of test problem 3

X-value	Exact Result	Computed Result	Error in our Method	Error Hybrid in Gbenga [19] Hybrid	Error in Olabode [16]
0.10000000	0.904837418036	0.904837418036	6.039613e-14	9.713674e-12	1.66845548e-09
0.20000000	0.818730753078	0.818730753078	4.719558e-13	6.212197e-11	1.07247812e-08
0.30000000	0.740818220682	0.740818220680	1.561751e-12	1.588829e-10	1459.00969e-08
0.40000000	0.670320046036	0.670320046032	3.620437e-12	2.856596e-10	5.60600832e-08
0.50000000	0.606530659713	0.606530659706	6.910916e-12	4.615921e-10	1.07616918e-07
0.60000000	0.548811636094	0.548811636082	1.166667e-11	6.875613e-10	1.80036532e-07
0.70000000	0.496585303791	0.496585303773	1.809769e-11	9.524539e-10	3.31228689e-06
0.80000000	0.449328964117	0.449328964091	2.638473e-11	1.269794e-09	3.46704102e-04
0.90000000	0.406569659741	0.406569659704	3.668321e-11	1.639399e-09	2.27964863e-03
1.00000000	0.367879441171	0.367879441122	4.912182e-11	2.052008e-09	6.38022004e-03
1.10000000	0.332871083698	0.332871083634	6.380341e-11	2.516401e-09	1.26418315e-02
1.20000000	0.301194211912	0.301194211831	8.080364e-11	3.030983e-09	2.10623676e-02

Table 4: Result of test problem 4

X-value	Exact	Computed	Error in	Error in
	Result	Result	our Method	Adesanya et al. [2]
0.10000000	0.109825086091	0.109825085979	1.117678e-10	1.54055(-09)
0.20000000	0.238536175113	0.238536174179	9.334808e-10	9.84550(-09)
0.30000000	0.384847228410	0.384847225133	3.277542e-09	2.36528(-08)
0.40000000	0.547296354303	0.547296346249	8.053440e-09	4.32732(-08)
0.50000000	0.724260414823	0.724260398574	1.624937e-08	3.90181(-08)
0.60000000	0.913971243576	0.913971214664	2.891198e-08	6.97008(-08)
0.70000000	1.114533312669	1.114533265544	4.712468e-08	5.20329(-08)
0.80000000	1.323942672205	1.323942600221	7.198464e-08	1.35224(-07)
0.90000000	1.540106973086	1.540106868507	1.045792e-07	4.74834(-07)
1.00000000	1.760866373072	1.760866227110	1.459613e-07	1.06936(-06)

5.2. Conclusion

We have proposed a new one-twelveth step hybrid block method for the direct numerical solution of third order initial value problems of ordinary differential equations in this paper. The method is consistent, convergent and zero stable. The method derived efficiently solved third order Initial Value Problems as can be seen in tables 1-4. In terms of accuracy, our method performs better than the existing methods compared with.

References

- [1] A. O. Adesanya and T. A. Udoh, Improved Continuous Method for Direct Solution of General Second Order Ordinary Differential Equation, *Journal of Nigeria Association of Mathematical Physics*, **13**, 2008, 59-62.
- [2] A. O. Adesanya, M. O. Udo, A. M. Alkali, A New Block-Predictor Corrector Algorithm for the Solution of $y''' = f(x, y, y', y'')$, *American Journal of Computational Mathematics*, 2012, 2, 341-344
- [3] D. O. Awoyemi, A P-Stable Linear Multistep Method for Solving Third Order Ordinary Differential Equation, *International Journal of Computer Mathematics*, **80**, No. 8, 2003, 85-991, **doi:** 10.1080/0020716031000079572.
- [4] D. O. Awoyemi and M. O. Idowu, A Class of Hybrid Collocation Method for Third Order Ordinary Differential Equation, *International Journal of Computer Mathematics*, **82**, No. 10, 2005, 1287-1293, **doi:** 10.1080/00207160500112902.
- [5] B. T. Olabode, Block Multistep Method for the Direct Solution of Third Order Ordinary Differential Equations, *FUTA Journal of Research in Sciences*, 2013 (2): 194-200
- [6] S. N. Jator, A Sixth Order Linear Multistep Method for Direct Solution of $y'' = f(x, y, y')$, *International Journal of Pure and Applied Mathematics*, **40**, No. 1, 2007, 457-472.
- [7] S. N. Jator and J. Li, A Self Starting Linear Multistep Method for the Direct Solution of the General Second Order Initial Value Problems, *International Journal of Computer Mathematics*, **86**, No. 5, 2009, 817-836, **doi:** 10.1080/00207160701708250.

- [8] D. O. Awoyemi, E. A. Adebile, A. O. Adesanya and T. A. Anake, Modified Block Method for Direct Solution of Second Order Ordinary Differential Equation, *International Journal of Applied Mathematics and Computation*, **3**, No. 3, 2011, 181-188.
- [9] J. D. Lambert, Computational Methods in Ordinary Differential Equation, *John Wiley and Sons Inc., Hoboken*, 1973.
- [10] S. O. Fatunla, M. N. O. Ikhile and F. O. Otunta, A Class of P-Stable Linear Multistep Numerical Method, *International Journal of Computer Mathematics*, **72**, 1999, 1-13, **doi:** 10.1080/00207169908804830.
- [11] K. M. Fasasi*, A. O. Adesanya, S. O. Adey, One Step Continuous Hybrid Block Method for the Solution of $y'' = f(x, y, y')$, *Journal of Natural Sciences Research*, **4**, No.10, 2014.
- [12] S. J. Kayode and D. O. Awoyemi, A Multi-Derivative Collocation Method for Fifth Order Ordinary Differential Equation, *Journal of Mathematics and Statistics*, **6**, No. 1, 2010, 60-63, **doi:** 10.3844/jmssp.2010.60.63.
- [13] Z. Omar and M. Suleiman, Parallel R-Point Implicit Block Method for Solving Higher Order Ordinary Differential Equation Directly, *Journal of ICT*, **3**, No. 1, 2003, 53-66.
- [14] L. F. Shampine and H. A. Watts, Block Implicit One Step Methods, *Journal of Math of Computation*, **23**, No. 108, 1969, 731-740, **doi:** 10.1090/S0025-5718-1969-0264854-5.
- [15] S. J. Kayode, A Zero Stable Method for Direct Solution of Fourth Order Ordinary Differential Equations, *American Journal of Applied Sciences*, **5**, No. 11, 2009, 1461-1466.
- [16] B. T. Olabode and Y. Yusuf, A New Block Method for Special Third Order Ordinary Differential Equation, *Journal of Mathematics and Statistics*, **5**, No. 3, 2009, 167-170.
- [17] D. O. Awoyemi, M. O. Udoh and A. O. Adesanya, Non-Symmetric Collocation Method for Direct Solution of General Third Order Initial Value Problems of Ordinary Differential Equation, *Journal of Natural and Applied Mathematics*, **7**, 2006, 31-37.
- [18] S. Abbas, Derivation of a New Block Method Similar to the Block Trapezoidal Rule for the Numerical Solution of First Order IVPs, *Science Echoes*, **2**, 2006, 10-24.
- [19] O. B. Gbenga, O. E. Olaoluwa, O. O. Olayemi, Hybrid and Non-hybrid Implicit Schemes for Solving Third Order ODEs Using Block Method as Predictors, *Mathematical Theory and Modeling*, **5**, 2015, 10-26