ONE-TWELVETH STEP CONTINUOUS BLOCK METHOD
FOR THE SOLUTION OF $y''' = f(x, y, y', y''$)

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Abstract: We consider direct solution to third order ordinary differential equations in this paper. Collocation and interpolation approach was adopted to generate a continuous hybrid multistep method. We adopted the use of power series as a basis function for approximate solution. We evaluated at off-grid points to get a continuous hybrid multistep method. Block method was later adopted to generate the independent solution at selected off-grid points. The properties of the block viz: order, zero stability and region of absolute stability are investigated. Our method was tested on third order ordinary differential equation and found to give better result when compared with existing methods.

Key Words: block method, approximate solutions, zero-stability, region of absolute stability, continuous hybrid multistep method

1. Introduction

In this paper, we considered the method of approximate solution of the general second order initial value problem of the form

We consider the solution to general third order initial value problem of the form

$$y''' = f(x, y, y', y''), \quad y^k(x_0) = y_0^k, \quad k = 0, 1, 2, \quad y \in \mathbb{R},$$

(1)

where $x_0$ is the initial point and $f$ is continuous within the interval of integration

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and satisfies the existence and uniqueness condition.

Many real life problems in sciences, engineering, biology and social sciences are model of third order ordinary differential equations. Some of these models do not always have theoretical solutions, thus numerical methods are often employed to solve them. Researchers in most cases always use method of reduction of higher order ODEs into system of first order ODEs to solve (1). This technique though quite good, is bedeviled with many problems such tediousness, complexity of the method, waste of time, and the need for large computer storage memories because of too many auxiliary functions, etc. Conventionally, higher order ordinary differential equations are solved directly by the predictor-corrector method where separate predictors are developed to implement the corrector and Taylor series expansion adopted to provide the starting values. Predictor-corrector methods are extensively studied by [1-12]. These authors proposed linear multistep methods with continuous coefficient, which have advantage of evaluation at all points within the grid over the proposed method in [4]. The major setbacks of predictor-corrector method are extensively discussed by [6]. Many research works have been grounded on Linear Multistep Method (LMM) and Linear Multistep Hybrid Method (LMHM) where Collocation and Interpolation are done at selected grid and off-grid points. There was then need to interpolate and collocate at grid and off-grid points with evaluation to be done at off-grid points only since this gives higher accuracy and large region of absolute stability. We apply the One-Twelveth Step Method introduced for the direct solution of Third Order Differential Equations with $x_n$ being the only grid point.

2. Mathematical Formulation of the Method

We consider the simple power series as a basis function for approximation:

$$Y(x) = \sum_{j=0}^{r+s-1} a_j \phi_j(x),$$  

(2)

where $\phi_j(x) = x^j$ and $x \in [a, b]$, $a_j s$ are coefficients to be determined and is a polynomial of degree $r + s - 1$. We construct a K-step Collocation Method by imposing the following conditions on (2)

$$Y(x_{n+j}) = y_{n+j}, \quad j = 0, 1, 2, ..., r - 1,$$  

(3)

$$Y^{(3)}(x_{n+j}) = f_{n+j}, \quad j = 0, 1, 2, ..., s - 1.$$  

(4)
Substituting (1) into (5) gives

\[ f(x, y, y', y'') = \sum_{j} j(j - 1)(j - 2)a_j x^{j-3}. \]  

(5)

We shall consider the grid point of step length \( \frac{1}{12} \) with constant step-size (h), where \( h = x_{n+i} - x_i, i = 0, 1, 2 \) and off-grid points at \( x_{n+\frac{1}{48}}, x_{n+\frac{1}{24}}, x_{n+\frac{1}{16}}, x_{n+\frac{1}{12}} \).

Interpolating (3) at \( x = x_n, x_{n+\frac{1}{48}}, x_{n+\frac{1}{24}}, x_{n+\frac{1}{16}}, x_{n+\frac{1}{12}} \) and collocating (6) at \( x = x_n, x_{n+\frac{1}{48}}, x_{n+\frac{1}{24}}, x_{n+\frac{1}{16}}, x_{n+\frac{1}{12}} \) to give a system of non-linear equation of the form

\[ AX = B. \]  

(6)

Here

\[ A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]^T, \]

\[ B = [y_n, y_{n+\frac{1}{48}}, y_{n+\frac{1}{24}}, y_{n+\frac{1}{16}}, f_n, f_{n+\frac{1}{48}}, f_{n+\frac{1}{24}}, f_{n+\frac{1}{16}}, f_{n+\frac{1}{12}}]^T, \]

and

\[
X = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\
1 & x_{n+\frac{1}{48}} & x_{n+\frac{1}{48}}^2 & x_{n+\frac{1}{48}}^3 & x_{n+\frac{1}{48}}^4 & x_{n+\frac{1}{48}}^5 & x_{n+\frac{1}{48}}^6 & x_{n+\frac{1}{48}}^7 \\
1 & x_{n+\frac{1}{24}} & x_{n+\frac{1}{24}}^2 & x_{n+\frac{1}{24}}^3 & x_{n+\frac{1}{24}}^4 & x_{n+\frac{1}{24}}^5 & x_{n+\frac{1}{24}}^6 & x_{n+\frac{1}{24}}^7 \\
0 & 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 \\
0 & 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{48}} & 20x_{n+\frac{1}{48}}^2 & 120x_{n+\frac{1}{48}}^3 & 210x_{n+\frac{1}{48}}^4 \\
0 & 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{24}} & 60x_{n+\frac{1}{24}}^2 & 120x_{n+\frac{1}{24}}^3 & 210x_{n+\frac{1}{24}}^4 \\
0 & 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{16}} & 60x_{n+\frac{1}{16}}^2 & 120x_{n+\frac{1}{16}}^3 & 210x_{n+\frac{1}{16}}^4 \\
0 & 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{12}} & 60x_{n+\frac{1}{12}}^2 & 120x_{n+\frac{1}{12}}^3 & 210x_{n+\frac{1}{12}}^4 \\
\end{bmatrix}. \]

Solving (6) for the \( a_j' \)s and substituting back into (3) and after much algebraic
simplification, we obtained

\[
a_0 = y_n
\]

\[
a_1 = \frac{793}{5806080} h^3 f_{n+\frac{1}{6}} - \frac{19}{645120} h^3 f_{n+\frac{1}{4}} + \frac{79}{5806080} h^3 f_{n+\frac{1}{2}} - \frac{29}{11612160} h^3 f_{n+\frac{1}{3}} + \frac{101}{11520} h^3 f_{n+\frac{1}{6}}
\]

\[
a_2 = -\frac{233}{69120} h^3 f_n - 2304 y_n + \frac{1}{3} + 1152 y_{n+\frac{1}{6}} + 1152 y_n
\]

\[
a_3 = \frac{1}{6} h^3 f_n
\]

\[
a_4 = 8 h^3 f_{n+\frac{1}{3}} - 6 h^3 f_{n+\frac{1}{4}} + \frac{8}{3} h^3 f_{n+\frac{1}{5}} - \frac{1}{2} h^3 f_{n+\frac{1}{6}} - \frac{25}{6} h^3 f_n
\]

\[
a_5 = -\frac{832}{5} h^3 f_{n+\frac{1}{6}} + \frac{912}{5} h^3 f_{n+\frac{1}{7}} - \frac{448}{5} h^3 f_{n+\frac{1}{8}} + \frac{88}{5} h^3 f_{n+\frac{1}{9}} + 56 h^3 f_n
\]

\[
a_6 = \frac{6912}{5} h^3 f_{n+\frac{1}{7}} - \frac{9216}{5} h^3 f_{n+\frac{1}{8}} + \frac{5376}{5} h^3 f_{n+\frac{1}{9}} - \frac{1152}{5} h^3 f_{n+\frac{1}{10}} - 384 h^3 f_n
\]

\[
a_7 = -\frac{147456}{35} h^3 f_{n+\frac{1}{8}} + \frac{221184}{35} h^3 f_{n+\frac{1}{9}} - \frac{147456}{35} h^3 f_{n+\frac{1}{10}} + \frac{36864}{35} h^3 f_{n+\frac{1}{11}} + \frac{36864}{35} h^3 f_n
\]

where

\[
q = \frac{x - x_n}{h}, \quad dq = \frac{1}{h}.
\]

We hereby present the continuous schemes

\[
y_{n+\frac{1}{8}} = y_n - 3 y_{n+\frac{1}{8}} + 3 y_{n+\frac{1}{7}} + \frac{1}{26542080} h^3 f_{n+\frac{1}{7}} - \frac{1}{6635520} h^3 f_{n+\frac{1}{6}} + \frac{7}{1474560} h^3 f_{n+\frac{1}{5}} + \frac{29}{6635520} h^3 f_{n+\frac{1}{4}} + \frac{1}{26542080} h^3 f_n
\]

\[
y_{n+\frac{1}{12}} = \frac{3}{7} y_n - 8 y_{n+\frac{1}{10}} + 6 y_{n+\frac{1}{9}} + \frac{1}{6635520} h^3 f_{n+\frac{1}{9}} + \frac{13}{3317760} h^3 f_{n+\frac{1}{8}} + \frac{7}{368640} h^3 f_{n+\frac{1}{7}} + \frac{43}{3317760} h^3 f_{n+\frac{1}{6}} + \frac{1}{6635520} h^3 f_n
\]

Evaluating \( y' \) at \( x = x_n, \ x_{n+\frac{1}{8}}, \ x_{n+\frac{1}{7}}, \ x_{n+\frac{1}{6}}, \ x_{n+\frac{1}{4}} \), we obtain
\[
hy_n = \frac{793}{5806080} h^3 f_{n + \frac{1}{12}} - \frac{19}{645120} h^3 f_{n + \frac{1}{6}} + \frac{79}{5806080} h^3 f_{n + \frac{1}{4}} - \frac{29}{11612160} h^3 f_{n + \frac{1}{3}} + \frac{307}{11612160} h^3 f_n + 96 y_{n + \frac{1}{6}} - 24 y_{n + \frac{1}{4}} - 72 y_n
\]

\[
hyn = -24 y_n + 24 y_{n + \frac{1}{6}} + \frac{1}{4644864} h^3 f_{n + \frac{1}{12}} - \frac{1}{1161216} h^3 f_{n + \frac{1}{6}} - \frac{1}{430080} h^3 f_{n + \frac{1}{4}} - \frac{383}{5806080} h^3 f_{n + \frac{1}{3}} - \frac{79}{23224320} h^3 f_n
\]

\[
hyn' = 24 y_n - 96 y_{n + \frac{1}{6}} + 72 y_{n + \frac{1}{4}} + \frac{13}{11612160} h^3 f_{n + \frac{1}{12}} - \frac{47}{5806080} h^3 f_{n + \frac{1}{6}} + \frac{97}{1935360} h^3 f_{n + \frac{1}{4}} + \frac{583}{5806080} h^3 f_{n + \frac{1}{3}} + \frac{13}{11612160} h^3 f_n
\]

\[
hyn'' = 72 y_n - 192 y_{n + \frac{1}{6}} + 120 y_{n + \frac{1}{4}} + \frac{1}{4644864} h^3 f_{n + \frac{1}{12}} + \frac{163}{5806080} h^3 f_{n + \frac{1}{6}} + \frac{13}{28672} h^3 f_{n + \frac{1}{4}} + \frac{1801}{5806080} h^3 f_{n + \frac{1}{3}} + \frac{89}{23224320} h^3 f_n
\]

\[
hyn''' = 120 y_n - 288 y_{n + \frac{1}{6}} + 168 y_{n + \frac{1}{4}} + \frac{391}{11612160} h^3 f_{n + \frac{1}{12}} + \frac{503}{11612160} h^3 f_{n + \frac{1}{6}} + \frac{569}{645120} h^3 f_{n + \frac{1}{4}} + \frac{3061}{5806080} h^3 f_{n + \frac{1}{3}} + \frac{11}{2322432} h^3 f_n
\]

(10)

Evaluating the second derivatives at \( y' \) at \( x = x_n, x_{n + \frac{1}{12}}, x_{n + \frac{1}{6}}, x_{n + \frac{1}{4}}, x_{n + \frac{1}{3}} \),

\[
h^2 y'' = -\frac{101}{5760} h^3 f_n + \frac{31}{5760} h^3 f_{n + \frac{1}{12}} - \frac{41}{17280} h^3 f_{n + \frac{1}{6}} + \frac{1}{2304} h^3 f_{n + \frac{1}{4}}
\]

\[
- \frac{233}{34560} h^3 f_n - 4608 y_{n + \frac{1}{12}} + 2304 y_{n + \frac{1}{6}} + 2304 y_n
\]

\[
h^2 y''' = 2304 y_n - 4608 y_{n + \frac{1}{12}} + 2304 y_{n + \frac{1}{6}} - \frac{1}{8640} h^3 f_{n + \frac{1}{12}} + \frac{1}{1440} h^3 f_{n + \frac{1}{6}}
\]

\[
- \frac{13}{5760} h^3 f_{n + \frac{1}{12}} + \frac{1}{864} h^3 f_{n + \frac{1}{6}} + \frac{1}{1920} h^3 f_n
\]

\[
h^2 y''' = 2304 y_n - 4608 y_{n + \frac{1}{12}} + 2304 y_{n + \frac{1}{6}} + \frac{7}{34560} h^3 f_{n + \frac{1}{12}}
\]

\[
- \frac{5}{3456} h^3 f_{n + \frac{1}{12}} + \frac{7}{640} h^3 f_{n + \frac{1}{6}} + \frac{193}{17280} h^3 f_{n + \frac{1}{4}} - \frac{1}{34560} h^3 f_n
\]

\[
h^2 y'''' = 2304 y_n - 4608 y_{n + \frac{1}{12}} + 2304 y_{n + \frac{1}{6}} - \frac{1}{2880} h^3 f_{n + \frac{1}{12}} + \frac{37}{4320} h^3 f_{n + \frac{1}{6}}
\]

\[
+ \frac{139}{5760} h^3 f_{n + \frac{1}{12}} + \frac{13}{1440} h^3 f_{n + \frac{1}{6}} + \frac{1}{3456} h^3 f_n
\]

\[
h^2 y''' = 2304 y_n - 4608 y_{n + \frac{1}{12}} + 2304 y_{n + \frac{1}{6}} + \frac{239}{34560} h^3 f_{n + \frac{1}{12}}
\]

\[
+ \frac{157}{5760} h^3 f_{n + \frac{1}{12}} + \frac{19}{1152} h^3 f_{n + \frac{1}{6}} + \frac{209}{17280} h^3 f_{n + \frac{1}{4}} - \frac{1}{3840} h^3 f_n
\]

(11)
2.1. Formation of the Block for One-Twelveth Step Block Method

We implement the block method proposed by Fatunla \[10\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
y_n + \frac{1}{6} \\
y_n + \frac{1}{2} \\
y_n + \frac{1}{3} \\
y_n + \frac{1}{4} \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_n - \frac{1}{6} \\
y_n - \frac{1}{2} \\
y_n - \frac{1}{3} \\
y_n - \frac{1}{4} \\
\end{bmatrix}
\]

\[\mathbf{y} = \begin{bmatrix} y_n \end{bmatrix} \]

\[\begin{bmatrix}
0 & 0 & 0 & \frac{1}{48} \\
0 & 0 & 0 & \frac{1}{24} \\
0 & 0 & 0 & \frac{1}{16} \\
0 & 0 & 0 & \frac{1}{12} \\
\end{bmatrix}
\begin{bmatrix}
y_n' + \frac{1}{6} \\
y_n' + \frac{1}{2} \\
y_n' + \frac{1}{3} \\
y_n' + \frac{1}{4} \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & \frac{1}{4608} \\
0 & 0 & 0 & \frac{1}{1152} \\
0 & 0 & 0 & \frac{1}{576} \\
0 & 0 & 0 & \frac{1}{288} \\
\end{bmatrix}
\begin{bmatrix}
y_n'' - \frac{1}{6} \\
y_n'' - \frac{1}{2} \\
y_n'' - \frac{1}{3} \\
y_n'' - \frac{1}{4} \\
\end{bmatrix}
\]

\[\mathbf{y}' = \begin{bmatrix} y_n' \end{bmatrix} \]

\[\mathbf{y}'' = \begin{bmatrix} y_n'' \end{bmatrix} \]

\[\mathbf{f} = \begin{bmatrix} f_n \end{bmatrix} \]

\[\begin{bmatrix}
107 \\
83 \\
17 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
y_n \\
y_n' \\
y_n'' \\
f_n \\
\end{bmatrix} +
\begin{bmatrix}
103 \\
8709129 \\
2293760 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
y_n' \\
y_n'' \\
f_n \\
\end{bmatrix} +
\begin{bmatrix}
43 \\
8709120 \\
5 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
y_n'' \\
f_n \\
\end{bmatrix} +
\begin{bmatrix}
47 \\
111476736 \\
1 \\
1 \\
\end{bmatrix}
\begin{bmatrix}
y_n'' \\
f_n \\
\end{bmatrix} =
\begin{bmatrix}
185794560 \\
8709120 \\
90720 \\
870912 \\
\end{bmatrix}
\begin{bmatrix}
y_n \\
y_n' \\
y_n'' \\
f_n \\
\end{bmatrix}
\]

\[\begin{bmatrix}
107 \\
83 \\
17 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
y_n \\
y_n' \\
y_n'' \\
f_n \\
\end{bmatrix} +
\begin{bmatrix}
103 \\
8709129 \\
2293760 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
y_n' \\
y_n'' \\
f_n \\
\end{bmatrix} +
\begin{bmatrix}
43 \\
8709120 \\
5 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
y_n'' \\
f_n \\
\end{bmatrix} +
\begin{bmatrix}
47 \\
111476736 \\
1 \\
1 \\
\end{bmatrix}
\begin{bmatrix}
y_n'' \\
f_n \\
\end{bmatrix} =
\begin{bmatrix}
185794560 \\
8709120 \\
90720 \\
870912 \\
\end{bmatrix}
\begin{bmatrix}
y_n \\
y_n' \\
y_n'' \\
f_n \\
\end{bmatrix}
\]

\[\mathbf{y} = \begin{bmatrix} y_n \end{bmatrix} \]

\[\mathbf{y}' = \begin{bmatrix} y_n' \end{bmatrix} \]

\[\mathbf{y}'' = \begin{bmatrix} y_n'' \end{bmatrix} \]

\[\mathbf{f} = \begin{bmatrix} f_n \end{bmatrix} \]

Writing (12) explicitly, gives

\begin{align*}
y_n + \frac{1}{12} & = y_n + \frac{1}{48} hy_n' + \frac{1}{4608} h^2 y_n'' + \frac{107}{111476736} h^3 f_n + \frac{1}{185794560} h^3 f_n + \frac{43}{185794560} h^3 f_n + \frac{1}{870912} h^3 f_n \\
& - \frac{47}{111476736} h^3 f_n + \frac{1}{12} + \frac{113}{123863040} h^3 f_n \\
y_n + \frac{1}{6} & = y_n + \frac{1}{24} hy_n' + \frac{1}{1152} h^2 y_n'' + \frac{83}{8709120} h^3 f_n + \frac{1}{290304} h^3 f_n + \frac{1}{8709120} h^3 f_n + \frac{13}{8709120} h^3 f_n + \frac{1}{1376256} h^3 f_n \\
& - \frac{19}{69672960} h^3 f_n + \frac{1}{6} + \frac{331}{90720} h^3 f_n \\
y_n + \frac{1}{3} & = y_n + \frac{1}{16} hy_n' + \frac{1}{512} h^2 y_n'' + \frac{69}{2293760} h^3 f_n + \frac{1}{2293760} h^3 f_n + \frac{5}{1376256} h^3 f_n + \frac{1}{1376256} h^3 f_n \\
& - \frac{3}{4587520} h^3 f_n + \frac{1}{3} + \frac{53}{90720} h^3 f_n \\
y_n + \frac{1}{2} & = y_n + \frac{1}{12} hy_n' + \frac{1}{288} h^2 y_n'' + \frac{17}{272160} h^3 f_n + \frac{1}{272160} h^3 f_n + \frac{1}{362880} h^3 f_n + \frac{1}{90720} h^3 f_n + \frac{1}{1451520} h^3 f_n \\
& - \frac{1}{870912} h^3 f_n + \frac{1}{12} + \frac{31}{1451520} h^3 f_n \\
\end{align*}
we obtained

Expanding the form

\[ C \]

the error constant. It implies that the local truncation error is given by

\[ C \]

be of order \( p \) if

\[ p \]

\[ △y = \frac{1}{48} hy'' + \frac{1}{144} h^2 f_n + \frac{1}{120} h^3 f_n + \frac{1}{18} - \frac{7}{552960} h^2 f_n + \frac{1}{18} + \frac{29}{829440} h^2 f_n + \frac{1}{18} - \frac{7}{110920} \]

\[ △y = \frac{1}{24} hy'' + \frac{1}{6144} h^2 f_n + \frac{1}{12960} h^2 f_n + \frac{1}{18} - \frac{1}{6912} h^2 f_n + \frac{1}{18} \]

\[ △y = \frac{1}{16} hy'' + \frac{1}{10240} h^2 f_n + \frac{1}{20480} h^2 f_n + \frac{1}{18} - \frac{3}{6144} h^2 f_n + \frac{1}{18} + \frac{1}{12960} h^2 f_n + \frac{1}{18} - \frac{1}{40960} h^2 f_n + \frac{1}{18} \]

\[ △y = \frac{1}{12} hy'' + \frac{1}{540} h^2 f_n + \frac{1}{2160} h^2 f_n + \frac{1}{18} + \frac{1}{1620} h^2 f_n + \frac{1}{18} + \frac{7}{12960} h^2 f_n \]

\[ △y = \frac{1}{18} hy'' + \frac{323}{17280} h^2 f_n + \frac{1}{18} - \frac{11}{1440} h^2 f_n + \frac{1}{18} + \frac{53}{17280} h^2 f_n + \frac{1}{18} - \frac{19}{34560} h^2 f_n + \frac{1}{18} + \frac{251}{34560} h^2 f_n \]

\[ △y = \frac{1}{12} hy'' + \frac{31}{1080} h^2 f_n + \frac{1}{18} + \frac{1}{180} h^2 f_n + \frac{1}{18} + \frac{1}{1080} h^2 f_n + \frac{1}{18} - \frac{1}{4320} h^2 f_n + \frac{1}{18} + \frac{29}{4320} h^2 f_n \]

\[ △y = \frac{1}{17} hy'' + \frac{17}{640} h^2 f_n + \frac{1}{18} + \frac{3}{160} h^2 f_n + \frac{1}{18} + \frac{7}{640} h^2 f_n + \frac{1}{18} - \frac{1}{1280} h^2 f_n + \frac{1}{18} + \frac{9}{1280} h^2 f_n \]

\[ △y = \frac{4}{135} h^2 f_n + \frac{1}{18} + \frac{1}{90} h^2 f_n + \frac{1}{18} + \frac{4}{135} h^2 f_n + \frac{1}{18} + \frac{7}{1080} h^2 f_n + \frac{1}{18} + \frac{7}{1080} h^2 f_n \]

3. Basic Properties of One-Twelveth Step Method

3.1. Order and Error Constant of the Block

\[ \triangle [y(x); h] = A^{(0)} Y_m^{(i)} - \sum_{i=0}^{k} \left( \frac{jh}{i} \right) y_n^{(i)} - h^{(3-i)} \left[ d_r f(y_n) + b_r F(Y_m) \right], \quad (16) \]

Expanding the form \( Y_m \) and \( F(Y_m) \) in Taylor series and comparing coefficients of \( h \), we obtained

\[ \triangle [y(x); h] = C_0 y(x) + C_1 hy'(x) + \ldots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \ldots \quad (17) \]

**Definition.** The linear operator and the associated block method are said to be of order \( p \) if \( C_0 = C_1 = \ldots = C_p = C_{p+1} = 0, C_{p+2} \neq 0 \). \( C_{p+2} \) is called the error constant. It implies that the local truncation error is given by \( T_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O(h^{p+3}) \).
Expanding the block in Taylor series expansion gives

\[
\begin{align*}
&\sum_{j=0}^{\infty} \frac{(\tan h)^j}{j!} y_j - y_n - \frac{1}{3} h y_n' - \frac{1}{24} h^2 y_n'' - \frac{113}{123888} h^3 y_n''' \\
&- \sum_{j=0}^{\infty} \frac{h^j+3}{j!} \left[ - \frac{107}{111476} \left( \frac{1}{2\pi} \right)^j + \frac{103}{18594} \left( \frac{1}{\pi} \right)^j - \frac{43}{18594} \left( \frac{1}{1\pi} \right)^j + \frac{47}{111476} \left( \frac{1}{\pi} \right)^j \right]
\end{align*}
\]

Comparing the coefficients of \( h \), the order of the block is \( p = 5 \), with error constant

\[
C_{p+3} = \left[ \begin{array}{c}
139 \\
1136188586899537920 \\
126806719307520 \\
319519241412160 \\
277389791723520 \\
\end{array} \right].
\]

### 3.2. Consistency

In numerical analysis, it is necessary that the method satisfies the necessary and sufficient conditions.

A numerical method is said to be consistent if the following conditions are satisfies

1. The order of the scheme must be greater than or equal to 1 i.e. \( p \geq 1 \).

2. \( \sum_{j=0}^{k} \alpha_j = 0 \)

3. \( \rho(r) = \rho'(r) = 0 \)

4. \( \rho'''(r) = 3! \sigma(r) \)

Where, \( \rho(r) \) and \( \sigma(r) \) are the first and second characteristics polynomials of our method. According to [3], the first condition is a sufficient condition for the associated block method to be consistent. Our method is order \( p \geq 1 \). Hence it is consistent

### 3.3. Zero Stability of the Method

The general form of block method is given as

\[
A^{(0)} Y_m = A^{(i)} Y_{m-1} + h^{mn} \left[ B^{(i)} Y_m + B^{(i)} Y_{m-1} \right]
\]
Applying (14) to (16) gives

\[
\left| \left[ \lambda A^{(0)} - A^{(i)} \right] \right| = \left| \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0
\]

\[\lambda^4 - \lambda^3 = 0, \lambda = 0, 0, 0, 1\]

Since no root has modulus greater than one and \(|\lambda| = 1\) is simple, the block method is zero stable in the \(h \to 0\)

4. Region of Absolute Stability of the Block Method

We express the stability matrix

\[M(z) = V + zB(M - zA)^{-1}U\] (20)

together with the stability function

\[p(\eta, z) = det(\eta I - M(z))\] (21)

Hence, we express the block method (16) in form of

\[
\begin{bmatrix} Y \\ - - - \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & U \\ - - - \\ B & V \end{bmatrix} \begin{bmatrix} h^3 f(y) \\ - - - \\ Y_{i-1} \end{bmatrix}
\] (22)

\[
A = \begin{bmatrix} 113 & 107 & 103 & 43 & 47 \\ 123863040 & 111476736 & 185794560 & 185794560 & 1114767360 \\ 331 & 83 & -1 & 13 & -19 \\ 69672960 & 8709120 & 290304 & 8709120 & 69672960 \\ 53 & 69 & -9 & 5 & -3 \\ 4587520 & 2293760 & 2293760 & 1376256 & 4587520 \\ 31 & 17 & 1 & 1 & -1 \\ 1451520 & 272160 & 362880 & 90720 & 870912 \end{bmatrix}
\]
$$B = \begin{bmatrix} 113 & 107 & -103 & 43 & -47 \\ 53 & 69 & 5 & -3 & 111476736 \\ 31 & 17 & 1 & 1 & 4587520 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \end{bmatrix}, \quad f(y) = \begin{bmatrix} f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \end{bmatrix},$$ (23)

$$Y_{i-1} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \end{bmatrix}, \quad Y_{i+1} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \end{bmatrix}.$$

The elements of the matrices $A$, $B$, $U$ and $V$ are substituted and computing the stability function with Maple software yield, the stability polynomial of the method which is then plotted in MATLAB environment with the Newton-Raphson Method where $N = 401$ and $tol = 10^{-11}$ to produce the required absolute stability region of the method, as shown by the figure below.

5. Implementation of the Method

In this section, we discuss the strategy for the implementation of the method. In addition, the performance of the method is tested on some examples of third order initial value problems in Ordinary Differential Equations. Absolute error of the approximate solution are then compared with the existing methods. In particular, the comparison are made with those proposed by Olabode [16] and Adesanya et al. [2]

Discussion of the results of the methods are also done here.

5.1. Numerical Experiments

The method is tested on some numerical problems to test the accuracy of the proposed methods and our results are compared with the results obtained using existing methods. The following problems are taken as test problems:

5.1.1. Problem 1

$$y''' = 3\sin x, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad h = 0.1$$ (24)

Exact solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$
5.1.2. Problem 2

\[ y''' = e^x, y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1 \]  \hspace{1cm} (25)

Exact solution: \( y(x) = 2 + 2x^2 + e^x \)

5.1.3. Problem 3

\[ y''' = -y, y(0) = 1, y'(0) = -1, y''(0) = 1, h = 0.1 \]  \hspace{1cm} (26)

Exact solution: \( y(x) = e^{-x} \)

5.1.4. Problem 4

\[ y''' + y' = 0, y(0) = 0, y'(0) = 1, y''(0) = 2, h = 0.05 \]  \hspace{1cm} (27)

Exact solution: \( y(x) = 2(1 - \cos x) + \sin x \)
Table 1: Result of test problem 1

<table>
<thead>
<tr>
<th>X-value</th>
<th>Exact Result</th>
<th>Computed Result</th>
<th>Error in our Method</th>
<th>Error in Hybrid Ghenga [19]</th>
<th>Error in Olabode [16]</th>
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Table 2: Result of test problem 2

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<th>Error in our Method</th>
<th>Error in Olabode [16]</th>
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Table 3: Result of test problem 3

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<th>Error in our Method</th>
<th>Error in Olabode [16]</th>
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Table 4: Result of test problem 4
We have proposed a new one-twelveth step hybrid block method for the direct numerical solution of third order initial value problems of ordinary differential equations in this paper. The method is consistent, convergent and zero stable. The method derived efficiently solved third order Initial Value Problems as can be seen in tables 1-4. In terms of accuracy, our method performs better than the existing methods compared with.

### References


