V-SHAPED TOEPLITZ OPERATOR

Preeti Dharmarha¹, Savita Rani^{2 §}

¹Department of Mathematics

Hansraj College, University of Delhi

Mahatma Hansraj Marg, Delhi, 110007, INDIA

²Department of Mathematics

Kurukshetra University

Kurukshetra, 136119, INDIA

Abstract: A V-shaped Toeplitz operator V'_ϕ is an operator on $L^2(\beta)$ defined as $V'_\phi = WM_\phi^*$, where M_ϕ^* is the weighted multiplication operator and W is an operator on $L^2(\beta)$ given by $We_{2n} = \frac{\beta_n}{\beta_{2n}}e_n$, $\{e_n\}_{n \in \mathbb{Z}}$ being the orthonormal basis. In this paper we will study its properties.

AMS Subject Classification: 47B35

Key Words: V-shaped Toeplitz operator

1. Introduction

The Toeplitz operators were introduced in 1911 by O. Toeplitz [5] and many mathematicians followed him with different generalizations of Toeplitz operators. The class of slant Toeplitz operators having the property that the matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Toeplitz operators

Received: December 23, 2016

Revised: February 25, 2017

Published: May 6, 2017

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url: www.acadpubl.eu

[§]Correspondence author

introduced by Ho[4] in 1995. These operators are very useful in many applications like prediction theory[2], solution of differential equation[6] and wavelet analysis[3]. However, these studies were made in context of the usual Hardy Spaces and Lorentz spaces. Motivated from these results and the various applications of Toeplitz operators, we introduced and studied V-shaped Toeplitz operators on $L^2(\beta)$. We now begin with terminology.

Let $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers such that $\beta_0 = 1$, $0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$, for all $n \geq 0$ and $0 < \frac{\beta_n}{\beta_{n-1}} \leq 1$ for all $n \leq 0$.

Consider the spaces (see [1],[7])

$$L^2(\beta) = \{f(z) = \sum_{n=-\infty}^{\infty} a_n z^n | a_n \in \mathbb{C},$$

$$\|f\|_{\beta}^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty\},$$

and

$$H^2(\beta) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n | a_n \in \mathbb{C},$$

$$\|f\|_{\beta}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty\}.$$

Then, $(L^2(\beta), \|\cdot\|_{\beta})$ is a Hilbert space with an orthonormal basis $\{e_n(z) = \frac{z^n}{\beta_n}\}_{n \in \mathbb{Z}}$ and with respect to the inner product defined by $\langle \sum a_n z^n, \sum b_n z^n \rangle = \sum a_n \overline{b_n} \beta_n^2$.

Also, $H^2(\beta)$ is a subspace of $L^2(\beta)$. If,

$$L^{\infty}(\beta) = \{\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n | \phi L^2(\beta) \subseteq L^2(\beta)\}$$

and $\exists c \in \mathbb{R}$ such that $\|\phi f\|_{\beta} \leq c \|f\|_{\beta}$ for all $f \in L^2(\beta)$, then $L^{\infty}(\beta)$ is a Banach space with a norm defined by

$$\|\phi\|_{\infty} = \inf\{c | \|\phi f\|_{\beta} \leq c \|f\|_{\beta},$$

for all $f \in L^2(\beta)\}$.

If $P: L^2(\beta) \longrightarrow H^2(\beta)$ is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$, then the weighted Toeplitz operator T_{ϕ} on $H^2(\beta)$ [7] with symbol ϕ in $L^{\infty}(\beta)$ is defined by $T_{\phi} f = P(\phi f)$ for all $f \in H^2(\beta)$

Let M_ϕ^* denote the weighted multiplication operator on $L^2(\beta)$. Then,

$$M_\phi^*(f) = \phi f,$$

for all $f \in L^2(\beta)$, and

$$M_\phi^* e_k(z) = \sum_{n=-\infty}^1 a_{n-k} \frac{\beta_n e_n(z)}{\beta_k} + \sum_{n=2}^{\infty} a_{2-k} \frac{\beta_n e_n(z)}{\beta_k},$$

for all $k \in \mathbb{Z}$.

Definition 1. The V-shaped Toeplitz operator V'_ϕ is an operator on $L^2(\beta)$ defined as $V'_\phi : L^2(\beta) \rightarrow L^2(\beta)$ such that $V'_\phi e_k(z) = \sum_{n=-\infty}^1 \frac{a_{2n-k}}{\beta_k} \beta_n e_n(z) + \sum_{n=2}^{\infty} \frac{a_{2-k}}{\beta_k} \beta_n e_n(z)$ for $k=0, \pm 1, \pm 2, \dots$

The matrix of V'_ϕ is a bilaterally infinite matrix given by

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & a_{-2} \frac{\beta_{-1}}{\beta_0} & a_{-3} \frac{\beta_{-1}}{\beta_1} & a_{-4} \frac{\beta_{-1}}{\beta_2} & \dots \\ \dots & a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & \dots \\ \dots & a_2 \frac{\beta_1}{\beta_0} & a_1 \frac{\beta_1}{\beta_1} & a_0 \frac{\beta_1}{\beta_2} & \dots \\ \dots & a_2 \frac{\beta_2}{\beta_0} & a_1 \frac{\beta_2}{\beta_1} & a_0 \frac{\beta_2}{\beta_2} & \dots \\ \dots & a_2 \frac{\beta_3}{\beta_0} & a_1 \frac{\beta_3}{\beta_1} & a_0 \frac{\beta_3}{\beta_2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Thus we see that this is the matrix obtained by simply eliminating every alternate row of the matrix of weighted multiplication operator M_ϕ^* and multiplying each i -th row by $\frac{\beta_i}{\beta_{2i}}$. If $W : L^2(\beta) \rightarrow L^2(\beta)$ is defined as

$$W e_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z)$$

and $W e_{2n-1}(z) = 0$, for all $n \in \mathbb{Z}$.

Then the matrix of W is

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \frac{\beta_0}{\beta_0} & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \frac{\beta_1}{\beta_2} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \frac{\beta_2}{\beta_4} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and $\|W\| = \sup_n \frac{\beta_n}{\beta_{2n}} \leq 1$.

Below, we will prove the following result.

Theorem 2. (i) $V'_\phi(f) = WM_\phi^*f$ for all $f \in L^2(\beta)$.

(ii) $W = V'_1$.

Proof. (i) We have

$$\begin{aligned} WM_\phi^*e_k(z) &= W \sum_{n=-\infty}^1 a_{n-k} \frac{\beta_n e_n(z)}{\beta_k} + \sum_{n=2}^{\infty} a_{2-k} \frac{\beta_n e_n(z)}{\beta_k} \\ &= \sum_{n=-\infty}^1 \frac{a_{2n-k}}{\beta_k} \beta_n e_n(z) + \sum_{n=2}^{\infty} \frac{a_{2-k}}{\beta_k} \beta_n e_n(z) \\ &= V'_\phi e_k(z). \end{aligned}$$

(ii) Take $\phi(z) = 1$ in the above result, then

$$V'_1 = WM_1^* = W.$$

Hence we can now give an alternate definition of V- Shaped Toeplitz operator V'_ϕ as $V'_\phi(f) = WM_\phi^*f = W(\phi f)$

Theorem 3. V'_ϕ is bounded.

Proof. We have

$$V'_\phi = WM_\phi^*.$$

Therefore

$$\begin{aligned} \|V'_\phi\| &= \|WM_\phi^*\| \\ &\leq \|W\| \|M_\phi^*\| \\ &\leq \|\phi\|_\infty. \end{aligned}$$

Since $\|M_\phi^*\| = \|\phi\|_\infty$ as shown by Shields in [1].

2. Algebraic Properties

In this section we will give algebraic properties of V-shaped Toeplitz operator.

Property 1. Determinant of the matrix of V-shaped Toeplitz operator V'_ϕ is zero.

Proof. As the matrix of V'_ϕ is

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & a_{-2} \frac{\beta_{-1}}{\beta_0} & a_{-3} \frac{\beta_{-1}}{\beta_1} & a_{-4} \frac{\beta_{-1}}{\beta_2} & \dots \\ \dots & a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & \dots \\ \dots & a_2 \frac{\beta_1}{\beta_0} & a_1 \frac{\beta_1}{\beta_1} & a_0 \frac{\beta_1}{\beta_2} & \dots \\ \dots & a_2 \frac{\beta_2}{\beta_0} & a_1 \frac{\beta_2}{\beta_1} & a_0 \frac{\beta_2}{\beta_2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

From the above matrix we observe that a_i 's in each column are same after some values of k 's. Now to verify that determinant is zero we will use the principle of mathematical induction to prove the result for $n \geq 3$. First of all we will prove the result for 3x3 and 4x4 matrix.

For 3x3 matrix

$$\begin{vmatrix} a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} \\ a_2 \frac{\beta_1}{\beta_0} & a_1 \frac{\beta_1}{\beta_1} & a_0 \frac{\beta_1}{\beta_2} \\ a_2 \frac{\beta_2}{\beta_0} & a_1 \frac{\beta_2}{\beta_1} & a_0 \frac{\beta_2}{\beta_2} \end{vmatrix}$$

$$= a_0 \frac{\beta_0}{\beta_0} (a_0 a_1 - a_{-1} a_2) - a_{-1} \frac{\beta_0}{\beta_1} (a_0 a_2 \frac{\beta_1}{\beta_0} - a_0 a_2 \frac{\beta_1}{\beta_0}) + a_{-2} \frac{\beta_0}{\beta_2} (a_1 a_2 \frac{\beta_2}{\beta_0} - a_1 a_2 \frac{\beta_2}{\beta_0})$$

$$= 0.$$

Hence the determinant of 3x3 V-shaped Toeplitz matrix is zero.

Now we will prove the result for 4x4 matrix.

$$\begin{vmatrix} a_{-2} \frac{\beta_{-1}}{\beta_0} & a_{-3} \frac{\beta_{-1}}{\beta_1} & a_{-4} \frac{\beta_{-1}}{\beta_2} & a_{-5} \frac{\beta_{-1}}{\beta_3} \\ a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & a_{-3} \frac{\beta_0}{\beta_3} \\ a_2 \frac{\beta_1}{\beta_0} & a_1 \frac{\beta_1}{\beta_1} & a_0 \frac{\beta_1}{\beta_2} & a_{-1} \frac{\beta_1}{\beta_3} \\ a_2 \frac{\beta_2}{\beta_0} & a_1 \frac{\beta_2}{\beta_1} & a_0 \frac{\beta_2}{\beta_2} & a_{-1} \frac{\beta_2}{\beta_3} \end{vmatrix} = a_{-2} \frac{\beta_{-1}}{\beta_0} \begin{vmatrix} a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & a_{-3} \frac{\beta_0}{\beta_3} \\ a_1 \frac{\beta_1}{\beta_1} & a_0 \frac{\beta_1}{\beta_2} & a_{-1} \frac{\beta_1}{\beta_3} \\ a_1 \frac{\beta_2}{\beta_1} & a_0 \frac{\beta_2}{\beta_2} & a_{-1} \frac{\beta_2}{\beta_3} \end{vmatrix}$$

$$- a_{-3} \frac{\beta_{-1}}{\beta_1} \begin{vmatrix} a_0 \frac{\beta_0}{\beta_0} & a_{-2} \frac{\beta_0}{\beta_2} & a_{-3} \frac{\beta_0}{\beta_3} \\ a_2 \frac{\beta_1}{\beta_0} & a_0 \frac{\beta_1}{\beta_2} & a_{-1} \frac{\beta_1}{\beta_3} \\ a_2 \frac{\beta_2}{\beta_0} & a_0 \frac{\beta_2}{\beta_2} & a_{-1} \frac{\beta_2}{\beta_3} \end{vmatrix}$$

$$+ a_{-4} \frac{\beta_{-1}}{\beta_2} \begin{vmatrix} a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-3} \frac{\beta_0}{\beta_3} \\ a_2 \frac{\beta_1}{\beta_0} & a_1 \frac{\beta_1}{\beta_1} & a_{-1} \frac{\beta_1}{\beta_3} \\ a_2 \frac{\beta_2}{\beta_0} & a_1 \frac{\beta_2}{\beta_1} & a_{-1} \frac{\beta_2}{\beta_3} \end{vmatrix}$$

$$- a_{-5} \frac{\beta_{-1}}{\beta_3} \begin{vmatrix} a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} \\ a_2 \frac{\beta_1}{\beta_0} & a_1 \frac{\beta_1}{\beta_1} & a_0 \frac{\beta_1}{\beta_2} \\ a_2 \frac{\beta_2}{\beta_0} & a_1 \frac{\beta_2}{\beta_1} & a_0 \frac{\beta_2}{\beta_2} \end{vmatrix}$$

$$= 0.$$

As we see above that if in 3x3 determinant a'_i s are same in two rows then the determinant is zero. Hence the determinant of 4x4 matrix is zero. By using induction hypothesis we can prove easily that if the determinant of mxm matrix is zero then the determinant of (m+1)x(m+1) matrix is zero. Hence the result is true for all $n \geq 3$.

From this we can generalize that the determinant of bilateraly infinite matrix of V-shaped Toeplitz operator is zero.

Remark 2. As we have seen above that the determinant of bilateraly infinite matrix of V-shaped Toeplitz matrix is zero but to obtain shape V the order of matrix must be odd.

Property 3. V'_ϕ does not commute with M_z^* .

Proof.

$$\begin{aligned} M_z^* V'_\phi e_k(z) &= M_z^* \left(\sum_{n=-\infty}^1 \frac{a_{2n-k}}{\beta_k} \beta_n e_n(z) + \sum_{n=2}^{\infty} \frac{a_{2-k}}{\beta_k} \beta_n e_n(z) \right) \\ &= z \left(\sum_{n=-\infty}^1 \frac{a_{2n-k}}{\beta_k} \beta_n \frac{z^n}{\beta_n} + \sum_{n=2}^{\infty} \frac{a_{2-k}}{\beta_k} \beta_n \frac{z^n}{\beta_n} \right) \\ &= \sum_{n=-\infty}^1 \frac{a_{2n-k}}{\beta_k} z^{n+1} + \sum_{n=2}^{\infty} \frac{a_{2-k}}{\beta_k} z^{n+1} \\ &= \sum_{n=-\infty}^1 \frac{a_{2n-k}}{\beta_k} \beta_{n+1} e_{n+1}(z) + \sum_{n=2}^{\infty} \frac{a_{2-k}}{\beta_k} \beta_{n+1} e_{n+1}(z) \dots (1) \end{aligned}$$

$$\begin{aligned} V'_\phi M_z^* e_k(z) &= V'_\phi z e_k(z) \\ &= V'_\phi \frac{\beta_{k+1}}{\beta_k} e_{k+1}(z) \\ &= \frac{\beta_{k+1}}{\beta_k} V'_\phi (e_{k+1}(z)) \\ &= \frac{\beta_{k+1}}{\beta_k} \left(\sum_{n=-\infty}^1 \frac{a_{2n-k+1}}{\beta_{k+1}} \beta_n e_n(z) + \sum_{n=2}^{\infty} \frac{a_{1-k}}{\beta_{k+1}} \beta_n e_n(z) \right) \\ &= \sum_{n=-\infty}^1 \frac{a_{2n-k+1}}{\beta_k} \beta_n e_n(z) + \sum_{n=2}^{\infty} \frac{a_{1-k}}{\beta_k} \beta_n e_n(z) \dots (2) \end{aligned}$$

From (1) and (2), we see that $M_z^* V'_\phi \neq V'_\phi M_z^*$

Property 4. *The mapping $\phi \longrightarrow V'_\phi$ is linear.*

Proof. $V'_{(\alpha\phi+\beta\psi)} = WM_{(\alpha\phi+\beta\psi)}^* = \alpha WM_\phi^* + \beta WM_\psi^* = \alpha V'_\phi + \beta V'_\psi$.
Hence the mapping is linear.

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