ON THE DIOPHANTINE EQUATION $3^x + 13^y = z^2$

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Abstract: In this paper we prove that the Diophantine equation $3^x + 13^y = z^2$ has exactly four non-negative integer solutions for x , y and z . The solutions are $(1, 0, 2)$, $(1, 1, 4)$, $(3, 2, 14)$ and $(5, 1, 16)$ respectively.

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1. Introduction

Diophantine equations of the form $a^x + b^y = c^z$ have been studied by numerous mathematicians for many years and by a variety of methods. In 2011, Suvarnamani [1] gave all solutions to the Diophantine equations of the type $2^x + p^y = z^2$ for primes $p = 2, 3, 2^{k+1} + 1$. He showed in his proof that $z - 2^{(k+1)/2} = p^u$ where k and u are non-negative integers and p is a positive odd prime number which is impossible. In this equation, although z and p^u are integers, yet $2^{(k+1)/2}$ is not as $2^{(k+1)/2} = (2^k)2^{1/2}$. However a modified correct proof for this equation was given by Sroysang [2] in 2013. Hadano [3] in 1976 gave solutions

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of Diophantine equations of the type $3^y + 11^z = 2^x$, $3^y + 13^z = 2^x$, $3^y + 17^z = 2^x$. Rabago [4] in 2013 solved the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ where x , y and z are non-negative integers. He found two solutions for each of the equations i.e., $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively. Sroysang [5, 6] in 2012 showed that the two Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$ have the unique solutions $(1, 0, 2)$ in non-negative integers (x, y, z) , respectively. Further, Sroysang [7] in 2014 solved the equation $3^x + 85^y = z^2$ and found that $(1, 0, 2)$ is a unique solution in non-negative integers x , y and z for this equation. The results on the related Diophantine equations have been found by several different scholars [8, 9, 10, 11, 12], employing a variety of methods.

In this paper, we prove: The Diophantine equation $3^x + 13^y = z^2$ has only four non-negative integer solutions $(1, 0, 2)$, $(1, 1, 4)$, $(3, 2, 14)$ and $(5, 1, 16)$ for x , y and z .

2. Preliminaries

Proposition 2.1. (*Catalan's conjecture*) $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a , b , x and y are integers such that $\min \{a, b, x, y\} > 1$. This was proved in 2004 by Mihailescu [13].

We shall use the following Lemma.

Lemma 2.2. [4] $(1, 2)$ is a unique solution (x, z) for the Diophantine equation $3^x + 1 = z^2$ where x and z are non-negative integers.

Proof. Let x and z be non-negative integers such that $3^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$, which is impossible. If $z = 0$, then $3^x + 1 = 0$ or $3^x = -1$, which is not possible. So we only consider the case when $x, z > 0$. We have $3^x + 1 = z^2$. or, $3^x = z^2 - 1$. or, $3^x = (z - 1)(z + 1)$. where, $3^u = (z - 1)$ and $3^{x-u} = (z + 1)$, $x > 2u$, u is a non-negative integer. It follows that $2 = (z + 1) - (z - 1) = 3^{x-u} - 3^u$ or $2 = 3^u(3^{x-2u} - 1)$. This implies that $3^u = 3^0$ and $3^{x-2u} - 1 = 2$. Thus $u = 0$ and $x = 1$. Hence, we have $x = 1$ and $z = 2$. Therefore, $(1, 2)$ is a unique solution (x, z) for the Diophantine equation $3^x + 1 = z^2$. $\square$

3. Main Result

Theorem 3.1. *The exponential Diophantine equation $3^x + 13^y = z^2$ has exactly four non-negative integer solutions $(1, 0, 2)$, $(1, 1, 4)$, $(3, 2, 14)$ and $(5, 1, 16)$ respectively for x , y and z .*

Proof. Let x , y and z be non-negative integers such that $3^x + 13^y = z^2$. We first consider the case when y is zero. By lemma 2.2, we have $(x, y, z) = (1, 0, 2)$. Now we consider the case when $x, y, z > 0$. Here two cases arise:

Case(1) If y is even. Let $y = 2k$, where k is a positive integer. Then $3^x + 13^{2k} = z^2$. Then $3^x = z^2 - 13^{2k} = (z - 13^k)(z + 13^k)$. Thus $z - 13^k = 3^u$, where u is a non-negative integer. Then $z + 13^k = 3^{x-u}$. Thus $2(13^k) = 3^{x-u} - 3^u = 3^u(3^{x-2u} - 1)$, $x > 2u$. Thus $u = 0$ and $3^x - 1 = 2 \cdot 13^k$. Since $k \geq 1$. Thus we have $3^x = 2 \cdot 13^k + 1 \geq 2 \cdot 13^1 + 1 = 27$. When $k = 1$, then $3^x = 3^3$. Hence $x = 3$. So $x = 3$ and $y = 2$. Hence the solution is $(3, 2, 14)$.

Case(2) If y is odd. Let $y = 2l + 1$ where l is a non-negative integer. For x , $3^x \equiv 3 \pmod{4}$ for odd integer x and $3^x \equiv 1 \pmod{4}$ for even integer x . Since $z^2 \equiv 0, 1 \pmod{4}$ then x must be odd and z is even.

(i) We take x odd positive integer and $z = 2m$, $m > 0$. We have $3^x + 13 \cdot 13^{2l} = (2m)^2$ which implies $3^x + 4 \cdot 13^{2l} = (4m^2) - 9 \cdot 13^{2l}$ or equivalently $3^x + 4 \cdot 13^{2l} = (2m - 3 \cdot 13^l)(2m + 3 \cdot 13^l)$. This equation has two possibilities.

$$\begin{cases} 2m - 3 \cdot 13^l = 1 \\ 2m + 3 \cdot 13^l = 3^x + 4 \cdot 13^{2l} \end{cases}$$

or

$$\begin{cases} 2m + 3 \cdot 13^l = 1 \\ 2m - 3 \cdot 13^l = 3^x + 4 \cdot 13^{2l} \end{cases}$$

Solving first set of equalities, we have $13^l(6 - 4 \cdot 13^l) = 3^x - 1$ which implies that $13^l = 13^0$ and $3^x - 1 = 6 - 4 \cdot 13^l$. This gives us the values $l = 0$, $x = 1$ and $m = 2$. Thus we obtain the solution $(x, y, z) = (1, 1, 4)$. From second set of equalities, we have $l = 0$ and $2m + 3 = 1$ which implies that $2m = -2$ or $m = -1$ which is impossible m is a natural number.

ii) Again we have $3^x + 13^{2l+1} = 2m^2$. So, $3^x + 13 \cdot 13^{2l} = (2m)^2$ which implies that $3^x - 212 \cdot 13^{2l} = (2m)^2 - 225 \cdot 13^{2l}$ or equivalently $3^x - 212 \cdot 13^{2l} = (2m - 15 \cdot 13^l)(2m + 15 \cdot 13^l)$. This equation has two possibilities.

$$\begin{cases} 3^x - 212 \cdot 13^{2l} = 2m + 15 \cdot 13^l \\ 2m - 15 \cdot 13^l = 1 \end{cases}$$

or

$$\begin{cases} 3^x - 212 \cdot 13^{2l} = 2m - 15 \cdot 13^l \\ 2m + 15 \cdot 13^l = 1 \end{cases}$$

From first set of equation, we have $13^l(30 + 212 \cdot 13^l) = 3^x - 1$ which implies that $l = 0$ and $30 + 212 \cdot 13^l = 3^x - 1$. This gives us values $x = 5$ and $m = 8$. Hence, the solution is $(x, y, z) = (5, 1, 16)$. From second set of equation, we have $l = 0$ and $212 \cdot 13^l - 30 = 3^x - 1$ which gives $3^x = 183$ which is not solvable. $\square$

Corollary 3.2. $(1, 1, 2)$ and $(5, 1, 4)$ are positive integer solutions (x, y, w) for the Diophantine equation $3^x + 13^y = w^4$.

Proof. Let x, y and w be positive integers such that $3^x + 13^y = w^4$. Suppose $z = w^2$. Then $3^x + 13^y = z^2$. By theorem 3.1, we have $(x, y, z) \in \{(1, 1, 4), (5, 1, 16)\}$. Then $w^2 = z \in \{4, 16\}$. Thus, $w \in \{2, 4\}$. Hence, $(1, 1, 2)$ and $(5, 1, 4)$ are positive integer solutions (x, y, w) for the Diophantine equation $3^x + 13^y = w^4$. $\square$

Corollary 3.3. The Diophantine equation $3^x + 13^y = v^8$ has unique solution in positive integers (x, y, v) as $(5, 1, 2)$.

Proof. Suppose that there are positive integers x, y , and v such that $3^x + 13^y = v^8$. Let $z = v^4$. Then $3^x + 13^y = z^2$. By Theorem 3.1, it follows that $(x, y, z) = (5, 1, 16)$. Thus, $z = v^4 = 16$. So, $v = 2$. Hence, the solution of Diophantine equation $3^x + 13^y = v^8$ in positive integers (x, y, z) is $(5, 1, 2)$. $\square$

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