ON WEAKLY $g(x)$ -NIL CLEAN RINGS

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Abstract: Let R be an associative ring with identity and let $g(x)$ be a fixed polynomial in $C(R)[x]$. We define R to be weakly $g(x)$ -nil ($g(x)$ -nil*) clean if every (non zero) element in R can be written as a sum or a difference of a nilpotent and a root of $g(x)$. For $b \in C(R) \cap U(R)$, we prove that R is weakly (nil) clean if and only if R is weakly $x(x-b)$ -(nil) clean. Moreover, for any even integer k , we characterize weakly $(x^k - 1)$ -nil* clean rings as the rings satisfying the nil k -involution property. Some other properties are also studied and several examples are given.

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1. Introduction

Let R be an associative ring with identity. Throughout this text, the notations $U(R)$, $J(R)$, $Id(R)$ and $N(R)$ will stand for the set of units, the Jacobson radical, the set of idempotents and the set of nilpotents of R , respectively. We denote by $M_n(R)$ and $T_n(R)$ the rings of all $n \times n$ matrices and all upper triangular matrices over R respectively. Following [15], we define an element r of a ring R to be clean if there is an idempotent $e \in R$ and a unit $u \in R$ such that $r = u + e$. A clean ring is defined to be one in which every element is

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clean. Similarly a nil clean ring was introduced by Diesel in [11]. An element r in a ring R is said to be nil clean if $r = e + n$ for some idempotent $e \in R$ and a nilpotent element $n \in R$. A ring R is nil clean if each element of R is nil clean. In [4], Breaz, Danchev and Zhou defined a ring R to be weakly nil clean if each element $r \in R$ can be written as $r = n + e$ or $r = n - e$ for $n \in N(R)$ and $e \in Id(R)$. We refer the reader to [3, 7, 8, 10, 19, 4] for a survey on nil clean and weakly nil clean rings.

Let $C(R)$ denotes the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. Following Camillo and Simón [5], R is called $g(x)$ -clean if for each $r \in R$, there exist a unit $u \in R$ and $s \in R$ such that $r = u + s$ and $g(s) = 0$. Of course $(x^2 - x)$ -clean rings are precisely the clean rings.

Nicholson and Zhou [17] proved that if $g(x) \in (x - a)(x - b)C(R)[x]$ where $a, b \in C(R)$ are such that b and $b - a$ are both units in R and ${}_R M$ is a semisimple left R -module, then $End({}_R M)$ is $g(x)$ -clean. Recently, Fan and Yang [12], studied more properties of $g(x)$ -clean rings. Among many conclusions, they proved that if $g(x) \in (x - a)(x - b)C(R)[x]$ where $a, b \in C(R)$ are such that $b - a$ is a unit in R , then R is a clean ring if and only if R is $(x - a)(x - b)$ -clean. Recently, Khashan and Handam [13] have generalized nil clean rings. They defined a ring R to be $g(x)$ -nil clean if every element in R is a sum of a nilpotent and a zero of $g(x)$. It is clear that a ring R is $g(x)$ -nil clean if and only if for each $r \in R$, $r = n + s$ where n is a nilpotent and s is a root of $g(x)$. Ashrafi and Ahmadi, [2] have introduced and study weakly $g(x)$ -clean rings as rings in which every element is a sum or a difference of a unit and a zero of $g(x)$.

The aim of our paper is to define and study weakly $g(x)$ -nil clean rings as a proper generalization of nil clean rings and a proper subclass of weakly $g(x)$ -clean rings. For a ring R and $g(x) \in C(R)[x]$, we say that $r \in R$ is weakly $g(x)$ -nil clean provided that there exist a nilpotent $n \in R$ and a root $s \in R$ of $g(x)$ such that $r = n + s$ or $r = n - s$. Moreover, R is called weakly $g(x)$ -nil clean if every element in R is weakly $g(x)$ -nil clean. If every non zero element in R is weakly $g(x)$ -nil clean, then R is called weakly $g(x)$ -nil* clean.

In the next section we introduce some fundamental facts concerning weakly $g(x)$ -nil clean rings and give some necessary examples. In Section 3, for $b \in C(R) \cap U(R)$, we characterize weakly $x(x - b)$ -clean (weakly $x(x - b)$ -nil clean) rings in terms of weakly clean (weakly nil clean) rings. Moreover, for any even integer k , we characterize weakly $(x^k - 1)$ -nil* clean rings as the rings satisfying the nil k -involution property.

2. Weakly $g(x)$ -nil Clean Rings

In this section, we study some of the basic properties of weakly $g(x)$ -nil clean rings. Moreover, we give some necessarily examples.

Definition 1. Let R be a ring and $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called weakly $g(x)$ -nil clean if there exist $n \in N(R)$ and a zero s of $g(x)$ such that $r = n + s$ or $r = n - s$. A ring R is called weakly $g(x)$ -nil clean (respectively, weakly $g(x)$ -nil* clean) if all of its elements (respectively, non zero elements) are weakly $g(x)$ -nil clean.

Let $g(x) \in C(R)[x]$, s a zero of $g(x)$ and $n \in N(R)$. For $r \in R$, we write $r = n \pm s$ if r is either a sum $n + s$ or a difference $n - s$. Obviously, every $g(x)$ -nil clean ring is weakly $g(x)$ -nil clean. Since the ring $\mathbb{Z}_6$ is a weakly nil clean ring that is not nil clean, then trivially $\mathbb{Z}_6$ is a weakly $(x^2 - x)$ -nil clean ring which is not $(x^2 - x)$ -nil clean. For a non trivial example, one can easily verify that the ring $\mathbb{Z}_3$ is weakly $(x^4 + x)$ -nil clean but not $(x^4 + x)$ -nil clean.

Moreover, if a ring R is a weakly $g(x)$ -nil clean, then it is weakly $g(x)$ -clean. Indeed, if we let $x \in R$, then $x - 1 = n + s$ or $x - 1 = n - s$ where $n \in N(R)$ and $g(s) = 0$. So, $x = (n + 1) + s$ or $x = (n + 1) - s$ where $n + 1 \in U(R)$.

The converse is not true in general. For example, simple computations show that the ring $R = T_2(\mathbb{Z}_3)$ is (weakly) $(x^4 - 1)$ -clean which is not weakly $(x^4 - 1)$ -nil clean.

For any two rings R and S , consider the ring homomorphism $\mu : C(R) \rightarrow C(S)$ with $\mu(1_R) = 1_S$. If $g(x) = \sum_{i=0}^n a_i x^i \in C(R)[x]$, we let $g_\mu(x) := \sum_{i=0}^n \mu(a_i) x^i \in C(S)[x]$. In particular, if $g(x) \in \mathbb{Z}[x]$, then $g_\mu(x) = g(x)$.

Next, we give some properties of the class of weakly $g(x)$ -nil clean rings. We start by a simple result.

Proposition 2. Let R and S be two rings, $\mu : R \rightarrow S$ be a ring epimorphism and $g(x) = \sum_{i=0}^n a_i x^i \in C(R)[x]$. If R is weakly $g(x)$ -nil clean, then S is weakly $g_\mu(x)$ -nil clean.

Let R and S be two rings such that R is weakly $g(x)$ -nil clean. If there is an epimorphism $\mu : R \rightarrow S$, then we say that S is a weakly $g^*(x)$ -nil clean. The verifications of the following assertions are left as exercises.

Corollary 3. Let R and S be ring. The following hold:

(1) If I is an ideal in R and R is weakly $g(x)$ -nil clean, then R/I is weakly $g^*(x)$ -nil clean. Moreover, the converse holds if I is nil and the roots of $g^*(x)$ lift modulo I .

(2) If the upper triangular matrix ring $T_n(R)$ is weakly $g(x)$ -nil clean, then R is weakly $g^*(x)$ -nil clean.

(3) If the skew formal power series $R[[x, \alpha]]$ (or in particular $R[[x]]$) over R is weakly $g(x)$ -nil clean, then R is weakly $g^*(x)$ -nil clean.

(4) Let M be an (R, S) -bimodule and $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be the formal triangular matrix ring. If T is weakly $g(x)$ -nil clean, then R and S are weakly $g^*(x)$ -nil clean.

(5) If R is commutative and M an R -module. Then the idealization $R(M)$ of R and M is weakly $g(x)$ -nil clean if and only if R is weakly $g^*(x)$ -nil clean.

Proposition 4. Let $R_1, R_2, \dots, R_m$ be rings, $R = \prod_{i=1}^m R_i$ and $g(x) \in \mathbb{Z}[x]$. Then R is weakly $g(x)$ -nil clean if and only if there exist $k \in \{1, 2, \dots, m\}$ such that R_k is weakly $g(x)$ -nil clean and R_j is $g(x)$ -nil clean for all $j \neq k$.

Proof. $\Rightarrow$) : For each $i \in \{1, 2, \dots, m\}$, R_i is a homomorphic image of R under the projection homomorphism. Hence, R_i is weakly $g(x)$ -nil clean by Proposition 2. Without loss of generality assume that neither R_1 nor R_2 are $g(x)$ -nil clean. Then there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that r_1 is not a sum of a nilpotent and a root of $g(x)$ and r_2 is not a difference of a nilpotent and a root of $g(x)$. It follows that (r_1, r_2) is not weakly $g(x)$ -nil clean in $R_1 \times R_2$, a contradiction.

$\Leftarrow$) : Assume that R_k is weakly $g(x)$ -nil clean for a fixed $k \in \{1, 2, \dots, m\}$ and R_j is $g(x)$ -nil clean for all $j \neq k$. Let $r = (r_i) \in R$. Then there exist $n_k \in N(R_k)$ and a zero s_k of $g(x)$ such that $r_k = n_k + s_k$ or $r_k = n_k - s_k$. If $r_k = n_k + s_k$, then for each $i \neq k$, write $r_i = n_i + s_i$ where $n_i \in N(R_i)$ and $g(s_i) = 0$. Therefore, $r = (n_i) + (s_i)$ is a sum of a nilpotent and a zero of $g(x)$. If $r_k = n_k - s_k$, then for $i \neq k$, write $r_i = n_i - s_i$ where $n_i \in N(R_i)$ and $g(s_i) = 0$. Therefore, $r = (n_i) - (s_i)$ is a difference of a nilpotent and a zero of $g(x)$. Therefore, R is weakly $g(x)$ -nil clean. $\square$

Generally, the ring of polynomials $R[t]$ is not weakly $g(x)$ -nil clean ring for non-zero polynomial $g(x) \in C(R)[x]$. For example, let R be any commutative ring and let $g(x) = x$. Then $t \in R[t]$ is not weakly $g(x)$ -nil clean. Indeed, if $t = n \pm s$ where $n \in N(R[t])$ and $g(s) = 0$, then it must be that $s = 0$ and $t = n$. But $t \notin N(R[t])$, so $R[t]$ is not weakly $g(x)$ -nil clean.

Let R be a ring with an identity and S be a ring (not necessary unitary) which is an (R, R) -bimodule such that $(s_1 s_2)a = s_1(s_2 a)$, $a(s_1 s_2) = (as_1)s_2$ and $(s_1 a)s_2 = s_1(as_2)$ for all $a \in R$, $s_1, s_2 \in S$. The ideal-extension $I(R, S)$ of R by S is defined as the additive abelian group $I(R, S) = R \oplus S$ with multiplication

$(a_1, s_1)(a_2, s_2) = (a_1a_2, a_1s_2 + s_1a_2 + s_1s_2)$. If $g(x) = (a_0, s_0) + (a_1, s_1)x + \dots + (a_n, s_n)x^n \in C(I(R, S))[x]$, then clearly $g_R(x) = a_0 + a_1x + \dots + a_nx^n \in C(R)[x]$.

Proposition 5. *Let R and S as above. If $I(R, S)$ is weakly $g(x)$ -nil clean, then R is weakly $g_R(x)$ -nil clean.*

Proof. If we define $\mu_R : I(R, S) \rightarrow R$ by $\mu_R(r, s) = r$, then μ_R is a ring epimorphism. The result follows by Proposition 2. $\square$

Let R be a ring and $\alpha : R \rightarrow R$ be a ring endomorphism. By $R[[x, \alpha]]$, we denote the ring of skew formal power series over R ; that is all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R . The skew polynomial ring $R[x, \alpha]$ can be defined in an analogous way. One can prove that $R[[x, \alpha]] \simeq I(R, \langle x \rangle)$ where $\langle x \rangle$ is the ideal generated by x .

Corollary 6. *Let R be a ring and $\alpha : R \rightarrow R$ be a ring endomorphism. If $R[[x, \alpha]]$ (or in particular $R[[x]]$) is weakly $g(x)$ -nil clean, then R is weakly $g_\mu(x)$ -nil clean where $\mu : R[[x, \alpha]] \rightarrow R$ is defined by $\mu(f) = f(0)$.*

A Morita Context (A, B, V, W, ψ, ϕ) consists of two rings A, B , two bimodules ${}_A V_B, {}_B W_A$ and a pair of bimodule homomorphisms $\psi : V \otimes_B W \rightarrow A$, $\phi : W \otimes_B V \rightarrow B$, such that $\psi(v \otimes w)v' = v\phi(w \otimes v')$, $\phi(w \otimes v)w' = w\psi(v \otimes w')$. With such a Morita context we associate the ring $T = \begin{bmatrix} A & V \\ W & B \end{bmatrix} =$

$\left\{ \begin{bmatrix} a & v \\ w & b \end{bmatrix} \mid a \in A, b \in B, v \in V, w \in W \right\}$ under the usual matrix addition and multiplication defined as:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' + \psi(v \otimes w') & av' + vb' \\ wa' + bw' & \phi(w \otimes v') + bb' \end{bmatrix}. \text{ We call } T \text{ a}$$

Morita Context ring. If $g(x) = \begin{bmatrix} a_0 & v_1 \\ m_0 & b_0 \end{bmatrix} + \begin{bmatrix} a_1 & v_2 \\ m_1 & b_1 \end{bmatrix} x + \dots + \begin{bmatrix} a_n & v_n \\ m_n & b_n \end{bmatrix} x^n \in C(T)[x]$, then clearly $g_A(x) = a_0 + a_1x + \dots + a_nx^n \in C(A)[x]$ and $g_B(x) = b_0 + b_1x + \dots + b_nx^n \in C(B)[x]$.

Proposition 7. *Let $T = \begin{bmatrix} A & V \\ W & B \end{bmatrix}$ be a Morita Context with $\psi, \phi = 0$.*

If T is weakly $g(x)$ -nil clean, then A is weakly $g_A(x)$ -nil clean and B is weakly $g_B(x)$ -nil clean.

Proof. Assume that T is weakly $g(x)$ -nil clean with $\psi, \phi = 0$. Let $I = \begin{bmatrix} 0 & V \\ W & B \end{bmatrix}$ and $J = \begin{bmatrix} A & V \\ W & 0 \end{bmatrix}$. Then clearly I and J are ideals of T and

moreover, $T/I \cong A$ and $T/J \cong B$. It follows by Proposition 3 that A is weakly $g_A(x)$ -nil clean and B is weakly $g_B(x)$ -nil clean. $\square$

Corollary 8. *Let A, B be two rings and M be an (A, B) -bimodule. Let $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be the formal triangular matrix ring. If T is weakly $g(x)$ -nil clean, then A is weakly $g_A(x)$ -nil clean and B is weakly $g_B(x)$ -nil clean.*

In the following Proposition, we consider a particular case of formal triangular matrix rings. Let R be a commutative ring and M an R -module. The trivial extension of R by M is the (commutative) ring

$$R(M) = \left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} : r \in R, m \in M \right\}$$

with the usual matrix addition and multiplication. We note that if $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \in N(R(M))$, then clearly $r \in N(R)$. We recall that R naturally embeds into $R(M)$ via $r \rightarrow \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$. Thus any polynomial $g(x) = \sum_{i=0}^n a_i x^i \in R[x]$ can be written as $g(x) = \sum_{i=0}^n \begin{bmatrix} r_i & 0 \\ 0 & r_i \end{bmatrix} x^i \in R(M)[x]$ and conversely.

Proposition 9. *Let R be a commutative ring and M an R -module. Then the idealization $R(M)$ of R and M is weakly $g(x)$ -nil clean if and only if R is weakly $g(x)$ -nil clean.*

Proof. $\Rightarrow$) : Note that $R \simeq R(M)/\widetilde{M}$ where $\widetilde{M} = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} : m \in M \right\}$. Hence, R is weakly $g(x)$ -nil clean by Proposition 2.

$\Leftarrow$) : Let $g(x) = \sum_{i=0}^n a_i x^i \in R[x]$ and $r \in R$. Since R is weakly $g(x)$ -nil clean, we have $r = n \pm s$ where $n \in N(R)$ and $g(s) = 0$. Then for $m \in M$, $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} = \begin{bmatrix} n & m \\ 0 & n \end{bmatrix} \pm \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$ where $\begin{bmatrix} n & m \\ 0 & n \end{bmatrix} \in N(R(M))$. Moreover, we have

$$\begin{aligned} g\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\right) &= a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + a_2 \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix} + \dots + a_n \begin{bmatrix} s^n & 0 \\ 0 & s^n \end{bmatrix} \\ &= \begin{bmatrix} a_0 + a_1 s + a_2 s^2 + \dots + a_n s^n & 0 \\ 0 & a_0 + a_1 s + a_2 s^2 + \dots + a_n s^n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $R(M)$ is weakly $g(x)$ -nil clean. $\square$

If R is a weakly $g(x)$ -nil clean and $\theta : R \rightarrow R$ is a ring endomorphism then for any n , the quotient $W = R[x; \theta] / \langle x^n \rangle$, where $R[x; \theta]$ is the Hilbert twist, is weakly $g(x)$ -nil clean. Indeed if $f = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} \in W$ and $a_0 = n \pm s$ where $n \in N(R)$ and $g(s) = 0$, then $f = (f - s) + s$ or $f = (f + s) - s$ is a weakly $g(x)$ -nil clean decomposition of f in W .

3. Weakly $x(x - b)$ -(nil) Clean Rings

In this section, we consider weakly $x(x - b)$ -(nil) clean rings for $b \in C(R) \cap U(R)$ and we investigate the relation between weakly $x(x - b)$ -(nil) clean rings and weakly (nil) clean rings.

Theorem 10. *Let R be a ring and $b \in C(R)$. Then R is weakly clean and $b \in U(R)$ if and only if R is weakly $x(x - b)$ -clean.*

Proof. $\Rightarrow$) : Suppose R is weakly clean with $b \in U(R)$ and let $r \in R$. Write $rb^{-1} = u \pm e$, where u is a unit and e is an idempotent. Then, $r = ub \pm eb$ where clearly $ub \in U(R)$ and eb is a root of $x(x - b)$. Hence, R is weakly $x(x - b)$ -clean.

$\Leftarrow$) : Suppose R is weakly $x(x - b)$ -clean. Write $0 = u \pm s$ where $u \in U(R)$ and $s(s - b) = 0$. Then $s = \pm u \in U(R)$ and $s - b = 0$, whence $b \in U(R)$. Now, let $r \in R$ and write $rb = u \pm s$ where $u \in U(R)$ and $s(s - b) = 0$. Then, $r = ub^{-1} \pm sb^{-1}$ where $ub^{-1} \in U(R)$. Moreover,

$$(sb^{-1})^2 = s(s - b + b)b^{-2} = s(s - b) + sbb^{-2} = sb^{-1}$$

Therefore, R is weakly nil clean. □

In [12], the authors proved that for a ring R with $b - a \in C(R)$, R is clean with $b - a \in U(R)$ if and only if R is $(x - a)(x - b)$ -clean. However, this equivalence can not be generalized for the weakly case. For example, the ring $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, a \neq 0, 3 \nmid b \text{ and } 5 \nmid b \right\}$ is weakly clean, [1]. On the other hand, one can see that R is not weakly $(x - 7)(x - 8)$ -clean.

The following corollary is a direct consequence of Theorem 10.

Corollary 11. *Let R be a ring and let $n \in \mathbb{N}$. The following are equivalent:*

- (1) R is weakly clean and $2 \in U(R)$.
- (2) R is weakly $(x^2 - 2^n x)$ -clean.
- (3) R is weakly $(x^2 + 2^n x)$ -clean.
- (4) R is weakly $(x^2 - 2x)$ -clean.
- (5) R is weakly $(x^2 + 2x)$ -clean.

Definition 12. Let R be a ring. Then R is said to have the involution property (nil involution property) if for every $r \in R$, we have $r = u + v$ where $u \in U(R)$ ($u \in N(R) + 1$) and $v^2 = 1$. More general, for a positive integer m , we say that R has the m -involution (nil m -involution) property if for every $r \in R$, we have $r = u + v$ where $u \in U(R)$ ($u \in N(R) + 1$) and $v^m = 1$.

The following characterization of the m -involution property is useful for the coming main theorem.

Lemma 13. Let R be a ring and let $k, m \in \mathbb{N}$. Then R has the m -involution property (nil m -involution property) if and only if for every $r \in R$, we have $r = u + v$ where $u \in U(R)$ ($u \in N(R) + 1$) and $(v - k)^m = 1$.

Proof. Suppose R has the m -involution property. If $r \in R$, then $r - k = u + w$ where $u \in U(R)$ and $w^m = 1$. Hence, $r = u + (w + k) = u + v$ where $(v - k)^m = w^m = 1$. Conversely, let $r \in R$ and write $r + k = u + w$ where $u \in U(R)$ and $(w - k)^m = 1$. Then $r = u + (w - k) = u + v$ where $v^m = 1$. A similar procedure can be used to prove the case of nil m -involution property. $\square$

It is clear that s is a zero of $x^2 - 1$ if and only if $-s$ is so. Hence, weakly $(x^2 - 1)$ -clean rings are the same as $(x^2 - 1)$ -clean rings. Thus, according to [12], we have

Corollary 14. Let R be a ring. The following are equivalent:

- (1) R is clean and $2 \in U(R)$.
- (2) R is weakly $(x^2 - 1)$ -clean.
- (3) R is $(x^2 - 2x)$ -clean.
- (4) R has the involution property.

Theorem 15. Let R be a ring and $b \in C(R)$. Then R is weakly nil clean and $b \in U(R)$ if and only if R is weakly $x(x - b)$ -nil clean.

Proof. Suppose R is weakly nil clean and $b \in U(R)$. For $r \in R$, choose $n \in N(R)$ and $e \in Id(R)$ such that $rb^{-1} = n \pm e$. Then, $r = nb \pm eb$ where clearly $nb \in N(R)$ and eb is a zero of $x(x - b)$. Therefore, R is weakly $x(x - b)$ -nil clean.

Conversely, suppose R is weakly $x(x - b)$ -nil clean. Write $1 = n \pm s$ where $n \in N(R)$ and $s(s - b) = 0$. Then $s = \pm(n - 1) \in U(R)$ and so $b = s \in U(R)$. Now, let $r \in R$ and write $rb = n \pm s$ where $n \in N(R)$ and $s(s - b) = 0$. Then, $r = nb^{-1} \pm sb^{-1}$. As in the proof Theorem 10, $sb^{-1} \in Id(R)$ and clearly $nb^{-1} \in N(R)$. It follows that R is weakly nil clean. $\square$

In particular, we conclude that a ring R is weakly nil clean and $2 \in U(R)$ if and only if R is weakly $(x^2 - 2x)$ -nil clean.

Lemma 16. [4]. *If a ring R is weakly nil clean, then 6 is a (central) nilpotent element in R .*

Since 6 is a central nilpotent in any weakly nil clean ring R , then $6b$ is nilpotent for any $b \in R$. It follows that $6b \pm 1 \in U(R)$. Hence, the following characterizations of weakly nil clean rings follow from Theorem 15

Corollary 17. *Let R be a ring and $b \in C(R)$. The following are equivalent*

- (1) *R is weakly nil clean.*
- (3) *R is weakly $x(x - (6b + 1))$ -nil clean.*
- (4) *R is weakly $x(x - (6b - 1))$ -nil clean.*

For example, weakly nil clean rings are equivalent to weakly $(x^2 \pm 5x)$ -nil clean and weakly $(x^2 \pm 7x)$ -nil clean rings.

It is clear that if s is a unit in a ring R with $0 = n \pm s$ for some $n \in N(R)$, then R is the zero ring. Thus, we need only to consider the weakly $g(x)$ -nil*clean property when the constant term of $g(x)$ is a unit. Indeed, suppose $g(x) = x^k + a_{k-1}x^{k-1} + \dots + a_1x + a_0 \in C(R)[x]$ with $a_0 \in U(R)$ and $g(s) = 0$. Then $s(s^{k-1} + a_{k-1}s^{k-2} + \dots + a_1) = -a_0 \in U(R)$ and so $s \in U(R)$. Examples of such polynomials are of the form $g(x) = x^k - 1$ and $g(x) = x^k + 1$ for $k \in \mathbb{N}$. Let p be a prime integer. By the well known Fermat theorem, the field Z_p is an examples of a weakly $(x^{p-1} - 1)$ -nil*clean ring which is not weakly $(x^{p-1} - 1)$ -nil clean. However weakly $g(x)$ -nil*clean and weakly $g(x)$ -nil clean coincide if all roots of $g(x)$ are non units.

Proposition 18. *A ring R is weakly $(x^2 - 1)$ -nil*clean if and only if it has the nil involution property.*

Proof. Suppose R is weakly $(x^2 - 1)$ -nil*clean and let $r \in R$. If $r = 2$, then $r = (0 + 1) + 1$ is a sum of a unipotent and an involution. For $r \neq 2$, we may write $r - 2 = n \pm s$ where $n \in N(R)$ and $s^2 = 1$. If $r - 2 = n + s$, then $r = (n + 1) + (s + 1)$ where $n + 1 \in N(R) + 1$ and $(s + 1 - 1)^2 = 1$. If $r - 2 = n - s$, then $r = (n + 1) + (1 - s)$ where $(1 - s - 1)^2 = 1$. Thus, R has the nil involution property by Lemma 13. Conversely, if R has the nil involution property and $0 \neq r \in R$, then $r + 1 = (n + 1) + v$ where $n \in N(R)$ and $(v - 1)^2 = 1$. Therefore, $r = (1 + n) + (v - 1)$ is the required decomposition of r . $\square$

Next, we generalize the equivalence in Proposition 18 to any arbitrary odd $k \in \mathbb{N}$.

Proposition 19. *Let R be a ring and k be an odd integer with $k \geq 3$. Consider the following statements:*

- (1) R has the nil $(k-1)$ -involution property.
- (2) R is weakly $(x^{k-1} - 1)$ -nil* clean.
- (3) R is weakly $(x^k - x)$ -nil clean.

Then (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3). If moreover, aR (or Ra) contains no non trivial idempotents for every $a \in R$, then the three statements are equivalent.

Proof. (1) $\Leftrightarrow$ (2) : Suppose R has the nil $(k-1)$ -involution property and let $0 \neq r \in R$. Write $r + 1 = u + v$ where $u \in N(R) + 1$ and $v^{k-1} = 1$. Then $r = (u - 1) + v$ where $u - 1 \in N(R)$ and v is a root of $x^{k-1} - 1$. Conversely, suppose R is weakly $(x^{k-1} - 1)$ -nil* clean. Let $r \in R$. If $r = 2$, then $r = (0+1)+1$. For $r \neq 2$, we write $r - 2 = n \pm s$ for $n \in N(R)$ and $s^{k-1} = 1$. If $r - 2 = n + s$, then $r = (n + 1) + (s + 1)$ where $n + 1 \in N(R) + 1$ and $(s + 1 - 1)^{k-1} = 1$. If $r - 2 = n - s$, then $r = (n + 1) + (1 - s)$ where $(1 - s - 1)^{k-1} = 1$ since $k - 1$ is even. In both cases, R has the nil $(k-1)$ -involution property by Lemma 13.

(2) $\Rightarrow$ (3) : The proof is obvious.

Now, we assume that for every $a \in R$, aR (or Ra) contains no non trivial idempotents and prove (3) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1) : suppose R is weakly $(x^k - x)$ -nil clean. Let $a \in R$ and write $a - 1 = n \pm s$ where $n \in N(R)$ and $s^k = s$. Then $as^{k-1} = (n + 1)s^{k-1} \pm s$ and so $a(1 - s^{k-1}) = (n + 1)(1 - s^{k-1}) = u(1 - s^{k-1})$ where $u \in U(R)$. Since clearly $u(1 - s^{k-1})u^{-1} = a(1 - s^{k-1})u^{-1} \in aR$ is an idempotent, then by assumption $u(1 - s^{k-1})u^{-1} = 0$. Therefore $s^{k-1} = 1$ and $a = (n + 1) + (\pm s)$ where $(\pm s)^{k-1} = 1$ again since $k - 1$ is even. The case when Ra contains no non trivial idempotent for every $a \in R$ is similar. Therefor, R has the nil $(k-1)$ -involution property. $\square$

It is easy to see that the ring $\mathbb{Z}_4$ is a (weakly) $(x^4 - x)$ -nil clean ring with $a\mathbb{Z}_4$ contains no non trivial idempotent for every $a \in \mathbb{Z}_4$. But, $\mathbb{Z}_4$ is not weakly $(x^3 - 1)$ -nil* clean. Therefore, the equivalence of the three statements in Proposition 19 does not hold for an even integer k .

Recall that for a ring R and $n \in \mathbb{N}$, we denote by $U_n(R)$ the set of elements in R that are either units or sum of n units. We now generalize the implication (2) $\Rightarrow$ (3) in the previous theorem to more general $g(x)$.

Proposition 20. *Let R be a ring, $k \in \mathbb{N}$ and $g(x) = x^k + a_{k-1}x^{k-1} + \dots + a_1x + a_0 \in C(R)[x]$ where $a_0 \in U(R)$. If R is weakly $g(x)$ -nil* clean, then $R = U_2(R)$. In particular, if R is weakly $(x^{k-2} + x^{k-3} + \dots + x + 1)$ -nil* clean, then $R = U_2(R)$ is weakly $(x^k - x)$ -nil clean.*

Proof. Suppose R is weakly $g(x)$ -nil* clean and let $r \in R$. If $r = 1$, then $r \in U_2(R)$. Suppose $r \neq 1$ and let $r - 1 = n \pm s$ where $n \in N(R)$ and $s^k + a_{k-1}s^{k-1} + \dots + a_1s + a_0 = 0$. Then $r = (n+1) \pm s$ where $n+1 \in U(R)$ and $s(s^{k-1} + a_{k-1}s^{k-2} + \dots + a_1) = -a_0 \in U(R)$. Therefore, $r = (n+1) + (\pm s)$ is a sum of two units in R and $R = U_n(R)$. where $s \in U(R)$ and so $s \in U(R)$. Thus, $r \in U_2(R)$. Since also $1 \in U_2(R)$, then $R = U_2(R)$. In particular if R is weakly $(x^{k-2} + x^{k-3} + \dots + x + 1)$ -nil* clean, then $R = U_2(R)$ by taking $a_0 = 1$. Now, let $r \in R$. If $r = 0$, then r is clearly a weakly $(x^k - x)$ -nil clean element. Let $0 \neq r \in R$ and write $r = n \pm s$ where $n \in N(R)$ and $s^{k-2} + s^{k-3} + \dots + s + 1 = 0$. Then $s^k - s = s(s-1)(s^{k-2} + s^{k-3} + \dots + s + 1) = 0$ and so R is weakly $(x^k - x)$ -nil clean. $\square$

In the next result we analogue Theorem 3.1 in [2]. The proof is almost similar.

Proposition 21. *Let R be a ring, k be an even positive integer and $a, b \in R$. Then R is weakly $(ax^k - bx)$ -nil clean if and only if R is weakly $(ax^k + bx)$ -nil clean.*

However, the equivalence in Proposition 21 does not hold for odd powers. For example, The ring $\mathbb{Z}_3$ is clearly a (weakly) $(x^3 - x)$ -nil clean which is not weakly $(x^3 + x)$ -nil clean.

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