SEPARATION AXIOMS IN
BITOPOLOGICAL ORDERED SPACES

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Abstract: In this paper, we introduce and study some new type of separation axioms in bitopological ordered spaces via \((i, j)\)-semiopen sets.

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1. Introduction

The concept of bitopological spaces was first introduced by Kelly [1]. After the introduction of the definition of a bitopological space by Kelly, a large number of topologists have turned their attention to the generalization of different concepts of a single topological space in this space. A bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) is a bitopological space \((X, \tau_1, \tau_2)\) equipped with a partial order \(\leq\) (that is, reflexive, transitive and antisymmetric). A subset \(A\) of \((X, \tau_1, \tau_2, \leq)\) is said to be increasing (resp. decreasing) if

\[
A = \{x \in X : a \leq x \text{ for } a \in A\} \quad \text{(resp. } A = \{x \in X : x \leq a \text{ for some } a \in A\})
\]

that is, if

\[
A = \bigcup_{a \in A} [a, \rightarrow] \quad \text{(resp. } A = \bigcup_{a \in A} [\leftarrow, a])
\]

where \([a, \rightarrow] = \{x \in X : a \leq x\}\) (resp.
In this paper, we introduce and study some new type of separation axioms in bitopological ordered spaces via \((i, j)\)-semiopen sets.

2. Preliminaries

For a subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\), \(i\,\text{Cl}(A)\) and \(i\,\text{Int}(A)\) denote the closure of \(A\) and the interior of \(A\), respectively with respect to \((X, \tau_i)\).

**Definition 1.** [5] A subset \(S\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be \((i, j)\)-semiopen if \(S \subset j\,\text{Cl}(i\,\text{Int}(S))\).

**Definition 2.** [5] The intersection of all \((i, j)\)-semiclosed sets containing \(S \subset X\) is called the \((i, j)\)-semiclosure of \(S\) and is denoted by \((i, j)-\text{SC}(X, x)\).

**Definition 3.** A subset \(M(x)\) of a bitopological space \((X, \tau_1, \tau_2)\) is called an \((i, j)\)-semineighbourhood of a point \(x \in X\) if there exists an \((i, j)\)-semiopen set \(S\) such that \(x \in S \subset M(x)\).

**Definition 4.** [4] A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is said to be \((i, j)\)-irresolute if \(f^{-1}(U)\) is \((i, j)\)-semiopen in \(X\) for every \((i, j)\)-semiopen set \(U\) of \(Y\).

**Definition 5.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be:

1. \((i, j)\)-semi-\(T_1\) if for every \(x, y \in X\), \(x \neq y\), there exist \(U, V \in (i, j)\)-SO such that \(x \in U, y \notin V\) and \(y \in V, x \notin U\).

2. \((i, j)\)-semi-\(T_2\) if for every \(x, y \in X\), \(x \neq y\), there exist \(U, V \in (i, j)\)-SO(X) such that \(x \in U, y \notin V\) and \(y \in V, x \notin U\) and \(U \cap V = \emptyset\).

3. \((i, j)\)-semiregular if for any closed set \(F\) in \(X\) and \(a \in X \setminus F\), there exist disjoint \((i, j)\)-semiopen sets \(U\) and \(V\) in \(X\) containing \(a\) and \(F\), respectively.

**Definition 6.** [2] A bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) is said to be upper (resp. lower) \(T_1\)-ordered if for each pair of elements \(a \nleq b\) (that is, \(a\) is not related to \(b\)) in \(X\) there exists a decreasing (resp. increasing) open set \(U\) containing \(b\) (resp. \(a\)) such that \(a \notin U\) (resp. \(b \notin U\)). \((X, \tau_1, \tau_2, \leq)\) is said to be \(T_1\)-ordered if it is both lower and upper \(T_1\)-ordered.
Definition 7. [2] A bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) is said to be \(T_2\)-ordered if for each pair of elements \(a \not\leq b\) in \(x\) there exist disjoint open sets \(U\) and \(V\) in \(X\) containing \(a\) and \(b\) respectively, \(U\) is increasing and \(V\) is decreasing.

3. On \((i, j)\)-Semi-\(T_1\) and \((i, j)\)-Semi-\(T_2\) Ordered Spaces

Definition 8. Let \((X, \tau_1, \tau_2, \leq)\) be a bitopological ordered space and \(A\) a subset of \(X\). Define:
\[
I^{(i,j)-s}(A) = \cap \{F : F\text{ is an increasing } (i, j)\text{-semiclosed subset of } X\text{ containing } A\},
\]
\[
D^{(i,j)-s}(A) = \cap \{K : K\text{ is a decreasing } (i, j)\text{-semiclosed subset of } X\text{ containing } A\},
\]
\[
I^{o(i,j)-s}(A) = \cup \{G : G\text{ is an increasing } (i, j)\text{-semiopen subset of } X\text{ contained in } A\},
\]
\[
D^{o(i,j)-s}(A) = \cup \{H : H\text{ is a decreasing } (i, j)\text{-semiopen subset of } X\text{ contained in } A\}.
\]
Clearly, \(I^{(i,j)-s}(A)\) (resp. \(D^{(i,j)-s}(A)\)) is the smallest increasing (resp. decreasing) \((i, j)\)-semiclosed subset of \(X\) containing \(A\) and \(I^{o(i,j)-s}(A)\) (resp. \(D^{o(i,j)-s}(A)\)) is the largest increasing (resp. decreasing) \((i, j)\)-semiopen subset of \(X\) contained in \(A\).

Proposition 9. For any subset \(A\) of a bitopological ordered space \((X, \tau_1, \tau_2, \leq)\), the following hold:

1. \(X \setminus I^{(i,j)-s}(A) = D^{o(i,j)-s}(X \setminus A)\).
2. \(X \setminus D^{(i,j)-s}(A) = I^{o(i,j)-s}(X \setminus A)\).
3. \(X \setminus I^{o(i,j)-s}(A) = D^{(i,j)-s}(X \setminus A)\).
4. \(X \setminus D^{o(i,j)-s}(A) = I^{(i,j)-s}(X \setminus A)\).

Proof. We shall prove (1) only, (2), (3) and (4) can be proved in a similar manner.

(1). Since \(I^{(i,j)-s}(A)\) is an \((i, j)\)-semiclosed increasing set containing \(A\), \(X \setminus I^{(i,j)-s}(A)\) is an \((i, j)\)-semiopen decreasing set such that \(X \setminus I^{(i,j)-s}(A) \subset X \setminus A\). Let \(U\) be an another \((i, j)\)-semiopen decreasing set such that \(U \subset X \setminus A\). Then \(X \setminus U\) is an \((i, j)\)-semiclosed increasing set such that \(X \setminus U \supset A\). It follows that \(I^{o(i,j)-s}(A) \subset X \setminus U\). That is \(U \subset X \setminus I^{(i,j)-s}(A)\). Thus, \(X \setminus I^{(i,j)-s}(A)\) is
the largest \((i, j)\)-semiopen decreasing set such that \(X \setminus I^{(i,j)s}(A) \subset X \setminus A\). That is \(X \setminus I^{(i,j)s}(A) = D^{o(i,j)−s}(X \setminus A)\).

**Lemma 10.** Let \((X, \tau_1, \tau_2, \leq)\) be a bitopological ordered space and \(A\) a subset of \(X\). Then \(x \in I^{(i,j)s}(A)\) (resp. \(x \in D^{o(i,j)−s}(A)\)) if and only if for every decreasing (resp. increasing) \((i, j)\)-semiopen subset \(U\) of \(X\) containing \(x\), \(U \cap A \neq \emptyset\).

**Proof.** Let \(U\) be a decreasing \((i, j)\)-semiopen subset of \(X\) containing \(x\) such that \(U \cap A = \emptyset\). Then \(X \setminus U\) is an increasing \((i, j)\)-semiclosed subset of \(X\) containing \(A\). Therefore, \(I^{(i,j)s}(A) \subset X \setminus U\). Thus \(x \notin I^{(i,j)s}(A)\). Conversely, if \(x \notin I^{(i,j)s}(A)\). Then \(x \setminus I^{(i,j)s}(A)\) is a decreasing \((i, j)\)-semiopen subset of \(X\) containing \(x\), but disjoint from \(A\). The case of \(D^{o(i,j)−s}(A)\) can be deal similarly.

**Definition 11.** A bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) is said to be upper (resp. lower) \((i, j)\)-semi-\(T_1\)-ordered if for each pair of elements \(a \nleq b\) (that is, \(a\) is not related to \(b\)) in \(X\) there exists a decreasing (resp. increasing) \((i, j)\)-semiopen set \(U\) containing \(b\) (resp. \(a\)) such that \(a \notin U\) (resp. \(b \notin U\)). \((X, \tau_1, \tau_2, \leq)\) is said to be \((i, j)\)-semi-\(T_1\)-ordered if it is both lower and upper \((i, j)\)-semi-\(T_1\)-ordered.

Clearly every \(T_1\)-ordered space is \((i, j)\)-semi-\(T_1\)-ordered and every \((i, j)\)-semi-\(T_1\)-ordered space is \((i, j)\)-semi-\(T_1\) but the converses are not true, in general.

**Example 12.** Let \(X = \{a, b, c\}\) be equipped with the topologies \(\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{\emptyset, X, \{a, b\}\}\) and with the partial order \(\leq\) defined as \(a \leq a, a \leq b, a \leq c, b \leq b, c \leq b, c \leq c\). Then \((X, \tau_1, \tau_2, \leq)\) is an \((i, j)\)-semi-\(T_1\) space but not \(\tau_1\)-\(T_1\)-ordered.

**Example 13.** Let \(X = \{a, b, c\}\) be equipped with the topologies \(\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{\emptyset, X, \{a, b\}\}\) and with the partial order \(\leq\) defined as \(a \leq a, a \leq b, a \leq c, b \leq b, c \leq b, c \leq c\). Then \((X, \tau_1, \tau_2, \leq)\) is an \((i, j)\)-semi-\(T_1\) space but not \((i, j)\)-semi-\(T_1\) ordered.

**Theorem 14.** For a bitopological ordered space \((X, \tau_1, \tau_2, \leq)\), the following statements are equivalent:

1. \((X, \tau_1, \tau_2, \leq)\) is lower (resp. upper) \((i, j)\)-semi-\(T_1\)-ordered,

2. for each pair \(a \nleq b\) of \(X\), there exists an \((i, j)\)-semiopen set \(U\) containing \(a\) (resp. \(b\)) such that \(x \nleq b\) (resp. \(a \nleq x\)) for all \(x \in U\),

3. for each \(x \in X, [\leftarrow, x]\) (resp. \([x, \rightarrow]\)) is \((i, j)\)-semiclosed,
4. when the net \( \{x_\alpha\}_{\alpha \in A} \) \((i,j)\)-semiconverges to \( a \) and \( x_\alpha \leq b \) (resp. \( b \leq x_\alpha \)) for all \( \alpha \in A \), then \( a \leq b \) (resp. \( b \leq a \)).

**Proof.** We shall prove the theorem for lower \((i,j)\)-semi-\(T_1\)-ordered spaces only. (1) \(\Rightarrow\) (2): Let \( a \not\in b \). Then by hypothesis, there exists an increasing \((i,j)\)-semiopen set \( U \) containing \( a \) such that \( b \not\in U \). If \( x \in U \) and \( x \leq b \), then \( b \in U \), a contradiction.

(2) \(\Rightarrow\) (3): Let \( y \in X \setminus \downarrow \). Then \( y \not\in x \). Then there exists an \((i,j)\)-semiopen set \( U \) containing \( y \) such that \( u \not\in x \) for all \( u \in U \). That is, \( y \in U \subset X \setminus \downarrow \).

Hence \( \downarrow \) is \((i,j)\)-semiclosed.

(3) \(\Rightarrow\) (1): Obvious.

(1) \(\Rightarrow\) (4): Let \( \{x_\alpha\}_{\alpha \in A} \) be a net in \( X \) \((i,j)\)-semiconverges to \( a \) and \( x_\alpha \leq b \) for all \( \alpha \in A \). If possible \( a \not\in b \), then by hypothesis, \( X \setminus \downarrow \) is an \((i,j)\)-semiopen set containing \( a \). Then there exists \( \lambda \in A \) such that \( x_\alpha \in X \setminus \downarrow \) for all \( \alpha \leq \lambda \). That is \( x_\alpha \not\in b \) for all \( \alpha \geq \lambda \), which is a contradiction.

(4) \(\Rightarrow\) (1): Let \( X \) be not lower \((i,j)\)-semi-\(T_1\)-ordered. Then there exists a pair \( a \not\in b \) in \( X \) such that for every \((i,j)\)-semiopen set \( U \) containing \( a \), there exists \( x_u \in U \) with \( x_u \leq b \). Let \( U \) be the collection of all \((i,j)\)-semiopen sets containing \( a \), then \( U \) is directed by inclusion and the net \( \{x_u\}_{u \in U} \) \((i,j)\)-semiconverges to \( a \). By hypothesis \( a \leq b \), which is a contradiction.

**Corollary 15.** If \((X, \tau_1, \tau_2, \leq)\) is lower (upper) \((i,j)\)-semi-\(T_1\)-ordered and \( \tau_1 \leq \tau_1^* \), then \((X, \tau_1^*, \tau_2^*, \leq)\) is also lower (upper) \((i,j)\)-semi-\(T_1\)-ordered.

**Theorem 16.** Every biopen subspace of an \((i,j)\)-semi-\(T_1\)-ordered space is \((i,j)\)-semi-\(T_1\)-ordered.

**Proof.** Let \((X, \tau_1, \tau_2, \leq)\) be an \((i,j)\)-semi-\(T_1\)-ordered space and \( A \) a biopen subspace of \( X \). Let \( x \in X \). Then by Theorem 9 (3), \( X \setminus \downarrow \) and \( X \setminus \rightarrow \) are \((i,j)\)-semiopen sets in \( X \). Since \( A \) is biopen, \( A \cap (X \setminus \downarrow) \) and \( A \cap (X \setminus \rightarrow) \) are \((i,j)\)-semiopen in \( A \). But \( A \cap (X \setminus \downarrow) = A \setminus \downarrow \) and \( A \cap (X \setminus \rightarrow) = A \setminus \rightarrow \). Hence, again by Theorem 9 (3) \( A \) is \((i,j)\)-semi-\(T_1\)-ordered.

**Theorem 17.** A bitopological space \((X, \tau_1, \tau_2, \leq)\) is \((i,j)\)-semi-\(T_1\)-ordered if and only if for each \( x \in X \), there exists an \((i,j)\)-semi-\(T_1\)-ordered \((i,j)\)-semiopen set in \( X \) which is both increasing and decreasing containing \( x \).

**Proof.** If \( X \) is \((i,j)\)-semi-\(T_1\)-ordered, then \( X \) is the required set for each \( x \in X \). Conversely, let \( a \not\in b \) in \( X \). Then, by hypothesis, there exist \((i,j)\)-semi-\(T_1\)-ordered \((i,j)\)-semiopen sets \( U_1 \) and \( U_2 \) in \( X \) containing \( a \) and \( b \), respectively, where \( U_1 \) and \( U_2 \) are both increasing and decreasing sets. If \( b \not\in U_1 \) and \( a \not\in U_2 \),
then there is nothing to prove. But if \(b \in U_1\) (resp. \(a \in U_2\)), then there exist \((i, j)\)-semiopen sets \(V\) and \(W\) in \(U_1\) (resp. \(U_2\)) containing \(a\) and \(b\), respectively. \(V\) is increasing and \(W\) is decreasing in \(U_1\) (resp. \(U_2\)) also \(b \notin V\) and \(a \notin W\).

Since \(U_1\) (resp. \(U_2\)) is both increasing and decreasing and it also an \((i, j)\)-semiopen subset of \(X\), \(V\) is an increasing and \(W\) is a decreasing \((i, j)\)-semiopen subsets of \(X\). Hence \((X, \tau_1, \tau_2, \leq)\) is \((i, j)\)-semi-\(T_1\)-ordered.

**Theorem 18.** Let \(f\) be an order preserving (that is, \(x \leq y \in X\) if and only if \(f(x) \leq f(y)\) in \(X^*\)) \((i, j)\)- irresolute function from a bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) to a bitopological ordered space \((X^*, \tau^*, \leq^*)\). If \((X^*, \tau^*, \leq^*)\) is \((i, j)\)-semi-\(T_1\)-ordered, then \((X, \tau_1, \tau_2, \leq)\) is also \((i, j)\)-semi-\(T_1\)-ordered.

**Proof.** Let \(a \notin b\) in \(X\). Then \(f(a) \notin f(b)\) in \(X\). Then there exists an increasing \((i, j)\)-semiopen set \(U^*\) in \(X^*\) containing \(f(a)\) but \(f(b) \notin U^*\). Since \(f\) is order preserving and \((i, j)\)- irresolute, \(f^{-1}(U^*) = U\) is an increasing \((i, j)\)-semiopen subset of \(X\). Also, \(a \in U\) and \(b \notin U\). Hence \((X, \tau_1, \tau_2, \leq)\) is lower \((i, j)\)-semi-\(T_1\)-ordered. Similarly, we can prove \((X, \tau_1, \tau_2, \leq)\) is upper \((i, j)\)-semi-\(T_1\)-ordered.

**Theorem 19.** The product of a family of \((i, j)\)-semi-\(T_1\)-ordered spaces is also \((i, j)\)-semi-\(T_1\)-ordered.

**Proof.** Let \(\{(X_\lambda, \tau_\lambda, \leq_\lambda) : \lambda \in \Lambda\}\) be a family of \((i, j)\)-semi-\(T_1\)-ordered space and \((X, \tau_1, \tau_2, \leq)\) be the product ordered space. Let \(a = (a_\lambda)\) and \(b = (b_\lambda)\) be two elements of \(X\) such that \(a \notin b\). Then there exists \(\mu \in \Lambda\) such that \(a_\mu \notin b_\mu\). Since \((X, \tau_1, \tau_2, \leq)\) is an \((i, j)\)-semi-\(T_1\)-ordered space, there exists an \((i, j)\)-semiopen set \(U_\mu\) such that \(a_\mu \in U_\mu\) but \(b_\mu \in U_\mu\). Define \(U = \prod U_\mu\), where \(U_\lambda = X_\lambda\) if \(\lambda \neq \mu\). Then \(U\) is an increasing \((i, j)\)-semiopen subset of \(X\) containing \(a\) but not \(b\). Therefore, \((X, \tau_1, \tau_2, \leq)\) is lower \((i, j)\)-semi-\(T_1\)-ordered. Similarly, we can prove \((X, \tau_1, \tau_2, \leq)\) is upper \((i, j)\)-semi-\(T_1\)-ordered.

**Definition 20.** A bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) is said to be \((i, j)\)-semi-\(T_2\)-ordered if for each pair of elements \(a \notin b\) in \(x\) there exist disjoint \((i, j)\)-semiopen sets \(U\) and \(V\) in \(X\) containing \(a\) and \(b\), respectively, \(U\) is increasing and \(V\) is decreasing.

**Example 21.** Let \(X\) be an infinite set, \(\tau_1 = tau_2\) a finite complement topology with discrete order. Then \((X, \tau_1, \tau_2, \leq)\) is an \((i, j)\)-semi-\(T_1\) ordered but not \((i, j)\)-semi-\(T_2\)-ordered.

**Example 22.** Let \(X = \{a, b, c\}\) be equipped with the topologies \(\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\), \(\tau_2 = \{\emptyset, X, \{a\}\}\) and with the partial order \(\leq\) defined
as \( a \leq a, \ a \leq b, \ a \leq c, \ b \leq b, c \leq b, c \leq c \). Then \((X, \tau_1, \tau_2, \leq)\) is an \((i, j)\)-semi-\(T_2\) space but not \(T_2\)-ordered.

**Example 23.** Let \(X = \{a, b, c\}\) be equipped with the topologies \(\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{\emptyset, X, \{a, b\}\}\) and with the partial order \(\leq\) defined as \(a \leq a, \ a \leq b, a \leq c, b \leq b, c \leq b, c \leq c\). Then \((X, \tau_1, \tau_2, \leq)\) is an \((i, j)\)-semi-\(T_2\) space but not \((i, j)\)-semi-\(T_2\) ordered.

**Example 24.** Let \(X = \{a, b, c\}\) be equipped with the topologies \(\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}\), \(\tau_2 = \mathcal{P}(X)\) and with the partial order \(\leq\) defined as \(a \leq a, \ a \leq b, a \leq c, b \leq b, c \leq b, c \leq c\). Then \((X, \tau_1, \tau_2, \leq)\) is an \((i, j)\)-semi-\(T_1\) ordered but not \((i, j)\)-semi-\(T_2\) ordered.

**Theorem 25.** A bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) is \((i, j)\)-semi-\(T_2\)-ordered if and only if for each \(x \in X\), there exists an increasing (resp. decreasing) \((i, j)\)-semiclopen subset of \(X\) containing \(x\) which is \((i, j)\)-semi-\(T_2\)-ordered.

**Proof.** If \(X\) is \((i, j)\)-semi-\(T_2\)-ordered, then \(X\) is the required increasing (resp. decreasing) \((i, j)\)-semiclopen subset of \(X\) for all \(x \in X\). Conversely, let \(x \notin y\) in \(X\). By hypothesis, there exists an increasing \((i, j)\)-semiclopen set \(V\) in \(X\) containing \(x\). If \(y \in V\), then nothing to prove. If \(y \notin V\). Then \(X \setminus V\) is a decreasing \((i, j)\)-semiclopen subset of \(X\) containing \(y\). Hence \((X, \tau_1, \tau_2, \leq)\) is \((i, j)\)-semi-\(T_2\)-ordered. Dually, we can prove the theorem for decreasing \((i, j)\)-semiclopen subsets of \(X\). \(\Box\)

**Theorem 26.** Let \(\{(X_\alpha, \tau_1^\alpha, \tau_2^\alpha, \leq_\alpha) : \alpha \in \Lambda\}\) be an arbitrary family of topological order spaces. Then the product space \(X = \prod_{\alpha \in \Lambda} X_\alpha\) equipped with the product topology \(\tau\) and with the partial order \(\leq\), defined as \((x_\alpha)_{\alpha \in \Lambda}\). Then, in the product ordered space,

\[
\begin{align*}
I^{(i,j)s}(\prod_{\alpha \in \Lambda} A_\alpha) &\subseteq \prod_{\alpha \in \Lambda} I^{(i,j)s}_{X_\alpha}(A_\alpha) \\
D^{(i,j)s}(\prod_{\alpha \in \Lambda} A_\alpha) &\subseteq \prod_{\alpha \in \Lambda} D^{(i,j)s}_{X_\alpha}(A_\alpha)
\end{align*}
\]

**Proof.** Let \(x = (x_\alpha) \notin I^{(i,j)s}(\prod_{\alpha \in \Lambda} A_\alpha)\) and \(U_\beta\) be a decreasing \((i, j)\)-semiopen subset of \(X_\beta\) containing \(x_\beta\) for some \(p \in \Lambda\). Define, \(U = \prod_{\alpha \in \Lambda} U_\alpha\), where \(U_\alpha = X_\alpha\) if \(\alpha \neq \beta\). Then, \(U\) is a decreasing \((i, j)\)-semiopen subset of \(\prod_{\alpha \in \Lambda} X_\alpha\) containing \(x\). Therefore, by Lemma 10, \(U \cap \prod_{\alpha \in \Lambda} A_\alpha \neq \emptyset\) that is, \((U_\beta \cap A_\beta) \times \prod_{\alpha \neq \beta} A_\alpha \neq \emptyset\).
Thus, \( U_\beta \cap A_\beta \neq \emptyset \). Again, by Lemma 10, \( x_\beta \in I_{X_\beta}^{(i,j)\alpha}(A_\beta) \). Similarly we can prove that \( D^{(i,j)\beta}(\prod_{\alpha \in \Lambda} A_\alpha) \subset \prod_{\alpha \in \Lambda} D^{(i,j)\alpha}(A_\alpha) \)

\[ \square \]

**Theorem 27.** The product of a family of \((i,j)\)-semi-\(T_2\)-ordered spaces is also \((i,j)\)-semi-\(T_2\)-ordered.

**Proof.** Let \( \{(X_\lambda, \tau_\lambda, \leq_\lambda) : \lambda \in \Lambda \} \) be a family of \((i,j)\)-semi-\(T_2\)-ordered space and \((X, \tau_1, \tau_2, \leq)\) be the product ordered space. If \((x_\lambda), (y_\lambda) \in X\) such that \((x_\lambda) \not\leq (y_\lambda)\). Then, there exists \( \mu \in \Lambda \) such that \( x_\mu \not\leq y_\mu \). Thus, there exist disjoint \((i,j)\)-semiopen sets \( U_\mu \) and \( V_\mu \) in \( X_\mu \) containing \( x_\mu \) and \( y_\mu \) respectively, where \( U_\mu \) is increasing and \( V_\mu \) is decreasing. Define, \( U = \prod_{\lambda \in \Lambda} U_\lambda \) such that \( U_\mu = X_\lambda \) if \( \lambda \neq \mu \) and \( U = \prod_{\lambda \in \Lambda} V_\lambda \) such that \( V_\mu = X_\lambda \) if \( \lambda \neq \mu \). Then \( U \) is an increasing \((i,j)\)-semiopen subset of \( X \) containing \((x_\lambda)\) and \( V \) is a decreasing \((i,j)\)-semiopen subset of \( X \) containing \((y_\lambda)\). Clearly, \( U \) and \( V \) are disjoint. Hence \((X, \tau_1, \tau_2, \leq)\) is \((i,j)\)-semi-\(T_2\)-ordered.

\[ \square \]

**Theorem 28.** If \((X, \tau_1, \tau_2, \leq)\) is \((i,j)\)-semi-\(T_2\)-ordered, then the graph of the order of \( X \) is \( \{(a, b) \in X \times X : a \leq b\} \) an \((i,j)\)-semiclosed subset of \( X \times X \).

**Proof.** Let \( G \) be the graph of the order of \( X \) and \((x, y) \in (X \times X) \setminus G\). Then \( x \not\leq y \). By hypothesis, there exist disjoint \((i,j)\)-semiopen sets \( U \) and \( V \) such that \( X \in U, U \) is increasing and \( y \in V, V \) is decreasing. Therefore, \( U \times V \) is an \((i,j)\)-semiopen subset of \( X \times X \) containing \((x, y)\). Also \( U \times V \cap G = \emptyset \). Thus, \((x, y) \in U \times V \subset (X \times X) \setminus G\). Hence \( F \) is \((i,j)\)-semiclosed.

\[ \square \]

**Theorem 29.** A bitopological ordered space \((X, \tau_1, \tau_2, \leq)\) is \((i,j)\)-semi-\(T_2\) if and only if for each \( x \in X \) the intersection of all increasing (resp. decreasing) \((i,j)\)-semiclosed \( i\)-(\(i,j\))-semineighbourhoods (resp. \( d\)-(\(i,j\))-semineighbourhoods) of \( x \) is \([x^*, \to]\) (resp. \([\leftarrow, x]\)).

**Proof.** Let \((X, \tau_1, \tau_2, \leq)\) be an \((i,j)\)-semi-\(T_2\)-ordered space and \( x \in X \). If \( G^* = \cap \{F : F \text{ is an increasing (i,j)-semiclosed i-(i,j)-semineighbourhood of } x\} \). Clearly, \([x^*, \to]\) \( \subset G^* \). Let \( y \notin [x^*, \to]\). Then \( x \not\leq y \). Then there exist disjoint \((i,j)\)-semiopen sets \( U \) and \( V \) containing \( x \) and \( y \), respectively such that \( U \) is increasing and \( V \) is decreasing. Hence \( x \in U \subset X \setminus V \). Therefore, \( X \setminus V \) is an increasing \((i,j)\)-semiclosed \( i\)-(\(i,j\))-semineighbourhood of \( x \) and \( y \notin X \setminus V \). Hence \( G^* = [x^*, \to]\). Similarly, we can show that intersection of all decreasing \((i,j)\)-semiclosed \( i\)-(\(i,j\))-semineighbourhoods of \( x \) is \([\leftarrow, x]\). Conversely, let \( x \leq y \)
in $X$. Then $y \notin [x^*, \rightarrow]$. By hypothesis, there exists an increasing $(i, j)$-semiclosed $i$-$(i, j)$-semineighbourhood $F$ of $x$ such that $y \notin F$. Then $y \in X \setminus F$, $X \setminus F$ is a decreasing $(i, j)$-semiopen set. Also, there exists an increasing $(i, j)$-semiopen set $U$ such that $x \in U \subset F$. Clearly, $U (X \setminus F) = \emptyset$. Hence $(X, \tau_1, \tau_2, \leq)$ is $(i, j)$-semi-$T_2$-ordered. By dual argument we can prove that $(i, j)$-semi-$T_2$-ordered if intersection of all decreasing $(i, j)$-semiclosed $d$-$(i, j)$-semineighbourhoods of $x$ is $[x^*, \rightarrow]$ for all $x \in X$.

References
