ON gar -CLOSED MAP IN TOPOLOGICAL SPACES

S. Sekar¹ §, G. Kumar²

¹Department of Mathematics

Government Arts College (Autonomous)

Salem, 636 007, Tamil Nadu, INDIA

²Department of Mathematics

A.E.T. College

Attur, 636 108, Tamil Nadu, INDIA

Abstract: In this paper, we introduce a new class of generalized α regular-closed map in topological spaces (briefly gar -closed map) and study some of their properties as well as inter relationship with other closed maps.

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Key Words: gar -closed set, b -closed map, gb -closed map, rgb -closed map and gp^* -closed map

1. Introduction

Different types of Closed and open mappings were studied by various researchers. In 1996, Andrijevic introduced new type of set called b -open set. A.A.Omari and M.S.M. Noorani [1] introduced and studied b -closed map.

The aim of this paper is to introduce generalized α regular-closed map and to continue the study of its relationship with various generalized closed maps. Through out this paper (X, τ) and (Y, σ) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all b - open sets X contained in A is called b -

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§Correspondence author

interior of A and it is denoted by $bint(A)$, the intersection of all b - closed sets of X containing A is called b - closure of A and it is denoted by $bcl(A)$.

2. Preliminaries

In this section, we referred some of the closed set definitions which was already defined by various authors.

Definition 2.1. Let a subset of A of a topological space (X, τ) is called

1. α - open set [9] if $A \subseteq int(cl(int(A)))$.
2. generalized closed set (briefly g - closed) [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
3. weakly closed set (briefly w - closed) [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open.
4. generalized $*$ closed set (briefly g - closed) [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g - open.
5. generalized α - closed set (briefly $g\alpha$ - closed) [8] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α - open.
6. An α - generalized closed set (briefly αg - closed) [7] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
7. generalized b - closed set (briefly gb - closed) [1] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .
8. semi generalized b - closed set (briefly sgb - closed) [4] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X .
9. generalized αb closed set (briefly gab - closed) [16] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is α open in X .
10. regular generalized b - closed set (briefly rgb - closed) [9] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X .
11. generalized pre regular closed set (briefly gpr - closed) [3] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X .
12. generalized α regular closed set (briefly gor - closed) [12] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X .

3. On Generalized α Regular-Closed Map

In this section, we introduce generalized α regular - closed map (gar - closed map) in topological spaces by using the notions of gar - closed sets and study some of their properties.

Definition 3.1. Let X and Y be two topological spaces. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called generalized α regular - closed (briefly, gar - closed map) if the image of every closed set in X is gar -closed in Y .

Theorem 3.2. Every closed map is gar - closed but not conversely.

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ is closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gar - closed in Y . Then f is gar - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.3. Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a$. The map is gar - closed but not closed as the image of and $\{a, c\}$ in X is $\{b, c\}$ is not closed in Y .

Theorem 3.4. Every $g\alpha$ - closed map is gar - closed set but not conversely.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ is $g\alpha$ - closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gar - closed in Y . Then f is gar - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.5. Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a, f(b) = c, f(c) = b$. The map is gar - closed but not $g\alpha$ - closed as the image of $\{c\}$ in X is $\{b\}$ is not $g\alpha$ - closed in Y .

Theorem 3.6. Every αg - closed map is gar - closed but not conversely.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be αg -closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gar - closed in Y . Then f is gar - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.7. Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{c\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. The map is gar - closed but not αg - closed as the image of $\{b, c\}$ in X is $\{b, c\}$ is not αg -closed in Y .

Theorem 3.8. Every gar - closed map is gpr - closed but not conversely.

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ be gar - closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gpr - closed in Y . Then f is gpr - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.9. Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c, f(b) = a, f(c) = b$. The map is gpr - closed but not gar - closed as the image of $\{c\}$ in X is $\{b\}$ is not gar - closed in Y .

Theorem 3.10. Every g - closed map is gar - closed but not conversely.

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ g - closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gar - closed in Y . Then f is gar - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.11. Consider $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a, f(b) = c, f(c) = b$. The map is gar -closed but not g -closed as the image of $\{b, c\}$ in X is $\{b, c\}$ is not g -closed in Y .

Theorem 3.12. Every w - closed map is gar - closed but not conversely.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be w - closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gar - closed in Y . Then f is gar - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.13. Consider $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c, f(b) = a, f(c) = b$. The map is gar - closed but not w - closed as the image of and $\{c\}$ in X is $\{b\}$ is not w - closed in Y .

Theorem 3.14. *Every g - closed map is gar - closed but not conversely.*

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ be g - closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gar - closed in Y . Then f is gar - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.15. Consider $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{b, c\}, \}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a$. The map is gar - closed but not g - closed as the image of $\{b, c\}$ in X is $\{a, c\}$ is not g - closed in Y .

Theorem 3.16. *Every $g\alpha$ - closed map is gar - closed but not conversely.*

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ be $g\alpha$ - closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence gar - closed in Y . Then f is gar - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.17. Consider $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = a, f(c) = c$. The map is gar - closed but not $g\alpha$ - closed as the image of $\{b, c\}$ in X is $\{a, c\}$ is not gar -closed in Y .

Theorem 3.18. *Every gar - closed map is rgb - closed but not conversely.*

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ be gar - closed map and V be an closed set in X then $f(V)$ is closed in Y . Hence rgb - closed in Y . Then f is rgb - closed. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 3.19. Consider $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = a, f(c) = c$. The map is rgb -closed but not gar - closed as the image of $\{a\}$ in X is $\{a\}$ is not gar -closed in Y .

Theorem 3.20. *A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous and gar - closed set A is gar -closed set of X then $f(A)$ is gar closed in Y .*

Proof. Let $f(A) \subseteq U$ where U is regular open set in Y . Since f is continuous, $f^{-1}(U)$ is open set containing A . Hence $\alpha cl(A) \subseteq f^{-1}(U)$ (as A is gar - closed). Since f is gar - closed $f(\alpha cl(A)) \subseteq U$ is gar closed set $\Rightarrow \alpha cl(f(bcl(A))) \subseteq U$, Hence $\alpha cl(A) \subseteq U$. So that $f(A)$ is gar - closed set in Y . $\square$

Theorem 3.21. *If a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous and closed set and A is gar - closed then $f(A)$ is gar - closed in Y .*

Proof. Let F be a closed set of A then F is gar - closed set. By theorem 3.20 $f(A)$ is gar - closed. Hence $f_A(F) = f(F)$ is gar - closed set of Y . Here f_A is gar - closed and also continuous. $\square$

Theorem 3.22. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is closed map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is gar - closed map, then the composition $g \cdot f : (X, \tau) \rightarrow (Z, \eta)$ is gar - closed map.*

Proof. Let F be any closed set in (X, τ) . Since f is closed map, $f(F)$ is closed set in (Y, σ) . Since g is gar - closed map, $g(f(F))$ is gar - closed set in (Z, η) . That is $g \cdot f(F) = g(f(F))$ is gar closed. Hence $g \cdot f$ is gar closed map. $\square$

Remark 3.23. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is gar - closed map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is closed map, then the composition need not gar - closed map.*

Theorem 3.24. *A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is gar - closed if and only if for each subset S of (Y, σ) and each open set U containing $f^{-1}(S) \subset U$, there is a gar - open set V of (Y, σ) such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. Suppose f is gar - closed. Let $S \subset Y$ and U be an open set of (X, τ) such that $f^{-1}(S) \subset U$. Now $X - U$ is closed set in (X, τ) . Since f is gar - closed, $f(X - U)$ is gar - closed set in (Y, σ) . There fore $V = Y - f(X - U)$ is an gar - open set in (Y, σ) . Now $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - V) = V$. (ie) $f^{-1}(V) \subset U$.

Conversely,

Let F be a closed set of (X, τ) . Then $f^{-1}(f(F^c)) \subset F^c$ and F^c is an open in (X, τ) . By hypothesis, there exist a gar - open set V in (Y, σ) such that $f(F^c) \subset V$ and $f^{-1}(V) \subset F^c \Rightarrow F \subset f^{-1}(V)^c$. Hence $V^c \subset f(F) \subset f(((f^{-1}(V))^c)^c) \subset V^c \Rightarrow f(V) \subset V^c$. Since V^c is gar - closed, $f(F)$ is gar - closed. (ie) $f(F)$ is gar - closed in Y . Therefore f is gar - closed. $\square$

Theorem 3.25. *If $f : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined as $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$, then $f : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is gar closed map.*

Proof. Let $U_1 \times U_2 \subset X_1 \times X_2$ where $U_i \in garcl(X_i)$, for $i = 1, 2$. Then $f(U_1 \times U_2) = f_1(U_1) \times f_2(U_2) \in garcl(X_1 \times X_2)$. Hence f is gar - closed set. $\square$

Theorem 3.26. *Let $h : X \rightarrow X_1 \times X_2$ be gar - closed map and Let $f_i : X \times X_i$ be define as $h(x) = (x_1, x_2)$ and $f_i(x) = x_i$, then $f_i : X \times X_i$ is gar - closed map for $i = 1, 2$.*

Proof. Let $U_1 \times U_2 \in X_1 \times X_2$, then $f_1(U_1) = h_1(U_1 \times X_2) \in garcl(X)$, there fore f_1 is gar - closed. Similarly we have f_2 is gar - closed. Thus f_i is gar - closed map for $i = 1, 2$. $\square$

Theorem 3.27. *For any bijection map $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

- (i) $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is gar - continuous.
- (ii) f is gar - open map.
- (iii) f is gar - closed map.

Proof. (i) $\Rightarrow$ (ii) Let U be an open set of (X, τ) . By assumption, $(f^{-1})^{-1}(U) = f(U)$ is gar - open in (Y, σ) and so f is gar - open.

(ii) $\Rightarrow$ (iii) Let F be a closed set of (X, τ) . Then F^c is open set in (X, τ) . By assumption $f(F^c)$ is gar - open in (Y, σ) . Therefore $f(F^c) = f(F)^c$ is gar - open in (Y, σ) . That is $f(F)$ is gar - closed in (Y, σ) . Hence f is gar - closed.

(iii) $\Rightarrow$ (i) Let F be a closed set of (X, τ) . By assumption, $f(F)$ is gar - closed in (Y, σ) . But $f(F) = (f^{-1})^{-1}(F) \Rightarrow (f^{-1})$ is continuous. $\square$

4. On Generalized α Regular-Open Map

In this section, we introduce generalized α regular - open map (gar - open map) in topological spaces by using the notions of gar - open sets and study some of their properties.

Definition 4.1. Let X and Y be two topological spaces. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called generalized α regular - open (briefly, gar - open) if the image of every open set in X is gar - open in Y .

Theorem 4.2. *Every open map is gar - open but not conversely.*

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ is open map and V be an open set in X then $f(V)$ is open in Y . Hence $g\alpha r$ - open in Y . Then f is $g\alpha r$ - open. $\square$

The converse of above theorem need not be true as seen from the following example.

Example 4.3. Consider $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a$. The map is $g\alpha r$ - open but not open as the image of and $\{a, c\}$ in X is $\{b, c\}$ is not open in Y .

Theorem 4.4. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is $g\alpha r$ - closed set if and only if for each subset S of Y and for each open set U containing $f^{-1}(S) \subset U$ there is a $g\alpha r$ - open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose f is $g\alpha r$ - closed set. Let $S \subset Y$ and U be an open set of (X, τ) such that $f^{-1}(S) \subset U$. Now $X - U$ is closed set in (X, τ) . Since f is $g\alpha r$ closed, $f(X - U)$ is $g\alpha r$ closed set in (Y, σ) . Then $V = Y - f(X - U)$ is $g\alpha r$ open set in (Y, σ) . There fore $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - V) = U$. (ie) $f^{-1}(V) \subset U$.

Conversely,

Let F be a closed set of (X, τ) . Then $f^{-1}(f(F^c)) \subset F^c$ and F^c is an open in (X, τ) . By hypothesis, there exists a $g\alpha r$ open set V in (Y, σ) such that $f(F^c) \subset V$ and $f^{-1}(V) \subset F^c \Rightarrow F \subset (f^{-1}(V))^c$. Hence $V^c \subset f(F) \subset f(((f^{-1}(V))^c)^c) \subset V^c \Rightarrow f(V) \subset V^c$. Since V^c $g\alpha r$ - closed, $f(F)$ is $g\alpha r$ - closed. (ie) $f(F)$ is $g\alpha r$ - closed in (Y, σ) and there fore f is $g\alpha r$ - closed. $\square$

Theorem 4.5. For any bijection map $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent.

- (i) $f^{-1} : (X, \tau) \rightarrow (Y, \sigma)$ is $g\alpha r$ - continuous.
- (ii) f is $g\alpha r$ open map.
- (iii) f is $g\alpha r$ - closed map.

Proof. (i) $\Rightarrow$ (ii) Let U be an open set of (X, τ) . By assumption $(f^{-1})^{-1}(U) = f(U)$ is $g\alpha r$ - open in (Y, σ) . There fore f is $g\alpha r$ - open map.

(ii) $\Rightarrow$ (iii) Let F be closed set of (X, τ) , Then F^c is open set in (X, τ) . By assumption, $f(F^c)$ is $g\alpha r$ - open in (Y, σ) . There fore $f(F)$ is $g\alpha r$ - closed in (Y, σ) . Hence f is $g\alpha r$ - closed.

(iii) $\Rightarrow$ (i) Let F be a closed set of (X, τ) , By assumption $f(F)$ is $g\alpha r$ - closed in (Y, σ) . But $f(F) = (f^{-1})^{-1}(F)$. Hence $f^{-1} : (X, \tau) \rightarrow (Y, \sigma)$ is $g\alpha r$ -continuous. $\square$

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