THE M -HYPERCYCLICITY CRITERION OF C_0 -SEMIGROUPS

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Abstract: The goal in this paper is to give a sufficient conditions for a C_0 -semigroup acting on complex separable infinite dimensional Banach space X to be M -hypercyclic, this is the M -hypercyclicity criterion. Moreover we characterize the C_0 -semigroup satisfying this criterion.

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1. Introduction and Preliminaries

A bounded operator T on a separable Banach space X is hypercyclic if there is a vector $x \in X$ such that $\text{Orb}(T, x) = \{T^n x, n \in \mathbb{N}\}$ is dense in X . For example, the derivation operators [5], the non-trivial translation operators on the space of entire functions are hypercyclic [16], and if B is the backward shift on $l^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$ [19]. We refer to [2], [12], [13] for much more detail on hypercyclicity.

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At 1997, Desch and al [8] introduced the hypercyclicity of C_0 -semigroups, a C_0 -semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ acting on X is called hypercyclic if there is a vector $x \in X$ such that $\text{Orb}(\mathcal{T}, x) = \{T_t x, t \geq 0\}$ is dense in X . Note that if there exist $t_0 > 0$ such that T_{t_0} is hypercyclic, then $\mathcal{T}$ is hypercyclic. It was shown by Oxtoby and Ulam [17] that for any C_0 -semigroups on Banach space X , there is a J interval G_δ dense in $\mathbb{R}^+$ such that, for all $t \in J$, T_t is hypercyclic. At 2007, Müller and al [6] showed that there is an equivalence between the hypercyclicity of $\mathcal{T} = (T_t)_{t \geq 0}$ and the hypercyclicity of T_t for all $t > 0$.

In [4] Bernal-Conzalez and al established a sufficient condition (namely Hypercyclicity Criterion) for a C_0 -semigroup to be hypercyclic. A C_0 -semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ on Banach space X satisfies the Hypercyclicity criterion if there is two dense subsets X_0 and Y_0 of X and increasing sequence $(t_n) \subset [0, +\infty[$ such that:

- i) $T_{t_n} x \rightarrow 0$ for all $x \in X_0$.
- ii) For every $y \in Y_0$ there is $u_n \rightarrow 0$ such that $T_{t_n} u_n \rightarrow y$.

In [9], [3] and [4] the authors proved that every C_0 -semigroups satisfies the hypercyclicity criterion is hypercyclic, recall that [4] Bernal and Grosse characterized this criterion, they showed that the C_0 -semigroup $(T_t)_{t \geq 0}$ satisfies the Hypercyclicity criterion if and only if the C_0 -semigroup $(T_t \oplus T_t)_{t \geq 0}$ is hypercyclic in $X \oplus X$, if and only if for every pair U, V of non-empty open sets of X and every neighborhood W of zero, there is some $t > 0$ such that $T_t(U) \cap W \neq \emptyset$ and $T_t(W) \cap V \neq \emptyset$.

At 2011, Madore and Martinez-Avendano [15] introduced a new concept of hypercyclicity, and studied the density of the orbit in a non trivial subspace instead of the whole space, and called this phenomena the subspace-hypercyclicity. Let a bounded operator T on separable Banach space X and $M \neq \{0\}$ a subspace of X , T is called subspace-hypercyclic (or M -hypercyclic) if there is a vector $x \in X$ such that $\text{Orb}(T, x) \cap M$ is dense in M . Madore, Martinez-Avendano [15] and Rezaei [18] proved that if T is M -transitive, then T is M -hypercyclic, recall that a operator is M -transitive if for each pair of non-empty open sets of M there is $n \in \mathbb{N}$ such that $T^n(U) \cap V$ contains a non-empty relatively open set in M . Madore, Martinez-Avendano [15] proved that the converse is not always true, there are subspace-hypercyclic operators that not subspace-transitive (see [10], [14] and [22]).

In [20] we investigated and studied the hypercyclicity in non trivial subspace for the C_0 -semigroups. Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on separable Banach space X , $\mathcal{T}$ is called M -hypercyclic for M a subspace of X if there is a vector

$x \in X$ such that $\text{Orb}(\mathcal{T}, x) \cap M$ is dense in M . Observe that the definition above reduces to the classical definition of hypercyclicity if $M = X$, again if $(A_t)_{t \geq 0}$ is hypercyclic C_0 -semigroup in X , then the C_0 -semigroup $T_t := A_t \oplus I$ is $(X \oplus \{0\})$ -hypercyclic but not hypercyclic, and if x is a hypercyclic vector of $(A_t)_{t \geq 0}$ then $x \oplus 0$ is a $(X \oplus \{0\})$ -hypercyclic vector of $(T_t)_{t \geq 0}$. On the other hand we proved that if $(T_t)_{t \geq 0}$ is M -hypercyclic then for every $\lambda \in \mathbb{C}$ and $t > 0$ we have $\ker(T_t^* - \lambda I) \subset M^\perp$ (T_t^* the adjoint of T_t), and we were giving a condition for a C_0 -semigroup to be M -hypercyclic. In [21] we showed that in every separable infinite dimensional Banach space there is a C_0 -semigroups M -hypercyclic for some subspace M , and any hypercyclic C_0 -semigroups is M -hypercyclic for some subspace M but the converse is not true in general (see also [15]).

In the following section we investigate the Hypercyclicity Criterion for a C_0 -semigroup, this is a given the sufficient condition for a C_0 -semigroups to be M -hypercyclic for some subspace M of X , where X is a infinite dimensional complex separable Banach space.

Next, We recall the following two notions and results (see [20])

Definition 1.1 (20, definition 2.3). Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on X . $\mathcal{T}$ is said M -hypercyclic if there is a vector $x \in X$ called vector M -hypercyclic for $\mathcal{T}$, such that $\text{Orb}(\mathcal{T}, x) \cap M = \{T_t x, t \geq 0\} \cap M$ is dense in M .

Definition 1.2 (20, definition 2.4). Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on X . $\mathcal{T}$ is called M -transitive if for every pair non-empty open subsets U, V of M there is $t > 0$ such that $T_t(U) \cap V$ contains a non-empty open set of M .

Theorem 1.1 (20, theorem 2.1). Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on X . The following conditions are equivalent:

1. $\mathcal{T}$ is transitive.
2. For every non-empty open U and V of M , there is $t_0 > 0$ such that $T_{t_0}(U) \cap V$ is a non-empty open of M .
3. For every non-empty open U and V of M , there is $t_0 > 0$ such that $T_{t_0}(U) \cap V$ is a non-empty and $T_{t_0}(M) \subset M$

Corollary 1.1 (20, corollary 2.3). Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on X . If $\mathcal{T}$ is M -transitive then $\mathcal{T}$ is M -hypercyclic.

2. The main results

We introduce the following notion.

Definition 2.1. Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on Banach space X , and M be a subspace of X ($M \neq \{0\}$). We say that $\mathcal{T}$ satisfies the M -hypercyclicity criterion if there is a sequence $(t_n) \subset [0, +\infty[$ such that:

1. $Z_0 = \{x \in M : T_{t_n}x \rightarrow 0\}$ and $Z_\infty = \{y \in M : \exists u_n \rightarrow 0 \text{ such that } T_{t_n}u_n \rightarrow y\}$ are dense in M .
2. $T_{t_n}(M) \subset M$ for all $n \geq 0$

Remark 1. The definition 2.1 reduces to the classical definition of hypercyclicity criterion if $M = X$, see [9].

Example 1. Let $(A_t)_{t \geq 0}$ be a C_0 -semigroup on X satisfies the criterion of hypercyclicity, then $(T_t)_{t \geq 0} := (A_t \oplus 0)_{t \geq 0}$ is a C_0 -semigroup satisfies the M -hypercyclicity criterion where $M = X \oplus \{0\}$. Indeed if $(A_t)_{t \geq 0}$ satisfy the hypercyclicity criterion, then there is a sequence (t_n) such that

$$Z'_0 = \{x \in X : \lim_{n \rightarrow +\infty} A_{t_n}x = 0\}$$

and

$$Z'_\infty = \{y \in X : \exists u_n \rightarrow 0 \text{ such that } A_{t_n}u_n \rightarrow y\}$$

are dense in X , if we put $M = X \oplus \{0\}$ then it follows that the spaces $Z_0 = Z'_0 \oplus \{0\}$ and $Z_\infty = Z'_\infty \oplus \{0\}$ are dense in M , since $T_{t_n}(M) \subset M$ for every $n \geq 0$, then $(T_t)_{t \geq 0}$ satisfies the M -hypercyclicity criterion.

The following theorem gives the M -hypercyclicity criterion for C_0 -semigroup.

Theorem 2.1. Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup, if there is $t_0 > 0$ and increasing sequence of positive integers $(n_k)_{k \geq 0}$ such that $\lim n_k = +\infty$, D and D' dense subsets of M such that:

- i) $T_{t_0}^{n_k}x \rightarrow 0$ for every $x \in D$.
- ii) For every $y \in D'$ there is $x_k \in M$, $x_k \rightarrow 0$ such that $T_{t_0}^{n_k}x_k \rightarrow y$
- iii) $T_{t_0}^{n_k}(M) \subset M$ for every $k \geq 0$.

then $\mathcal{T}$ satisfies the M -hypercyclicity criterion.

Proof. Using Definition 2.1, it is obvious by setting $t_k = t_0 n_k$ that $D \subset Z_0$ and $D' \subset Z_\infty$. □

Theorem 2.2. *Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on Banach space X . If $\mathcal{T}$ satisfies the M -hypercyclicity criterion then $\mathcal{T}$ is M -hypercyclic.*

Proof. By corollary 1.1 it suffices to prove that $\mathcal{T}$ is M -transitive.

Let U and V be two non-empty open subset of M . Since Z_0 and Z_∞ are dense in M , then $Z_0 \cap U \neq \emptyset$ and $Z_\infty \cap V \neq \emptyset$.

Let $a \in Z_0 \cap U$ and $b \in Z_\infty \cap V$, then there is $(u_n)_{n \geq 0} \subset M$ such that $u_n \rightarrow 0$ and $T_{t_n} u_n \rightarrow b$, we put $x_n = a + u_n$ for every $n \geq 0$, it follows that $x_n \in M$ and $x_n \rightarrow a \in U$. Hence there exists $n_0 \geq 0$ such that $x_n \in U$ for all $n \geq n_0$.

On the other hand $T_{t_n} x_n = T_{t_n} a + T_{t_n} u_n \rightarrow b \in V$, then there exists $n_1 \geq 0$ such that $T_{t_n}(x_n) \in V$ for every $n \geq n_1$. Consequently if $n \geq \max(n_0, n_1)$ we have $T_{t_n}(U) \cap V \neq \emptyset$, since $T_{t_n}(M) \subset M$ then $\mathcal{T}$ is M -transitive. Therefore $\mathcal{T}$ is M -hypercyclic. $\square$

Remark 2. If $\mathcal{T} = (T_t)_{t \geq 0}$ is M -hypercyclic, then there is $x \in X$ such that $\text{Orb}(\mathcal{T}, x) \cap M$ is dense in M . Then, for all $y \in M$, there is $t_n \mapsto +\infty$ such that $T_{t_n} x \in M$ and $T_{t_n} x \rightarrow y$, hence there is $(t_n)_{n \geq 0} \subset [0, +\infty[$ such that Z_∞ is dense in M , however we are not sure that with same $(t_n)_{n \geq 0}$ the subset Z_0 is dense in M .

We know that if $(T_t)_{t \geq 0}$ and $(S_t)_{t \geq 0}$ are C_0 -semigroups on X , then $(T_t \oplus S_t)_{t \geq 0}$ is also a C_0 -semigroup on $X \oplus X$, with $X \oplus X$ is the sum direct of X which defined by $X \oplus X := \{x \oplus y = (x, y) : x \in X; y \in X\}$, for more information see [7] and [11].

Proposition 2.1. *Let M_1 and M_2 be two closed subspaces of X and $(T_t)_{t \geq 0}, (S_t)_{t \geq 0}$ are two C_0 -semigroups on X .*

If $(T_t \oplus S_t)_{t \geq 0}$ is $M_1 \oplus M_2$ -hypercyclic then $(T_t)_{t \geq 0}$ is M_1 -hypercyclic and $(S_t)_{t \geq 0}$ is M_2 -hypercyclic.

Proof. Suppose that $(T_t \oplus S_t)_{t \geq 0}$ is $M_1 \oplus M_2$ -hypercyclic. Then there is $x \oplus y \in X \oplus X$ such that $\text{Orb}((T_t \oplus S_t)_t, x \oplus y) \cap (M_1 \oplus M_2)$ is dense in $M_1 \oplus M_2$.

Let $a \oplus b \in M_1 \oplus M_2$ where $a \in M_1$ and $b \in M_2$, then for every $\varepsilon > 0$ there is a sequence $(t_n)_{n \geq 0}$, such that $\|(T_{t_n} \oplus S_{t_n})(x \oplus y) - (a \oplus b)\| < \varepsilon$.

Consequently we have $\|(T_{t_n} x - a) \oplus (S_{t_n} y - b)\| < \varepsilon$, and therefore

$$\|T_{t_n} x - a\| + \|S_{t_n} y - b\| < \varepsilon,$$

hence $\|T_{t_n} x - a\| < \varepsilon$ and $\|S_{t_n} y - b\| < \varepsilon$.

It follows that $a \in \text{Orb}((T_t)_{t \geq 0}, x) \cap M_1$ and $b \in \text{Orb}((S_t)_{t \geq 0}, y) \cap M_2$. This gives the result. $\square$

We have the following result, the proof is Similar to those of proposition 2.1.

Proposition 2.2. *Let M_1 and M_2 be tow closed subspaces of X , and $(T_t)_{t \geq 0}, (S_t)_{t \geq 0}$ are tow C_0 -semigroups on X . If $(T_t \oplus S_t)_{t \geq 0}$ satisfies the $M_1 \oplus M_2$ -hypercyclicity criterion, then $(T_t)_{t \geq 0}$ satisfies the M_1 -hypercyclicity criterion and $(S_t)_{t \geq 0}$ satisfies the M_2 -hypercyclicity criterion.*

Theorem 2.3. *Let $(T_t)_{t \geq 0}$ be a C_0 -semigroup on Banach space X , and M be a subspace ($M \neq \{0\}$) of X . If $(T_t)_{t \geq 0}$ satisfies the M -hypercyclicity criterion then $(T_t \oplus T_t)_{t \geq 0}$ is $(M \oplus M)$ -hypercyclic.*

Proof. Suppose that $(T_t)_{t \geq 0}$ satisfies the M -hypercyclicity criterion, then there is a sequence $(t_n)_{n \geq 0}$ such that $Z_0 = \{x \in M : T_{t_n}x \rightarrow 0\}$ and $Z_\infty = \{y \in M : \exists u_n \rightarrow 0 \text{ such that } T_{t_n}u_n \rightarrow y\}$ are both dense in M and $T_{t_n}(M) \subset M$ for every $n \geq 0$

Let U_1, U_2, V_1, V_2 be a non-empty open sets of M , then there is $n_0 \geq 0$ such that $T_{t_n}(U_i) \cap V_i \neq \emptyset$ for all $n \geq n_0 \geq 0$ and $i = 1, 2$.

We obtain that $(T_{t_n} \oplus T_{t_n})(U_1 \oplus U_2) \cap (V_1 \oplus V_2) \neq \emptyset$ for all $n \geq n_0 \geq 0$, and since $(T_{t_n} \oplus T_{t_n})(M \oplus M) \subset (M \oplus M)$ then $(T_t \oplus T_t)_{t \geq 0}$ is $(M \oplus M)$ -transitive. Finally by corollary 1.1 $(T_t \oplus T_t)_{t \geq 0}$ is $(M \oplus M)$ -hypercyclic. $\square$

Remark 3. If there is $x \oplus y \in M \oplus M$ a vector $M \oplus M$ -hypercyclic of $(T_t \oplus T_t)_{t \geq 0}$ then the converse of the theorem 2.3 is true.

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