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ON VERTEX PRODUCT CORDIAL LABELING

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Abstract: In this paper, we define the vertex product cordial labeling for some path related graphs like P_n^2, path union of k copies of P_n^2 graph and $P_n \odot K_1$. We also discuss vertex product cordial labeling of helm graph and some graphs related to gear graph. In the present paper, we also show that the graph obtained from cycle C_n after switching of a vertex admits vertex product cordial labeling. Moreover we show that corona product $C_n \odot K_1$, ladder graph $P_n \times P_2$ with one edge path to one vertex, $C_n \odot \bar{K}_m$ and the banana tree $BT(n_1, n_2, n_3)$ also admits vertex product cordial labeling with certain conditions.

AMS Subject Classification: 05C78

Key Words: vertex product cordial graph, gear graphs, Banana tree, Ladder graph, corona product

1. Introduction

All graphs considered are finite, connected, undirected and simple. The graphs has the vertex set $V(G)$ and edge set $E(G)$ where $|V(G)| = p$ and $|E(G)| = q$. Graph labeling is an assignment of integers to vertices or edges or both vertices and edges of the graph subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling). All terms and definitions, not defined specifically in this paper, we refer to Rosa and Harary [4,5].

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The origin of graph labeling is graceful labeling which was introduced by Rosa [5] in 1967. The concept of cordial labeling was introduced by Cahit [1] in 1987 as a weaker version of graceful and harmonious labeling. He write down a paper in which he investigated several results on this newly defined concept. Some labeling with variations in cordial labeling have been introduced such as prime cordial labeling, E-cordial labeling, product cordial labeling and total product cordial labeling. Cordiality behavior of numerous graph were studied by several authors [10, 11, 12, 13]. The notion of product cordial labeling was introduced by Sundaram [10] in which the absolute difference in cordial labeling is replaced by product of the vertex labels.

The present work is focused on vertex product cordial labeling of graphs. S.M. Lee and A. Liu [2] proved that all complete bipartite and all fans are cordial. Further, they proved that, the cycle C_n is cordial if and only if $n \not\equiv 2 \pmod 4$, the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$, $n \geq 3$. In this section we provide brief summary of definitions and other required information for our investigation.

Definition 1.1. A labeling $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.

Notation. If for any edge $e = uv$, the induced edge labeling $f^{\square} : E(G) \to \{0, 1\}$ is given by $f^{\square}(e) = |f(u) - f(v)|$. Then

$$v_f(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f,$$
$$e_{f^*}(i) = \text{number of vertices of } G \text{ having label } i \text{ under } f^{\square},$$

where $i = 0, 1$.

Definition 1.2. A binary vertex labeling f of a graph G is called a cordial labeling if $|v_f(1) - v_f(0)| \leq 1$ and $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$. A graph G is cordial if it admits cordial labeling.

In 2004, the notion of product cordial labeling was originated by Sundaram, Ponraj and Somasundaram et al. [10].

Definition 1.3. A binary vertex labeling of a graph G with induced edge labeling $f^{\square} : E \to \{0, 1\}$ defined by $f^{\square}(e = uv) = f(u)f(v)$ is called a product cordial or vertex product cordial labeling if $|v_f(1) - v_f(0)| \leq 1$ and $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$. A graph G is cordial if it admits cordial labeling.

In 2004 Sundaram, Ponraj and Somsundaram [10] proved that unicyclic graph of odd order, triangular snakes, dragons, helms and union of two paths are product cordial. They also proved that a graph with p vertices and q edges with $p \geq 4$ is product cordial then $q < \frac{p^2-1}{4} + 1$. In 2010 Vaidya and Kanani investigated product cordial labeling for the shadow graph of cycle C_n. The following type of problems can be considered in the area of vertex product cordial labeling.

- How vertex product cordial labeling is affected under various graph operations.
- Construct new families of vertex product cordial graphs by finding suitable labeling.

Vaidya, Dani, Kanani, Barasara and Vyas have investigated vertex product cordial labeling for cycle related graphs, friendship graph, middle graph of path and tensor graph. Vaidya and Barasara proved that friendship graph, cycle with one chord, cycle with twin chord are product cordial graphs [3]. Vaidya and Dani [12] proved that the graph obtained by joining apex vertices of two stars is product cordial and similar results for shell and wheel graphs. The path union of k copies of cycle C_n (the graph obtained by joining two copies of cycle C_n by path P_k) is proved product cordial by Vaidya, Dani, Kanani [12,15]. The graph obtained by joining the connected components of respective graphs by a path of arbitrary length is proved product cordial by Vaidya and Vyas [19].

Definition 1.4. A vertex of degree one is called a pendant vertex.

Definition 1.5. $G^+ = G \odot K_1$ is the graph obtained by joining exactly one pendant edge to every vertex of a graph G.

Definition 1.6. The corona product $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as a graph obtained by one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and joining ith vertex of G_1 with an edge to every vertex in the ith copy of G_2.

Definition 1.7. C_n^t is the graph constructed from a cycle C_n by joining two vertices whose distance in the cycle is t. Similarly, for the gear graph G_n^t is the graph constructed from the gear graph G_n by joining two vertices whose distance in the gear graph is t.

Definition 1.8. Let $G_1, G_2, \ldots, G_n$ be a family of disjoint stars. Let v be a new vertex. The tree obtained by joining v to one pendant vertex of each star is called a banana tree. The class of all such trees is denoted by $BT(G_1, G_2, \ldots, G_n)$.

Definition 1.9. A vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.

In this paper, we prove that vertex product cordial labeling of path P_n^2, gear graph G_n. We also discuss vertex product cordial labeling of gear graph G_n^4, $P_n \odot K_1$, the banana tree $BT(n_1, n_2, n_3)$, corona product $C_n \odot K_1$, ladder graph $P_n \times P_2$ with one edge path to one vertex and $C_n \odot \bar{K}_m$.

2. Main Results

With the definition of vertex product cordial graph in the previous section in hand, we are ready to study vertex product cordial labeling for certain graphs. First we discuss vertex product cordial labeling for some path related graphs.

Theorem 2.1. *P_n^2 is vertex product cordial graph for odd n.*

Proof. Let P_n^2 be the graph with vertices $v_1, v_2, \ldots, v_n$. We have $|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n - 3$. Define the labeling $f : V(P_n^2) \to \{0, 1\}$ as follows:

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \le i \le \left[\frac{n}{2}\right] + 1, \\ 0 & \text{for } \left[\frac{n}{2}\right] + 1 < i \le n. \end{cases}$$

In view of the above labeling pattern, we have

$$v_f(1) = \left[\frac{n}{2}\right] + 1, \quad v_f(0) = \left[\frac{n}{2}\right]$$

and

$$e_{f^*}(1) = n - 2 \quad \text{and} \quad e_{f^*}(0) = n - 1 \quad \text{and the result holds.}$$ □

Example 2.1. The graph P_5^2 and its vertex product cordial labelling is shown in Figure 1.

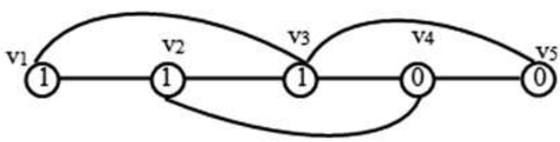

Figure 1: Vertex product cordial labeling of P_5^2.

Theorem 2.2. *P_n^2 is not vertex product cordial graph for even n.*

Proof. The graph P_n^2 has n vertices and $2n-3$ edges. Assign 1 to $n/2$ vertices, so that the vertex condition form the product graph is satisfied. The vertices with label 1 cause $n-3$ edges to have label 1 and at most n edges to have label 0. Therefore $|e_{f^*}(1) - e_{f^*}(0)| = 3$. So the edge condition for vertex product cordial graph is not satisfied.

Hence P_n^2 is not a vertex product cordial graph for even n. □

Theorem 2.3. *The path union of k copies of graph P_n^2 admits vertex product cordial labeling.*

Proof. Let G be the path union of k copies $G_1, G_2, \ldots, G_k$ of P_n^2. Let $\{v_{i1}, v_{i2}, \ldots, v_{in}\}$ be the vertex set and $v_{in}v_{i+1,1}$ be the edge joining G_i and G_{i+1}. $|V(G)| = nk$ and $|E(G)| = 2k(n-1)-1$. We define the labeling as follows $f : V(G) \to \{0,1\}$.

Case 1: k is even, $\forall\ n$.

$$\begin{aligned} f(v_{ij}) &= 1 \quad \text{for } 1 \le i \le k/2,\ 1 \le j \le n, \\ f(v_{ij}) &= 0 \quad \text{for } \frac{k}{2}+1 \le i \le k \text{ and } 1 \le j \le n. \end{aligned}$$

In view of the above labeling pattern, we have

$$v_f(1) = v_f(0) = \frac{nk}{2}$$

and

$$e_{f^*}(1) = \left\lceil \frac{2k(n-1)-1}{2} \right\rceil,\ e_{f^*}(0) = \left\lceil \frac{2k(n-1)-1}{2} \right\rceil + 1.$$

So the vertex condition and edge conditions of vertex product cordial labeling are satisfied.

Case 2: k is odd and n is also odd.

$$\begin{aligned} f(v_{ij}) &= 1 \text{ for } 1 \le i \le \left\lceil \frac{k}{2} \right\rceil \ \forall\ j, \\ f(v_{ij}) &= 1 \text{ for } i = \left\lceil \frac{k}{2} \right\rceil + 1 \text{ and } 1 \le j \le \left\lceil \frac{n}{2} \right\rceil + 1, \\ f(v_{ij}) &= 0 \text{ for } i = \left\lceil \frac{k}{2} \right\rceil + 1 \text{ and } \left\lceil \frac{n}{2} \right\rceil + 2 \le j \le n, \\ f(v_{ij}) &= 0 \text{ for } \left\lceil \frac{k}{2} \right\rceil + 1 \le i \le k \text{ and } 1 \le j \le n. \end{aligned}$$

It can be easily seen that the vertex conditions and edge conditions of product cordial labeling are satisfied either $|v_f(1) - v_f(0)| \leq 1$ and $|v_{f^*}(1) - v_{f^*}(0)| \leq 1$. □

Remark 2.1. The path union of k copies of P_n^2 does not admits vertex product cordial labeling for odd k and even n. Assign 1 to $\frac{nk}{2}$ vertices, so that the vertex condition form the product cordial graph is satisfied. The vertices with label 1 cause $nk - k - 2$ edges to have label 1 and at most $nk - k + 1$ edge to have label 0. Therefore $|v_{f^*}(1) - v_{f^*}(0)| = 3$. So the edge condition for product cordial graph is not satisfied.

Example 2.2. The product cordial labeling of path union of 2-copies of P_5^2 in Figure 2. It is the case related k even and n odd. Product cordial labeling of path union of 2 copies of P_5^2.

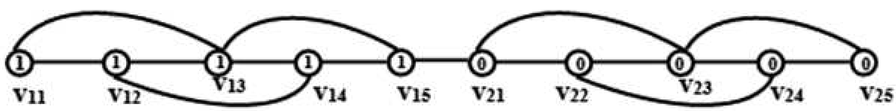

Figure 2: Vertex product cordial labeling of path union of 2 copies of P_5^2.

Theorem 2.4. *$P_n \odot K_1$ admits vertex product cordial labeling.*

Proof. Let $G = P_n \odot K_1$, with $v_1, v_2, \ldots, v_n,\ u_1, u_2, \ldots, u_n$ vertices and $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_1u_1, v_2u_2, \ldots, v_nu_n$ edges. Consider the vertex labeling $f : V(G) \to \{0, 1\}$ defined by

$$f(v_i) = 1 \quad \text{for } 1 \leq i \leq n,$$
$$f(u_i) = 0 \quad \text{for } 1 \leq i \leq n.$$

In view of the above defined labeling we have $v_f(1) = v_f(0) = n$ and $e_{f^*}(1) = n - 1$, $e_{f^*}(0) = n$.

Thus, $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$. So $P_n \odot K_1$ admits vertex product cordial labeling for all n. □

Example 2.3. Vertex product cordial labeling for $P_4 \odot K_1$ is shown in Figure 3.

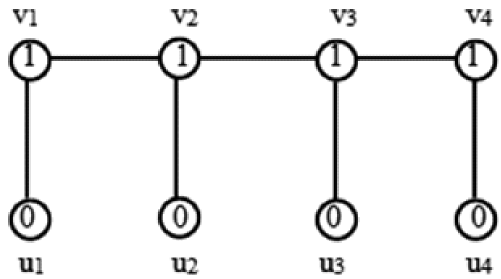

Figure 3: Vertex product cordial labeling of $P_4 \odot K_1$

3. Vertex Product Cordial Labeling of Helm Graph and Gear Related Graphs

Definition 3.1. The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n pendant vertices.

Definition 3.2. The gear graph G_n is obtained from the wheel W_n by subdividing each of its rim edge.

Theorem 3.1. *H_n is a vertex product cordial graph.*

Proof. Let v be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the pendant vertices of H_n. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. We define vertex labeling $f : V(G) \to \{0,1\}$ as follows:

Case 1: n is even.
$f(v) = 1$,

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \le i \le \frac{n}{2}+1, \\ 0 & \text{for } \frac{n}{2}+2 \le i \le n, \end{cases}$$

and

$$f(u_i) = \begin{cases} 1 & \text{for } 1 \le i \le \frac{n}{2}-1, \\ 0 & \text{for } \frac{n}{2} \le i \le n. \end{cases}$$

In view of the above defined labeling pattern we have $v_f(1) = n+1$ and $v_f(0) = n$ and $e_{f^*}(1) = e_{f^*}(0) = \frac{3n}{2}$. So $|v_f(1) - v_f(0)| \le 1$ and $|e_{f^*}(1) - v_{f^*}(0)| \le 1$. Hence, H_n admits vertex product cordial labeling for even n.

Case 2: n is odd.
$f(v) = 1$,

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \le i \le \left[\frac{n}{2}\right] + 1, \\ 0 & \text{for } \left[\frac{n}{2}\right] + 2 \le i \le n, \end{cases}$$

and

$$f(u_i) = \begin{cases} 1 & \text{for } 1 \le i \le \left[\frac{n}{2}\right], \\ 0 & \text{for } \left[\frac{n}{2}\right] + 1 \le i \le n. \end{cases}$$

In view of the above labeling pattern we have $v_f(1) = n+1$, $v_f(0) = n$ and $e_{f^*}(1) = \left[\frac{3n}{2}\right]$ and $e_{f^*}(0) = \left[\frac{3n}{2}\right] + 1$.

Thus, $|v_f(1) - v_f(0)| \le 1$ and $|e_{f^*}(1) - v_{f^*}(0)| \le 1$. □

Example 3.1. Vertex product cordial labeling for H_8 is shown in Figure 4.

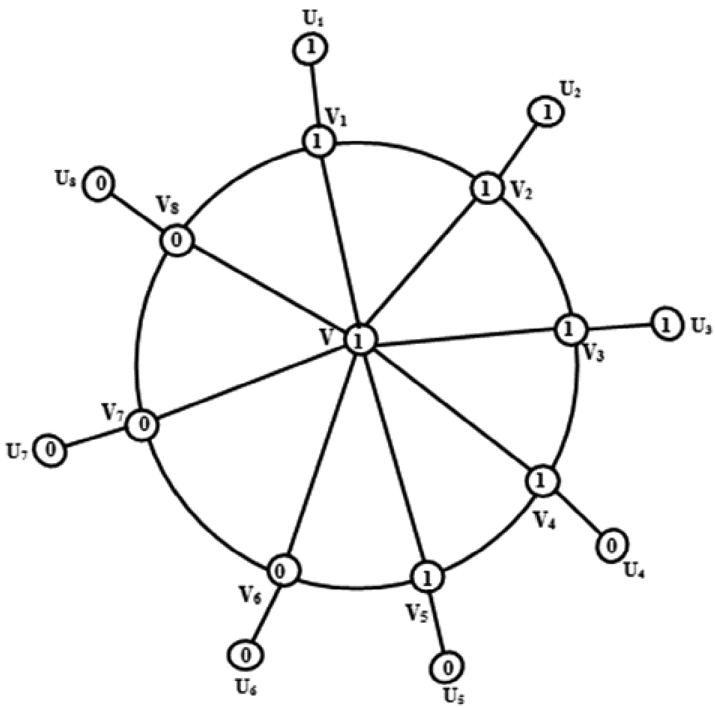

Figure 4: Vertex product cordial labeling of H_8.

Theorem 3.2. *The gear graph G_n is vertex product cordial graph for odd n.*

Proof. Let u be the apex vertex of gear graph G_n and $v_1, v_2, \dots, v_n$ and $u_1, u_2, \dots, u_n$ be the rim vertices. Consider the vertex labeling $f : V(G_n) \to \{0, 1\}$ defined by
$f(u) = 1$,

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \left[\frac{n}{2}\right] + 1, \\ 0 & \text{for } \left[\frac{n}{2}\right] + 1 < i \leq n, \end{cases}$$

and

$$f(u_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \left[\frac{n}{2}\right], \\ 0 & \text{for } \left[\frac{n}{2}\right] + 1 \leq i \leq n. \end{cases}$$

In view of the above defined labeling we have

$$v_f(1) = n + 1, \qquad v_f(0) = n$$
$$e_{f^*}(1) = \left[\frac{3n}{2}\right], \quad e_{f^*}(0) = \left[\frac{3n}{2}\right] + 1.$$

Hence the gear graph G_n is vertex product cordial graph for odd n. □

Example 3.2. The gear graph G_7 with its vertex cordial labelling is shown in Figure 5.

Remark 3.1. In the above theorem we can write $v_2, v_4, \dots$ in place of $u_1, u_2, \dots$ so on.

Theorem 3.3. *The gear graph G_n^4 with a chord admits vertex product cordial labelling for even $n > 4$.*

Proof. Let u be the apex vertex of the gear graph G_n and $v_1, v_2, \dots, v_{2n}$ be the rim vertices of G_n. Let $v_1 v_5$ be the chord in G_n. Then $|V(G_n)| = 2n + 1$ and $|E(G)| = 3n + 1$, we define
$f : V(G_n) \to \{0, 1\}$ as follows

$$\begin{aligned} &f(u) = 1, \\ &f(v_i) = 1, \quad \text{for } 1 \leq i \leq n - 1, \; f(v_n) = 0, \\ &f(v_i) = 1, \quad \text{for } i = n + 1, \\ &f(v_i) = 0, \quad \text{for } n + 2 \leq i \leq 2n. \end{aligned}$$

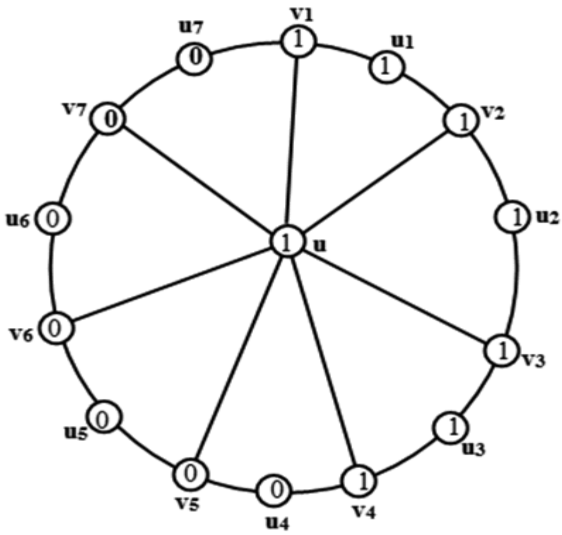

Figure 5: Vertex product cordial labeling of G_7.

In view of the above defined labeling pattern we have $v_f(1) = n+1$, $v_f(0) = n$ and $e_{f^*}(1) = \frac{3n}{2}$, $e_{f^*}(0) = \frac{3n}{2} + 1$. Hence, then graph G_n^4 admits vertex product cordial labeling. □

Example 3.3. For better understanding of the above defined labeling pattern the vertex product cordial labeling of G_8^4 is shown in Figure 6. In this labeling n is even.

4. Switching of a Vertex and Vertex Product Cordial Labeling

Theorem 4.1. *Switching of a vertex in cycle C_n admits vertex product cordial labeling.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the successive vertices of C_n and G_{v_1} denotes the graph obtained by switching of a vertex v_1 of $G = C_n$. Then $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 5$. We define the vertex labeling $f : V(G_{v_1}) \to \{0, 1\}$ as follows:

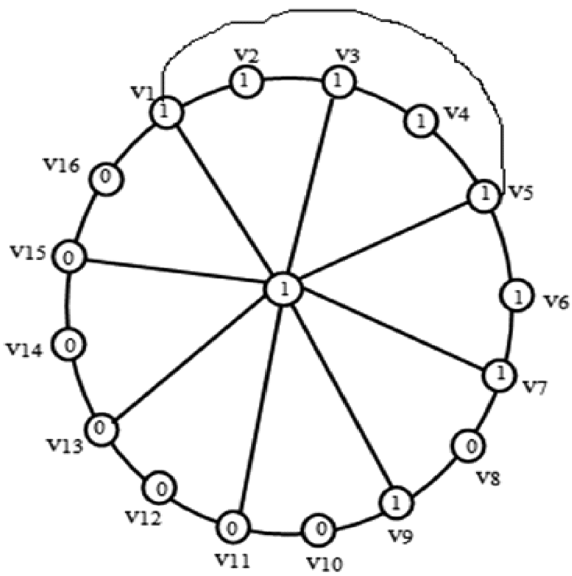

Figure 6: Vertex product cordial labeling of G_8^4.

Case 1: when n is even.
$f(v_1) = 1,\ f(v_2) = 0,$

$$f(v_i) = \begin{cases} 1 & \text{for } 3 \leq i \leq \frac{n}{2} + 1, \\ 0 & \text{for } \frac{n}{2} + 2 \leq i \leq n. \end{cases}$$

In view of the above defined labeling pattern we have $v_f(1) = v_f(0) = \frac{n}{2}$ and $e_{f^*}(1) = n-3$, $e_{f^*}(0) = n-2$. Thus $|v_f(1) - v_f(0)| \leq 1$ and $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$
.

Case 2: when n is odd.
$f(v_1) = 1,\ f(v_2) = 0,$

$$f(v_i) = \begin{cases} 1 & \text{for } 3 \leq i \leq \left[\frac{n}{2}\right] + 2, \\ 0 & \text{otherwise.} \end{cases}$$

In view of the above defined labeling pattern we have $v_f(1) = \left[\frac{n}{2}\right] + 1$, $v_f(0) = \left[\frac{n}{2}\right]$ and $e_{f^*}(1) = \left[\frac{2n-5}{2}\right] + 1$, $e_{f^*}(0) = \left[\frac{2n-5}{2}\right]$. Thus the required condition of vertex product cordial labeling is satisfied. □

Example 4.1. Switching of a vertex in cycle C_9 namely G_9 with its vertex product cordial graph is shown in Figure 7.

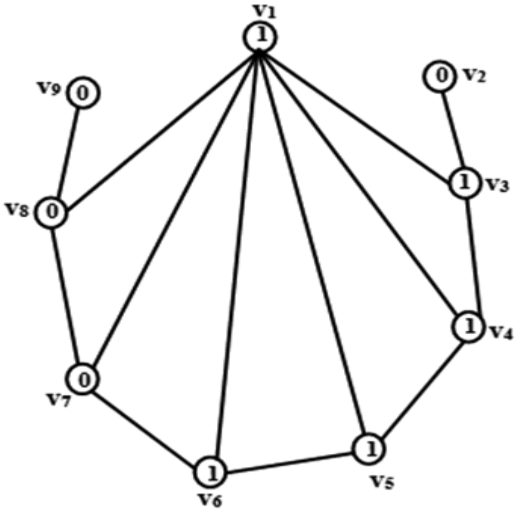

Figure 7: G_9 with its vertex product cordial labeling

5. Vertex Product Cordial Labeling for some Product Related Graphs

In this section, we exhibit vertex product cordial labeling for some product related graphs like $C_n \odot K_1$, ladder graph $P_n \times P_2$ with a 1-edge path attached to its one vertex and for $C_n \odot \bar{K}_m$.

Theorem 5.1. *The corona product $C_n \odot K_1$ admits vertex product cordial labeling for all n.*

Proof. Let $G = C_n \odot K_1$ and $\{v_1, v_2, \ldots, v_n\}$ be the set of cycle vertices and $\{u_1, u_2, \ldots, u_n\}$ be the set of pendant vertices for the graph $C_n \odot K_1$ where u_i is adjacent to v_i, $v_i v_{i+1}$ if $i \in \{1, 2, 3, \ldots, n-1\}$. For Corona product $C_n \odot K_1$, $|V(C_n \odot K_1)| = 2n$ and $|E(C_n \odot K_1)| = 2n$ we define $f : V(C_n \odot K_1) \to \{0, 1\}$ as follows:

$$f(u_i) = 1 \quad \text{for } 1 \le i \le n,$$
$$f(v_i) = 1 \quad \text{for } 1 \le i \le n.$$

In view of the above defined labeling pattern we have $v_f(1) = v_f(0) = n$ and $e_{f^*}(1) = e_{f^*}(0) = n$. So $|v_f(1) - v_f(0)| \le 1$ and $|e_{f^*}(1) = e_{f^*}(0)| \le 1$. Hence the corona product $C_n \odot K_1$ admits vertex product cordial labeling. □

Example 5.1. The Corona product $C_6 \odot K_1$ with its vertex product cordial labeling is shown in Figure 8.

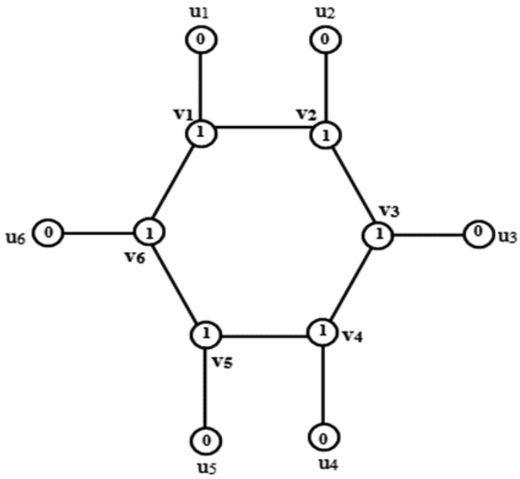

Figure 8: $C_6 \odot K_1$ with its vertex product cordial labeling.

Theorem 5.2. *The graph G obtained by attaching a 1-edge path to one-vertex of a ladder graph $P_n \times P_2$ is vertex product cordial.*

Proof. Let G be the graph obtained by attaching 1-edge path to one-vertex of a ladder graph $P_n \times P_2$. Let $V(G) = \{v, v_1, v_2, \dots v_n, u_1, u_2, \dots, u_n\}$ and the edge set $E(G) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\} \cup \{v_n v\}$. Clearly $|V(G)| = 2n+1$ and $|E(G)| = 3n-1$. We define the vertex labeling $f : V(G) \to \{0,1\}$ as follows:

Case 1. n is odd. Then

$$f(v_i) = 1 \quad \text{for } 1 \le i \le \left[\frac{n}{2}\right] + 1,$$

$$f(v_i) = 0 \quad \text{for } \left[\frac{n}{2}\right] + 2 \le i \le n,$$

$$f(u_i) = 1 \quad \text{for } 1 \le i \le \left[\frac{n}{2}\right] + 1,$$

$$f(u_i) = 0 \quad \text{for } \left[\frac{n}{2}\right] + 2 \le i \le n,$$

and

$$f(v) = 0.$$

In view of the above defined labeling $v_f(1) = \left[\frac{2n+1}{2}\right] + 1$, $v_f(0) = \left[\frac{2n+1}{2}\right]$ and $e_{f^*}(0) = e_{f^*}(1) = \frac{3n-1}{2}$.

Case 2. n is even. Then

$$\begin{aligned} f(v_i) &= 1 \quad \text{for } 1 \le i \le \frac{n}{2} + 1, \\ f(v_i) &= 0 \quad \text{for } \frac{n}{2} + 2 \le i \le n, \\ f(u_i) &= 1 \quad \text{for } 1 \le i \le \frac{n}{2}, \\ f(u_i) &= 0 \quad \text{for } \frac{n}{2} + 1 \le i \le n, \\ &\text{and} \\ f(v) &= 0. \end{aligned}$$

In view of the above defined labeling $v_f(1) = n + 1$, $v_f(0) = n$ and $e_{f^*}(0) = \left[\frac{3n-1}{2}\right] + 1$, $e_{f^*}(1) = \left[\frac{3n-1}{2}\right]$.

One can observe that in each case the labeling defined above satisfies the conditions of vertex product cordial labeling and the graph under consideration is vertex product cordial.

Illustration. For better understanding of the above defined labeling pattern the vertex product cordial labeling of ladder graph $P_5 \times P_2$ with one chord is shown in Figure 9. In this labeling n is odd.

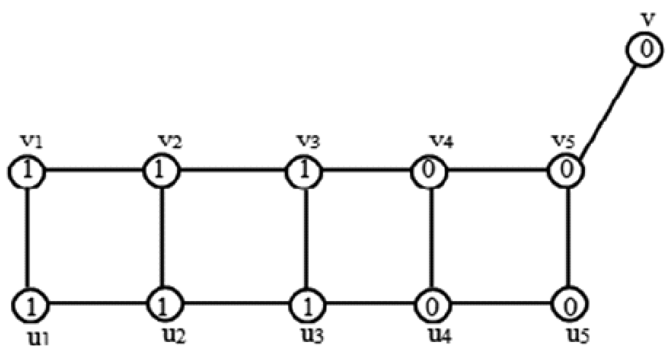

Figure 9: Vertex product cordial labeling of $P_5 \times P_2$ with one chord

Theorem 5.3. *$C_n \odot \bar{K}_m$ admits vertex product cordial labeling for odd n and even m.*

Proof. Let the vertex set $V = \{u_i : 1 \le i \le n\} \cup \{u_{ij} : 1 \le i \le n, 1 \le j \le m\}$ and the edge set $E = \{u_i u_{i+1}, u_n u_1 : 1 \le i \le n-1\} \cup \{u_i u_{ij} : 1 \le i \le n, 1 \le$

$j \leq m\}$. Here $|V| = mn + n$, $|E| = mn + n$. We define $f : V(G) \to \{0, 1\}$

$$
\begin{aligned}
&f(u_i) = 1 \quad \text{for } 1 \leq i \leq \left[\frac{n}{2}\right] + 1;\\
&f(u_i) = 0 \quad \text{for } \left[\frac{n}{2}\right] + 2 \leq i \leq n;\\
&f(u_{ij}) = 1 \quad \text{for } i = 1, 2, \dots, \left[\frac{n}{2}\right],\ 1 \leq j \leq m;\\
&f(u_{ij}) = 1 \quad \text{for } i = \left[\frac{n}{2}\right] + 1,\ 1 \leq j \leq m/2;\\
&f(u_{ij}) = 0 \quad \text{for } i = \left[\frac{n}{2}\right] + 1,\ \frac{m}{2} + 1 \leq j \leq m;\\
&f(u_{ij}) = 0 \quad \text{for } i = \left[\frac{n}{2}\right] + 2, \dots, n;\ 1 \leq j \leq m.
\end{aligned}
$$

In view of the above defined labeling pattern we have $v_f(0) = \left[\frac{mn+n}{2}\right]$, $v_f(1) = \left[\frac{mn+n}{2}\right] + 1$ and $e_{f^*}(1) = \left[\frac{mn+n}{2}\right]$, $e_{f^*}(0) = \left[\frac{mn+n}{2}\right] + 1$. Hence, the graph $C_n \odot \bar{K}_m$ is vertex product cordial graph. □

Example 5.2. Graph $C_6 \odot K_1$ with its vertex product cordial labeling is shown in Figure 10.

Theorem 5.4. *Let $G = BT(n_1, n_2, n_3)$. If $n_1 = n_2 = n_3 = n$, then G admits vertex product cordial labeling.*

Proof. Let the vertex set

$$V(G) = \{s, v, v_1, v_2, \dots, v_n, u, u_1, u_2, \dots, u_n, w, w_1, w_2, \dots, w_n\}$$

and the edge set $E(G) = \{sv_n, su_n, sw_n\} \cup \{vv_i : 1 \leq i \leq n\} \cup \{uu_i : 1 \leq i \leq n\} \cup \{ww_i : 1 \leq i \leq n\}$. Clearly $|V(G)| = 3n + 4$ and $|E(G)| = 3n + 3$. Define $f : V(G) \to \{0, 1\}$ as follows:

$$
\begin{aligned}
&f(s) = f(v) = f(u) = 1,\ f(w) = 0,\\
&f(v_n) = f(wn) = f(u_n) = 1,\\
&f(v_i) = 1 \quad \text{for } 1 \leq i \leq n - 1,\\
&f(u_i) = 0 \quad \text{for } 1 \leq i \leq n - 1,\\
&f(w_i) = 0 \quad \text{for } 1 \leq i \leq n - 1.
\end{aligned}
$$

Case 1: When n is even.
In view of the above defined labeling

$$v_f(1) = \frac{3n + 4}{2} = v_f(0) \text{ and}$$

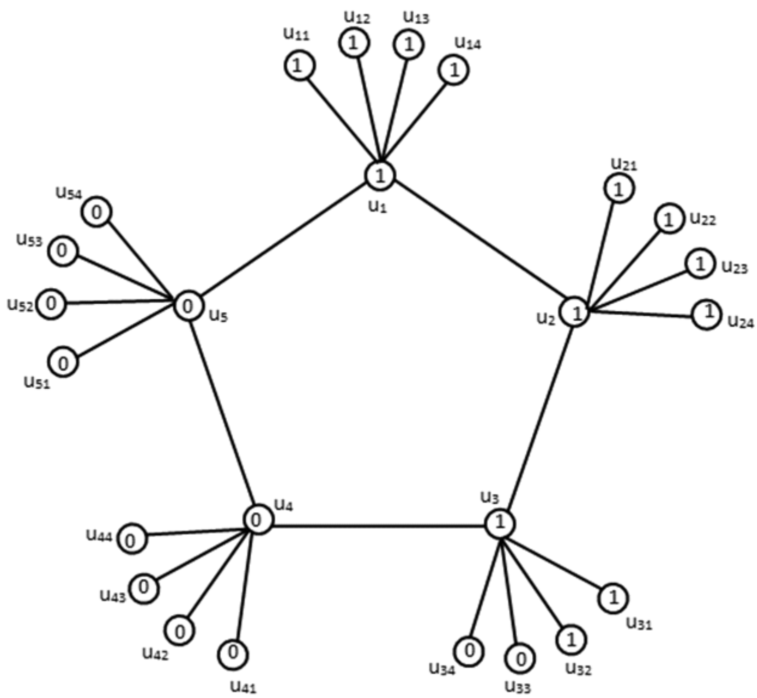

Figure 10: $C_6 \odot K_1$ with its vertex product cordial labeling

$$e_f(1) = \left[\frac{3n+3}{2}\right], \ e_f(0) = \left[\frac{3n+3}{2}\right] + 1.$$

Hence the result holds for even n.

Case 2: When n is odd.
In case 2 we have

$$v_f(1) = \left[\frac{3n+4}{2}\right] + 1, \quad v_f(0) = \left[\frac{3n+4}{2}\right] \quad \text{and}$$
$$e_{f^*}(1) = e_{f^*}(0) = \frac{3n+3}{2}.$$

Hence the Banana tree $BT(n_1, n_2, n_3)$, $n_1 = n_2 = n_3 = n$ is a vertex product cordial graph. □

Example 5.3. The Banana tree $BT(6, 6, 6)$ with its vertex product cordial graph is shown in Figure 11.

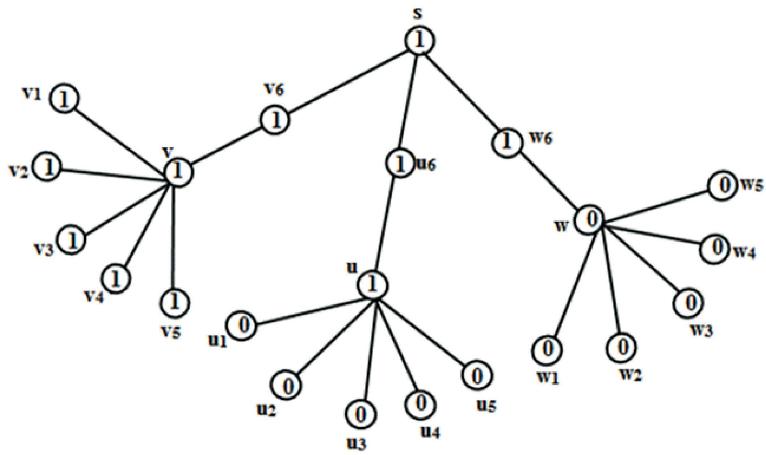

Figure 11: Vertex product cordial labeling of $BT(6,6,6)$.

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