International Journal of Pure and Applied Mathematics
Volume 114 No. 3 2017, 603-609
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v114i3.15

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CONVERGENCE ANALYSIS OF SAA METHOD FOR A STOCHASTIC EXTENDED VERTICAL LINEAR COMPLEMENTARITY PROBLEM

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Abstract: In this paper, we investigate convergence properties of sample average approximation (SAA) method for a stochastic extended vertical linear complementarity problem, where the underlying function is the expected value of stochastic function. In particular, using the implicit function theorem, we develop sufficient conditions for the convergence of SAA method, namely if the solution of the true problem exists, then so does the SAA problem with probability one when the sample size is large enough and a sequence of SAA solutions converges to a solution of the true problem with probability one with the sample size tends to infinity. At last, an example is illustrated to show the application of the analysis.

AMS Subject Classification: 90C30
Key Words: sample average approximation, stochastic extended vertical linear complementarity problem, convergence analysis

1. Introduction

Our concern in this paper is the following stochastic extended vertical linear complementarity problem (SEVLCP)

$$\min\{\mathbb{E}[M_1(\xi(\omega))]x+\mathbb{E}[q_1(\xi(\omega))],\ \mathbb{E}[M_2(\xi(\omega))]x+\mathbb{E}[q_2(\xi(\omega))],\ x\}=0, \quad (1)$$

where $M_i:\Re^k\to\mathcal{S}^{n\times n}$ and $q_i:\Re^k\to\Re^n, i=1,2$ are random mappings, $\mathcal{S}^{n\times n}$

Received: February 15, 2017
Revised: April 27, 2017
Published: May 23, 2017

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denotes the $n \times n$ symmetric matrices space, $\xi : \Omega \to \Xi \subset \Re^k$ is a random vector defined on probability space $(\Omega, \mathcal{F}, \mathrm{P})$, $\mathbb{E}$ denotes the mathematical expectation. Throughout the paper, we assume that $\mathbb{E}[M_i(\xi(\omega))]$ and $\mathbb{E}[q_i(\xi(\omega))], i = 1, 2$ are well defined and finite. To ease the notation, we write $\xi(\omega)$ as ξ and this should be distinguished from ξ being a deterministic vector of Ξ in a context.

SEVLCP is a natural extension of deterministic extended vertical linear complementarity problem[10, 11, 12, 13], which contains some expectation of stochastic vectors to reflect uncertain factors in real world. When $M_1(\xi(\omega)) = M_2(\xi(\omega))$ and $q_1(\xi(\omega)) = q_2(\xi(\omega))$, the SEVLCP is the stochastic complementarity problem (SCP), some examples of SCP arising from the areas of economics engineering and operations management, can be found in [3] and [2]. Exact evaluation of the expected value is impossible or prohibitively expensive, see [3]. Many authors have suggested the sample average approximation (SAA) method to solve SCP, see for example [4, 5, 6, 7, 8, 9]. The basic idea of SAA is to generate an independent identically distributed (iid) sample $\xi^1, \cdots, \xi^N$ of ξ and then approximate the expected value with sample average. In this context, SEVLCP (1) is approximated by

$$\min\{\widehat{M}_1^N x + \widehat{q}_1^N,\ \widehat{M}_2^N x + \widehat{q}_2^N,\ x\} = 0, \tag{2}$$

where

$$\widehat{M}_i^N = \frac{1}{N}\sum_{i=1}^{N} M_i(\xi^i),\ \widehat{q}_i^N = \frac{1}{N}\sum_{i=1}^{N} q_i(\xi^i), i = 1, 2 \tag{3}$$

are the sample-average mapping of $M_i(\xi(\omega))$ and $q_i(\xi(\omega)), i = 1, 2$ respectively. We refer to (1) as the true problem and (2) as the SAA problem to (1).

When using SAA method, an important issue is whether or not the sequence of solutions of SAA problem converges to a solution of true problem almost surely as the sample size tends to infinity. To answer this question, we need study the convergence of the sequence of solutions of SAA problem. In this paper we obtain the sufficient conditions ensure the existence of the solution of SAA problem when the sample size is large enough and the almost sure convergence of the sequence of solutions of SAA problem.

This paper is organized as follows: Section 2 gives preliminaries needed throughout the whole paper. In Section 3, convergence analysis of the SAA method is obtained. In Section 4, an example is illustrated to show the application of the convergence analysis.

Throughout this paper we use the following notations. Let $\|\cdot\|$ denote the Euclidean norm of a vector or the Frobenius norm of a matrix. We denote $B(x, \delta)$ the closed ball around x with radius $\delta > 0$ and I the identity matrix. For a continuously differentiable map $F : \Re^n \to \Re^m$, $\mathcal{J}F(z)$ denotes the Jacobian of F at $z \in \Re^n$. For Lipschitz vector-valued functions, Clarke [1] introduced the notation of generalized Jacobian as follows.

Definition 1. Let $F : \Re^n \to \Re^m$ be Lipschitz near $\bar{z} \in \Re^n$ and let w_F denote the set where F fails to be differentiable. The set

$$\partial^c F(\bar{z}) := \mathrm{co}\{\lim_{i\to\infty} \mathcal{J}F(z_i) \mid z_i \to \bar{z}, z_i \notin w_F\}$$

is called the generalized Jacobian of F at $\bar{z}$.

2. Main Results

In this section, we derive conditions ensuring the existence and convergence of solutions of SAA estimator (2).

In what follows, we make an assumption to facilitate the analysis.

Assumption 1 $M_i(\xi(\omega)), i = 1, 2$ are positive definite for a.e. $\omega \in \Omega$.

We provide the following lemma.

Lemma 2. *Suppose a, b, c, d, e, f are real numbers, then we have*

$$|\min\{a, b, c\} - \min\{d, e, f\}| \le |a - d| + |b - e| + |c - f|.$$

Proof.

$$\begin{aligned}
& |\min\{a, b, c\} - \min\{d, e, f\}| \\
= & \left|\tfrac{\min\{a,b\}+c-|\min\{a,b\}-c|}{2} - \tfrac{\min\{d,e\}+f-|\min\{d,e\}-f|}{2}\right| \\
\le & |\min\{a, b\} - \min\{d, e\}| + |c - f| \\
= & \left|\tfrac{a+b-|a-b|}{2} - \tfrac{d+e-|d-e|}{2}\right| + |c - f| \\
\le & |a - d| + |b - e| + |c - f|.
\end{aligned}$$

□

Next we show the consistency of convergence property of SAA problem to that of corresponding true problem.

Theorem 3. *Let $\bar{x} \in \Re^m$ be a solution of SEVLCP (1). Suppose Assumption 1 holds, then the SAA problem (2) with N samples has a solution x_N almost surely when N is large enough and $x_N \to \bar{x}$ w.p.1 as N tends to infinity.*

Proof. Let

$$F(x) = \min\{\mathbb{E}[M_1(\xi(\omega))]x + \mathbb{E}[q_1(\xi(\omega))],\ \mathbb{E}[M_2(\xi(\omega))]x + \mathbb{E}[q_2(\xi(\omega))],\ x\},$$

then $F(x) = 0$ and by Definition 1, we have

$$\partial^c F(\bar{x}) = \mathrm{co}\,\{\mathbb{E}[M_1(\xi(\omega))], \mathbb{E}[M_1(\xi(\omega))], I\}.$$

Under Assumption 1, the elements in $\partial^c F(\bar{x})$ are nonsingularity, therefore according to the implicit function theorem on nonsmooth equation[1, Theorem 7.1.1], there exist constant numbers $\delta > 0, \varepsilon > 0$ and a Lipschitz function with Lipschitz constant C,

$$\phi : B(0, \delta) \to B(\bar{x}, \varepsilon)$$

such that $\phi(0) = \bar{x}$ and $F(\phi(\delta')) = \delta'$ holds for every $\delta' \in B(0, \delta)$.

Let

$$\widetilde{H}_N(x) = F(x) - \widetilde{F}_N(x)$$

and $\hat{\delta} = \min\{\delta, (2c)^{-1}\varepsilon\}$, where

$$\widetilde{F}_N(x) = \min\{\widehat{M}_1^N x + \widehat{q}_1^N, \widehat{M}_2^N x + \widehat{q}_2^N, x\}.$$

By Lemma 1, we have

$$\begin{aligned} \|\widetilde{H}_N(x)\| &= \|F(x) - \widetilde{F}_N(x)\| \\ &\le \|\widehat{M}_1^N - \mathbb{E}[M_1(\xi(\omega))]\|\|x\| + \|\widehat{M}_2^N - \mathbb{E}[M_2(\xi(\omega))]\|\|x\|. \end{aligned}$$

By the Large Number Law, we obtain

$$\widehat{M}_i^N - \mathbb{E}[M_i(\xi(\omega))] \to 0 \text{ w.p.1 as } N \to \infty, i = 1, 2.$$

Hence it holds that

$$\lim_{N\to\infty} \sup_{x\in B(\bar{x},\varepsilon)} \|\widetilde{H}_N(x)\| = 0 \quad \text{w.p.1.} \tag{4}$$

Therefore when N is large enough, with probability one,

$$\sup_{x\in B(\bar{x},\varepsilon)} \|\widetilde{H}_N(x)\| < \hat{\delta}.$$

For any fixed N, define

$$\varphi_N(x) = \phi(\widetilde{H}_N(x)).$$

Since for N large enough,

$$\begin{aligned}\|\phi(\widetilde{H}_N(x)) - \bar{x}\| &= \|\phi(\widetilde{H}_N(x)) - \phi(0)\| \\ &\leq C\|\phi(\widetilde{H}_N(x))\| < c\hat{\delta} \\ &\leq C(2C)^{-1}\varepsilon < \varepsilon.\end{aligned}$$

Consequently, when N is large enough, $\varphi_N(\cdot)$ is a continuous mapping from the compact convex set $B(\bar{x}, \varepsilon)$ to itself. By Brouwer's fixed point theorem, when N is large enough, for each N, with probability one, there exists $x_N \in B(\bar{x}, \varepsilon)$ such that

$$\varphi_N(x_N) = x_N,$$

that is,

$$\phi(\widetilde{H}_N(x_N)) = x_N,$$

w.p.1. When N is large enough, for each fixed N,

$$\widetilde{H}_N(x_N) \in B(0, \delta)$$

w.p.1. Hence by the definition of $\phi(\cdot)$, we have w.p.1,

$$F(x_N) = F(\phi(\widetilde{H}_N(x_N))) = \widetilde{H}_N(x_N) = F(x_N) - \widetilde{F}(x_N),$$

that is, $\widetilde{F}(x_N) = 0$ w.p.1. On the other hand, by (4), we have

$$x_N = \varphi_N(x_N) = \phi(\widetilde{H}_N(x_N)) \to \phi(0) = \bar{x}$$

w.p.1 as $N \to \infty$. □

3. Numerical Test

In this section, we use projection and contraction method in [14] to solve the SAA problem. Under Assumption 1 in Theorem 3, we can easily show that convergence of the projection and contraction method for solving SAA problem.

Next we present a simple test obtained by the SAA projection and contraction method. Our numerical experiments are carried out in Matlab 7.1 running on a PC with Intel Pentium M of 1.60 GHz CPU. In our experiments, we employed the random number generator unifrnd, normrnd and exprnd in Matlab 7.1 to generate independently and identically distributed random samples $\{\xi^1, \xi^2, \cdots, \xi^N\}$. We solved problems (1.9) with different N by the contraction method in [14] to obtain the approximated optimal solution x_N. We recorded number of iterations (Iter).

Example 1. Consider the stochastic extended vertical linear complementary problem (1) in which $\xi = (\xi_1, \xi_2, \xi_3)$, ξ_1, ξ_2, ξ_3 are independent random variables, ξ_1 has an exponential distribution $\mathrm{EXP}(\lambda = 2)$, ξ_2 has a uniform distribution on $[0,1]$ and ξ_3 has a normal distribution $N(\mu, \sigma^2)$ with $\mu = 0.5, \sigma = 0.1$, $M_i(\xi(\omega))$ and $q_i(\xi(\omega)), i = 1,2$ are given by

$$M_1(\xi(\omega)) = \begin{pmatrix} 6\xi_1(\omega) & \xi_2(\omega) \\ 1 & 2 \end{pmatrix}, \quad q_1(\xi(\omega)) = (1, 2\xi_3(\omega))^T,$$

$$M_2(\xi(\omega)) = \begin{pmatrix} 4\xi_1(\omega) & \xi_2(\omega) \\ 2 & 3 \end{pmatrix}, \quad q_2(\xi(\omega)) = (-1, -2\xi_3(\omega))^T,$$

respectively. This problem has a unique solution $x^* = (0,0,0,0)^T$. The numerical results are shown in Table 1.

Table 1: The computational results

N	x_N	Iter
50	$10^{-7} \times (0.4773, 0.9546)^T$	320
100	$10^{-7} \times (0.2450, 0.3842)^T$	324
1000	$10^{-7} \times (0.1465, 0.2122)^T$	315

Our preliminary numerical results shown in Tables 1 reveal that the solutions of true problem can be obtained by its SAA problem. In the above example, we solve a SEVLCP which can be calculated an analytic solution in order to show the effect of increasing the sample size clearly. In practice, when we do not know the solution, it is hopeful that we could still observe convergence and solve the problem. This needs more tests and could be a subject of further investigation.

Acknowledgements

This work is supported by the National Natural Science Foundation of China under project No. 11671183 and No. 11671184 and Program for Liaoning Excellent Talents in University under project No. LJQ2015059.

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