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OSCILLATION CRITERIA FOR SECOND ORDER LINEAR GENERALIZED DIFFERENCE EQUATION

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Abstract: In this paper, we obtain some oscillation criteria for the second order generalized linear difference equation

$$\Delta_\ell^2 u(k) + p(k)u(k) = 0 \tag{1}$$

where $k \in [0,\infty)$, $p \in L'([u_0,\infty))$ and ℓ is a positive real.

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1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors [1] and [14]-[16] have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\}, \tag{2}$$

no significant progress took place on this line. But recently, when we took up the definition of Δ as given in (2) we developed the theory of difference equations in a different direction ([8]-[9]). For convenience, we labelled the operator Δ defined by (2) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory were established (see [8], [10], [11], [12], [13]).

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In 1957, Z. Nehari [4] and Jiqin Deng [3] worked on Oscillation criteria for second order linear differential equations. In fact, in the last few years, several monographs and hundreds of papers have been written; see for example the monographs [see [1], [2]] and the papers [[5] - [7]]. But only very few papers are available on the qualitative properties of solutions of difference equations involving Δ_ℓ. To move forward on this area, in this paper, we study the oscillation property of the second order generalized linear difference equation (1).

Throughout this paper we use the following notations:

(i) $\mathbb{N} = \{0, 1, 2, 3, \dots\},\ \mathbb{N}(a) = \{a, a+1, a+2, \dots\}$,

(ii) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \dots\}$,

(iii) $\lceil x \rceil$ upper integer part of x.

2. Preliminaries

Definition 1. [8] Let $u(k),\ k \in [0,\infty)$ be a real or complex valued function and $\ell \in (0,\infty)$. Then, the generalized difference operator Δ_ℓ is defined as

$$\Delta_\ell u(k) \ = \ u(k+\ell) - u(k). \tag{3}$$

Similarly, the generalized difference operator of the r^{th} kind is defined as

$$\Delta_\ell^r u(k) = \underbrace{\Delta_\ell(\Delta_\ell(\dots(\Delta_\ell u(k))\dots))}_{r\ times}. \tag{4}$$

Definition 2. [8] Let $u(k),\ k \in [0,\infty)$ be a real or complex valued function and $\ell \in (0,\infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows.

$$\text{If } \Delta_\ell v(k) \ = \ u(k), \ \text{ then } v(k) \ = \ \Delta_\ell^{-1} u(k) + c_j, \tag{5}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j),\ j = k - \left[\frac{k}{\ell}\right]\ell$.

In general, $\Delta_\ell^{-n} u(k) = \Delta_\ell^{-1}(\Delta_\ell^{-(n-1)} u(k))$ for $n \in \mathbb{N}(2)$.

Lemma 3. *[8] If the real valued function $u(k)$ is defined for all $k \in [a,\infty)$, then*

$$\Delta_\ell^{-1} u(k) = \sum_{r=1}^{\left[\frac{k-a}{\ell}\right]} u(k - r\ell) + c_j, \tag{6}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$.

Theorem 4. *If* $\Delta_\ell v(k) = u(k)$ *for* $k \in [k_2, \infty)$ *and* $j = k - k_2 - \left[\frac{k-k_2}{\ell}\right]\ell$, *then* $v(k) - v(k_2 + j) = \sum_{r=0}^{\left[\frac{k-k_2-j-\ell}{\ell}\right]} u(k_2 + j + r\ell)$.

Lemma 5. *[9]*

$$\begin{aligned}\Delta_\ell\{u(k)v(k)\} &= u(k+\ell)\Delta_\ell v(k) \ + v(k)\Delta_\ell u(k)\\ &= v(k+\ell)\Delta_\ell u(k) \ + u(k)\Delta_\ell v(k), \forall\ k \in \mathbb{N}_\ell(a).\end{aligned} \tag{7}$$

Definition 6. The solution $u(k)$ of (1) is called oscillatory if for any $k_1 \in [a, \infty)$ there exists a $k_2 \in \mathbb{N}_\ell(k_1)$ such that $u(k_2)u(k_2 + \ell) \leq 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(k)$ is not oscillatory, then it is said to be nonoscillatory (i.e. $u(k)u(k + \ell) > 0$ for all $k \in [k_1, \infty)$).

3. Main Results

Theorem 7. *If for large* $k \in \mathbb{N}_\ell(a)$,

$$\sum_{s=k}^{\infty} p(s) \geq \frac{\alpha_0}{k} \tag{8}$$

where $\alpha_0 > \frac{1}{4}$, *then (1) is oscillatory.*

Proof. Assume that (8) holds. Let us assume that $u(k)$ be a non oscillatory solution of (1). Without loss of generality we assume that $u(k) > 0$ for all $k \geq k_0 > 0$. By (8) it is easy to see that there exists $k_1 \geq k_0$ such that

$$\sum_{s=k}^{\infty} p(s) \geq \frac{\alpha_0}{k} \qquad \text{for} \qquad k \geq k_1, \tag{9}$$

which yields that there exists an integer $n(k) \geq k$, such that

$$\sum_{s=k}^{\infty} p(s) - \sum_{s=k'}^{\infty} p(s) \geq \frac{\alpha_1}{k} \quad \text{and} \quad \frac{{\alpha_1}^2}{k} - \frac{{\alpha_1}^2}{k'} \geq \frac{\alpha^2}{k} \quad \text{for} \quad k' \geq n(k)$$

$$\sum_{s=k}^{k'} p(s) \geq \frac{\alpha_1}{k}, \tag{10}$$

where $\alpha_0 \geq \alpha_1 \geq \alpha > \dfrac{1}{4}$.

$$\begin{aligned}
\text{Let } v(k) &= \frac{\Delta_\ell u(k)}{u(k)} \quad \text{for} \quad k \geq k_0,\\
\Delta_\ell u(k) &= v(k)u(k) \quad \text{for} \quad k \geq k_0, \quad \text{then}\\
\Delta_\ell^2 u(k) &= v(k+\ell)\Delta_\ell u(k) + \Delta_\ell v(k)u(k)\\
&= \left[v(k+\ell)v(k) + \Delta_\ell v(k)\right] u(k).
\end{aligned}$$

Substituting this into (1), we get

$$[\Delta_\ell v(k) + v(k+\ell)v(k) + p(k)] = 0 \quad \text{for} \quad k \geq k_0. \tag{11}$$

Summing (11) form k to k', and by (10) we get

$$v(k) - v(k') = \sum_{s=k}^{k'} v(s+\ell)v(s) + \sum_{s=k}^{k'} p(s) \geq 0, \tag{12}$$

for $k' \geq n(k)$ and $k \geq k_1$. If there exists $k_2 \geq k_1$, such that $v(k_2) < 0$, then from (12) for $k \geq n(k_2), v(k) < 0$. Therefore, $v(k)$ is eventually positive (or) eventually negative. If $v(k)$ is eventually negative, then there exists $k_3 \geq k$ such that $v(k) \leq 0$ for $k \geq k_3$ and

$$|v(k)| \geq \sum_{s=k_3}^{k} v(s+\ell)v(s) + \sum_{s=k_3}^{k} p(u) \qquad \text{for} \qquad k \geq n(k_3) \tag{13}$$

which, with (10) yields

$$|v(k)| \geq \sum_{s=k_3}^{k} v(s+\ell)v(s) + \sum_{s=k_3}^{k} p(s) \geq \sum_{s=k_3}^{k} v(s+\ell)v(s) + \frac{\alpha_1}{k_3} \tag{14}$$

i.e., $\qquad |v(k)| \geq \dfrac{\alpha_1}{k_3} \geq \dfrac{\alpha}{k} \quad$ for $\quad k \geq n(k_3)$.

Substituting this into (12), we obtain

$$|v(k)| \geq \sum_{s=n(k_3)}^{k} \frac{{\alpha_1}^2}{(s+\ell)s} + \frac{\alpha_1}{n(k_3)} \geq \frac{{\tau_0}^2 + \tau_0}{k} \tag{15}$$

for $k \geq n(n(k_3))$, where $\tau_0 = \alpha > \dfrac{1}{4}$.

If $v(k)$ is eventually positive, then there exists $k_4 \geq k_1$ such that $v(k) > 0$ for $k \geq k_4$, and from (10) and (12), we obtain

$$v(k) \geq \sum_{s=k}^{k'} v(s+\ell)v(s) + \sum_{s=k}^{k'} p(s) \geq \frac{\alpha}{k} \text{ for } k' \geq n(k) \text{ and } k \geq k_4 \tag{16}$$

Applying similar method as applied in getting (15), we obtain

$$|v(k)| \geq \frac{{\tau_0}^2 + \tau_0}{k} \quad \text{for} \quad k \geq k_4. \tag{17}$$

Setting $\tau_i = \tau_{i-1}^2 + \tau_0$, $i = 1, 2, 3, \ldots$ and taking $k_5 = \max\{n(n(k_3)), k_4\}$ from (15) and (17), we get

$$|v(k)| \geq \frac{\tau_1}{k} \quad \text{for} \quad k \geq k_5.$$

By induction, from (14) and (16), we can prove that

$$|v(k)| \geq \frac{\tau_i}{k} \quad \text{for} \quad k \geq k_5, \quad i = 1, 2, 3, \ldots \tag{18}$$

It is easy to see that, $|v(k)| = \infty$ for $k \geq k_5$. This contradiction yields that the proof is complete. □

Theorem 8. *If for large $k \in \mathbb{N}_\ell(a)$,*

$$\sum_{s=k}^{\infty} p(s) \leq \frac{1}{4k} \tag{19}$$

then (1) has an eventually positive solution.

Proof. Take $k_0 > 0$ such that

$$\sum_{s=k}^{\infty} p(s) \leq \frac{1}{4k} \quad k \geq k_0. \tag{20}$$

Define

$$v_0(k) = v_1(k) = \frac{1}{2k} \quad \text{for} \quad k \geq k_0, \tag{21}$$

$$v_{i+2}(k) = \sum_{s=k}^{\infty} v_i(s)v_{i+1}(s+\ell) + \sum_{s=k}^{\infty} p(s) \quad \text{for} \quad k \geq k_0. \tag{22}$$

which,together with (20) and (21), yield

$$v_2(k) = \sum_{s=k}^{\infty} v_0(s)v_1(s+\ell) + \sum_{s=k}^{\infty} p(u)$$
$$= \frac{1}{4}\sum_{s=k}^{\infty} \frac{1}{s(s+\ell)} + \sum_{s=k}^{\infty} p(s) \le \frac{1}{4}\frac{1}{k} + \frac{1}{4k} = \frac{1}{2k} = v_0(k).$$

By induction, one can easily prove in-general that,

$$\sum_{s=k}^{\infty} p(s) \le v_{i+1}(k) \le v_i(k) \le \frac{1}{2k} \quad \text{for} \quad k \ge k_0, i = 0, 1, 2, 3, \ldots$$

Therefore, the sequence $\{v_i(k)\}$ has a limiting function $v(k)$ on i with

$$\sum_{s=k}^{\infty} p(s) \le \lim_{i\to\infty} v_{i+1}(k) = v(k) \le \frac{1}{2k} \quad \text{for} \quad k \ge k_0.$$

From (16), we find

$$v(k) = \sum_{s=k}^{\infty} v(s+\ell)v(s) + \sum_{s=k}^{\infty} p(s) \quad \text{for} \quad k \ge k_0.$$

Operating both sides on Δ_ℓ, it follows that

$$\Delta_\ell v(k) = \Delta_\ell \left[\sum_{s=k}^{\infty} v(s+\ell)v(s) + \sum_{s=k}^{\infty} p(s)\right] = -v(k+\ell)v(k) - p(k).$$

$$\Delta_\ell v(k) + v(k+\ell)v(k) + p(k) = 0 \quad \text{for} \quad k \ge k_0. \tag{23}$$

Set $u(k_0) = 1$, $u(k) = u_0 \prod_{r=1}^{\lceil \frac{k-a}{\ell} \rceil} [v(k - r\ell) + 1]$ for $k \ge k_0$.

Then $u(k) > 0$, $\Delta_\ell u(k) = v(k)u(k)$ for $k \ge k_0$. Hence,

$$\Delta_\ell^2(u(k)) = v(k+\ell)\Delta_\ell(u(k)) + \Delta_\ell v(k)u(k) = -p(k)u(k)$$

$$\Delta_\ell^2 u(k) + p(k)u(k) = 0 \quad \text{for} \quad k \ge k_0. \tag{24}$$

Clearly, for $k \ge k_0, u(k) > 0$, is a positive solution of (1).

The proof is completed. □

Example 9. The second order linear generalized difference equation $\Delta_\ell^2 u(k) + \dfrac{3\ell^2}{(4k-\ell)(4k+3\ell)} u(k) = 0$, and for $p(k) = \dfrac{3\ell^2}{(4k-\ell)(4k+3\ell)}$, satisfies Theorem 8 and hence it has a positive solution. In fact $u(k) = \prod_{r=1}^{\lceil \frac{k-\ell}{\ell} \rceil} \dfrac{4(k-r\ell)}{4(k-r\ell)-\ell}$ with $u(0) = 1$ is a positive solution.

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