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FIXED POINT FOR CYCLIC MULTI-VALUED MAPPING IN COMPLETE DISLOCATED QUASI-b-METRIC SPACES

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Abstract: This paper aims to extend the concept of cyclic single-valued mapping to the case of multi-valued mapping in dislocated quasi-b-metric spaces. The existence of fixed point in dislocated quasi-b-metric spaces.

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Key Words: fixed point, multi-valued mapping, cyclic multi-valued mapping, dislocated quasi-b-metric spaces

1. Introduction and Preliminaries

Currently, fixed point theorem (we say that $x \in X$ is a *fixed point* of self-map T if $Tx = x$) has been widely studied. There are many results have been published. A fundamental result of fixed point theory is Banach contraction principle, which introduced in 1922 by Banach [1] as the following theorem:

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Theorem 1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Banach contraction mapping i.e., there exists a constant $a \in [0, 1)$ such that*

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

In 1969, Kannan [2] introduced the concept of Kannan mapping as follows:

Theorem 2. *Let (X, d) be a complete metric space and let T be a Kannan mapping i.e., there exists a constant $b \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$. Then T has a unique fixed point.

Chatterjea [7] introduced the concept of Chatterjea mapping in 1972, as follows:

Theorem 3. *Let (X, d) be a complete metric space and let T be a Chatterjea mapping i.e., there exists $c \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$. Then T has a unique fixed point.

The concept of Zamfirescu mapping was introduced in 1972 by Zamfirescu [4] as the following theorem:

Theorem 4. *Let (X, d) be a complete metric space and let T be a Zamfirescu mapping i.e., there exist real numbers $a \in [0, 1)$, $b \in [0, \frac{1}{2})$ and $c \in [0, \frac{1}{2})$ such that at least one of the following conditions is true:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$,

(z_3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

for all $x, y \in X$. Then T has a unique fixed point.

Next, we recall the concept of cyclic map as the following definition:

Definition 5. Let A and B be nonempty subsets of metric space (X, d) and $T : A \cup B \to A \cup B$. Then T is called a *cyclic mapping* if and only if $T(A) \subseteq B$ and $T(B) \subseteq A$.

In 2003, Kirk et al. [5] extended Banach contraction to a case of cyclic mapping as follows:

Theorem 6. *Let A and B be nonempty closed subsets of metric space (X, d) and let a cyclic mapping T be a cyclic-Banach contraction i.e., there exists $a \in [0, 1)$ such that*

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x \in A$ and $y \in B$. Then T has a unique fixed point in $A \cap B$.

In 2010, Karapinar and Erhan [6] extended the concept of Kannan mapping to cyclic mapping as the following theorem:

Theorem 7. *Let A and B be nonempty closed subsets of metric space (X, d) and let a cyclic mapping T be a cyclic-Kannan contraction i.e., there exists $b \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$$

for all $x \in A$ and $y \in B$. Then T has a unique fixed point in $A \cap B$.

The concept of metric space has been extended, improved and generalized in many different ways (see, [7],[8],[9]). Let X be a nonempty set, a function $d : X \times X \to [0, \infty)$ is called *distance* on X. The pair (X, d) is called a *distance space*. We need the following conditions:

(d_1) $d(x, y) = d(y, x) = 0$ implies $x = y$,

(d_2) $d(x, y) = d(y, x) = 0$ iff $x = y$,

(d_3) $d(x, y) = d(y, x)$,

(d_4) $d(x, y) \leq s\,[d(x, z) + d(z, y)]$ with constant $s \geq 1$,

for all $x, y, z \in X$. If (X, d, s) satisfies conditions $(d_2), (d_3)$ and (d_4), then it is called a *b-metric space*. If (X, d, s) satisfies conditions (d_2) and (d_4), then it is called a *quasi-b-metric space*. If it satisfies conditions (d_1), (d_3) and (d_4), then it is called a *dislocated b-metric space*. If (X, d, s) satisfies conditions (d_1) and (d_4), then it is called a *dislocated quasi-b-metric space* (dqb-metric space for short) , which was introduced by Klin-eam and Suanoom [10] in 2015. They also gave some examples of dqb-metric spaces:

Example 8. [10] Let $X = \mathbb{R}$ and let

$$d(x, y) = |x - y|^2 + \frac{|x|}{m} + \frac{|y|}{n},$$

where $m, n \in \mathbb{N} \setminus \{1\}$ with $m \neq n$. Then (X, d, s) is a dqb-metric space with the coefficient $s = 2$, but (X, d, s) is neither a quasi-b-metric space nor dislocated b-metric space.

Example 9. [10] Let $X = \mathbb{R}$ and let

$$d(x, y) = |x - y|^2 + 3|x|^2 + 2|y|^2,$$

then (X, d, s) is a dqb-metric space with the coefficient $s = 2$, but (X, d, s) is neither a quasi-b-metric space nor dislocated b-metric space.

Moreover, they introduced the notion of dqb-cyclic-Banach and dqb-cyclic-Kannan mapping and derived the fixed point theorems for such space as follows:

Theorem 10. *[10] Let A and B be nonempty closed subsets of a complete dqb-metric space (X, d, s). If T is a cyclic-Banach contraction, then T has a unique fixed point in $A \cap B$.*

Theorem 11. *[10] Let A and B be nonempty closed subsets of a complete dqb-metric space (X, d, s). If T is a cyclic-Kannan mapping,then T has a unique fixed point in $A \cap B$.*

In 1967, Nadler initiated the concept of multi-valued mapping. Let (X, d, s) be a dqb-metric space and $A, B \in \mathcal{CB}(X)$ where $\mathcal{CB}(X)$ denotes the family of non-empty closed and bounded subsets of X, define the functional $H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max \{h(A, B), h(B, A)\}.$$

where

$$h(A, B) = \sup \{d(a, B) : a \in A\},$$

with

$$d(a, B) = \inf \{d(a, b) : b \in B\}.$$

In the following lemma, we list some basic properties of H:

Lemma 12. *Let (X, d, s) be a dqb-metric space. For any $A, B \in \mathcal{CB}(X)$ and any $x, y \in X$, the following statements are true:*

1. $d(x, B) \leq d(x, b)$ *for any* $b \in B$,
2. $h(A, B) \leq H(A, B)$,
3. $d(a, B) \leq H(A, B)$ *for any* $a \in A$,
4. $H(A, A) = 0$,

5. $H(A,B) = H(B,A)$,

6. $d(x,A) \leq s\,[d(x,y) + d(y,A)]$.

Later, in 1969, he studied the existence of a fixed point of a contraction multi-valued mapping. An element $x \in X$ such that $x \in Tx$ is called a *fixed point* of a multi-valued mapping $T : X \longrightarrow \mathcal{CB}(X)$. We denote by F_T the set of all fixed point of T, i.e., $F_T = \{x \in X | x \in Tx\}$. The following result was proved by Nadler [11]:

Theorem 13. *[11] Let (X,d) be metric space and $T : X \longrightarrow \mathcal{CB}(X)$ is said to be a multi-value contraction mapping, i.e., there exists a real number $a \in [0,1)$ such that*

$$H(Tx,Ty) \leq ad(x,y)$$

for all $x, y \in X$. Then T has a fixed point.

The concept of Zamfirescu multi-valued mapping was introduced by Kaewkhao and Neammanee in 2010 [12] as follows:

Theorem 14. *[12] Let (X,d) be a complete metric space and $T : X \longrightarrow \mathcal{CB}(X)$ be a multi-valued Zamfirescu mapping, i.e., there exist real numbers a, b and c satisfying $a \in [0,1), b \in [0,\frac{1}{2})$ and $c \in [0,\frac{1}{2})$ such that, for all $x, y \in X$, at least one of the following conditions is satisfied:*

1. $H(Tx,Ty) \leq ad(x,y)$,

2. $H(Tx,Ty) \leq b\,[d(x,Tx) + d(y,Ty)]$,

3. $H(Tx,Ty) \leq c\,[d(x,Ty) + d(y,Tx)]$.

Then T has fixed point and it is complete.

In 2011, Kaewkhao and Neammanee [13] extended a concept of cyclic single-value to the multi-valued and also studied the existence of a fixed point which is stated in the following theorem:

Theorem 15. *[13] Let A and B be non-empty closed subsets of a complete metric space (X,d) and let T be a cyclic multi-valued mapping with closed and bounded value i.e., there exists a constant $a \in [0,1)$ such that*

$$H(Tx,Ty) \leq ad(x,y),$$

for all $x \in A, y \in B$. Then T has at least one fixed point in $A \cap B$.

Recently, Jinakul et al. [14] have studied the conditions for existence of a common fixed point of any two multi-valued mappings on a complete b-metric space. The result was presented as follows:

Theorem 16. *Let (X, d, s) be a complete b-metric space and let $S, T; X \to \mathcal{CB}(X)$ be multi-valued mappings satisfying*

$$H(Tx, Sy) \leq ad(x, Ty) + b\,(d(x, Sy) + d(Ty, Tx))$$

, where $a + 2bs < 1$, $a, b \geq 0$, $b < \frac{1}{s^2}$ for all $x, y \in X$. Then $F(T) = F(S) \neq \phi$ and $Tx = Sx = F(T)$ for all $x \in F(T)$.

In this paper, we extend the concept of cyclic single-valued mapping to the case of multi-valued mapping and derive the existence of fixed point in dqb-metric.

Now, we recall definitions about convergence of sequences, Cauchy sequence, closeness, boundedness and completeness in a dqb-metric space as follows:

Definition 17. Let (X, d, s) be a dqb-metric space.

(1) A sequence $\{x_n\}$ *dqb-converges* to $x \in X$ if and only if for all $\epsilon > 0$ there exist $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, $d(x, x_n) < \epsilon$. In this case, we write $\lim_{n\to\infty} d(x, x_n) = 0$.

(2) A sequence $\{x_n\}$ is called a *dqb-Cauchy sequence* if and only if for all $\epsilon > 0$ there exist $n_\epsilon \in \mathbb{N}$ such that for all $m > n \geq n_\epsilon$, $d(x_n, x_m) < \epsilon$. In this case, we write $\lim_{n\to\infty} d(x_n, x_m) = 0$.

(3) A dqb-metric space (X, d, s) is *complete* if every Cauchy sequence in X is convergent in X.

(4) A subset Y of X is *closed* if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$.

(5) A subset Y of X is *bounded* if there exists $M \in (0, \infty)$ such that $d(x, y) \leq M$ for all $x, y \in Y$.

The following results are useful for some of the proof in the paper:

Lemma 18. *Let (X, d, s) be a dqb-metric space and let A be a nonempty closed subset of X and $x \in X$. If $d(x, A) = 0$, then $x \in A$.*

Proof. Suppose $d(x, A) = 0$, we have $\inf\{d(x, a) : a \in A\} = 0$. For $n \in \mathbb{N}$, we have $\frac{1}{n} > 0$, then there exists a sequence $\{a_n\}$ in A such that $d(x, a_n) < \frac{1}{n}$.

Next, we will show that a sequence $\{a_n\}$ dqb-converges to x. Let $\epsilon > 0$, by the Archimedean property there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then for each $n \in \mathbb{N}$ such that $n \geq N$, we arrive at $\frac{1}{n} \leq \frac{1}{N}$. Thus $d(x, a_n) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Hence a sequence $\{a_n\}$ dqb-converges to x. Since A is closed, then we have $x \in A$. □

Lemma 19. *Let (X, d, s) be a dqb-metric space and $A, B \in \mathcal{CB}(X)$. If $H(A, B) = 0$, then $A = B$.*

Proof. Suppose that $H(A, B) = 0$, we have $\max\{h(A, B), h(B, A)\} = 0$, which implies that $h(A, B) = 0$ and $h(B, A) = 0$. Since $h(A, B) = 0$, then $d(a, B) = 0$, for all $a \in A$. By Lemma 18, we have $a \in B$, i.e., $A \subset B$. Similarly, we also have $A \subset B$. Thus $A = B$. □

Lemma 20. *Let (X, d, s) be a dqb-metric space and $A, B \in \mathcal{CB}(X)$. Then for each $q > 1$ and for all $x \in A$ there exists $y \in B$ such that $d(x, y) \leq qH(A, B)$.*

Proof. Let $q > 1$ and $x \in A$. By the definition of infimum, we have there exists $y \in B$ such that

$$d(x, y) \leq qd(x, B) \leq qh(A, B) \leq qH(A, B).$$

□

2. Main Results

In this section, we prove the existence of fixed point for cyclic multi-valued Banach contraction mappings and cyclic multi-valued Kannan mappings in dqb-metric spaces.

Theorem 21. *Let A and B be non-empty closed subsets of a complete dqb-metric space (X, d, s) and $T : A \cup B \to \mathcal{CB}(X)$ be a cyclic multi-valued mapping. If there exists a constant $a \in (0, 1)$ with $sa < 1$ such that*

$$H(Tx, Ty) \leq ad(x, y),$$

for all $x \in A$ and $y \in B$. Then T has at least one fixed point in $A \cap B$.

Proof. Let $1 < q < \frac{1}{a}$ and let $x_0 \in A$ be fixed. Choose $x_1 \in Tx_0 \subseteq B$. By Lemma 20, there exists $x_2 \in Tx_1 \subseteq A$ such that

$$\begin{aligned} d(x_1, x_2) &\leq qH(Tx_0, Tx_1) \\ &\leq qad(x_0, x_1). \end{aligned}$$

Again by Lemma 20, there exists $x_3 \in Tx_2 \subseteq B$ such that

$$\begin{aligned} d(x_2, x_3) &\le qH(Tx_1, Tx_2) \\ &\le qad(x_1, x_2) \\ &\le (qa)^2 d(x_0, x_1). \end{aligned}$$

Continuing this process, for $n \in \mathbb{N}$ we have $x_{n+1} \in Tx_n$ such that

$$\begin{aligned} d(x_n, x_{n+1}) &\le (qa)^n d(x_0, x_1) \\ &= \alpha^n d(x_0, x_1), \text{ where } \alpha = qa < 1. \end{aligned}$$

Next, we will show that the sequence $\{x_n\}$ is dqb-Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m = n + k$ for some $k \in \mathbb{N}$, by using the triangular inequality, we have

$$\begin{aligned} d(x_n, x_m) &= d(x_n, x_{n+k}) \\ &\le sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^k d(x_{n+k-1}, x_{n+k}) \\ &\le s\alpha^n d(x_0, x_1) + s^2 \alpha^{n+1} d(x_0, x_1) + \cdots + s^k \alpha^{n+k-1} d(x_0, x_1) \\ &= \left[1 + (s\alpha) + (s\alpha)^2 + \cdots + (s\alpha)^{k-1}\right] s\alpha^n d(x_0, x_1) \\ &= \left[\frac{1 - (s\alpha)^k}{1 - s\alpha}\right] s\alpha^n d(x_0, x_1). \end{aligned}$$

Since $\alpha < 1$, then, we get $\lim_{n\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is dqb-Cauchy sequence. Since (X, d, s) is complete, then sequence $\{x_n\}$ dqb-converges to some $x \in X$. We note that $\{x_{2n}\}$ is a sequence in A and $\{x_{2n-1}\}$ is a sequence in B in a way that both sequences tend to the same limit x. Since A and B are closed, so $x \in A \cap B$. Next, we show that $x \in Tx$. Consider

$$\begin{aligned} d(x, Tx) &\le s\left[d(x, x_{n+1}) + d(x_{n+1}, Tx)\right] \\ &= sd(x, x_{n+1}) + sd(x_{n+1}, Tx) \\ &\le sd(x, x_{n+1}) + sH(Tx_n, Tx) \\ &= sd(x, x_{n+1}) + sH(Tx, Tx_n) \\ &\le sd(x, x_{n+1}) + sad(x, x_n). \end{aligned}$$

Since sequence $\{x_n\}$ dqb-converges to x, then, we get $\lim_{n\to\infty} d(x, Tx) = 0$. Hence $d(x, Tx) = 0$. Since Tx is closed and by Lemma 18, we have $x \in Tx$. Therefore T has at least one fixed point in $A \cap B$ as required. □

Finally, we introducing the concept of cyclic multi-valued Kannan mapping in dqb-metric space:

Theorem 22. *Let A and B be non-empty closed subsets of a complete dqb-metric space (X, d, s) and $T : A \cup B \to \mathcal{CB}(X)$ be a cyclic multi-valued mapping. If there exists a constant $b \in (0, \frac{1}{2})$ with $sb < \frac{1}{2}$ such that*

$$H(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)],$$

for all $x \in A$ and $y \in B$. Then T has at least one fixed point in $A \cap B$.

Proof. Let $1 < q < \frac{1}{2b}$ and let $x_0 \in A$ be fixed. Choose $x_1 \in Tx_0 \subseteq B$. By Lemma 20, there exists $x_2 \in Tx_1 \subseteq A$ such that

$$\begin{aligned} d(x_1, x_2) &\leq qH(Tx_0, Tx_1) \\ &\leq qb\,[d(x_0, Tx_0) + d(x_1, Tx_1)] \\ &\leq qbd(x_0, x_1) + qbd(x_1, x_2), \end{aligned}$$

so

$$d(x_1, x_2) \leq \frac{qb}{1 - qb} d(x_0, x_1),$$

Again by Lemma 20, there exists $x_3 \in Tx_2 \subseteq B$ such that

$$\begin{aligned} d(x_2, x_3) &\leq qH(Tx_1, Tx_2) \\ &\leq qb\,[d(x_1, Tx_1) + d(x_2, Tx_2)] \\ &\leq qbd(x_1, x_2) + qbd(x_2, x_3), \end{aligned}$$

so

$$\begin{aligned} d(x_2, x_3) &\leq \frac{qb}{1 - qb} d(x_1, x_2) \\ &\leq \left(\frac{qb}{1 - qb}\right)^2 d(x_0, x_1). \end{aligned}$$

Continuing this process, for $n \in \mathbb{N}$ we have $x_{n+1} \in Tx_n$ such that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \left(\frac{qb}{1 - qb}\right)^n d(x_0, x_1) \\ &= \beta^n d(x_0, x_1) \text{ where } \beta = \frac{qb}{1 - qb} < 1. \end{aligned}$$

Next, we will show that the sequence $\{x_n\}$ is dqb-Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m = n + k$, by using the triangular inequality, we have

$$\begin{aligned} d(x_n, x_m) &= d(x_n, x_{n+k}) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^k d(x_{n+k-1}, x_{n+k}) \\ &\leq s\beta^n d(x_0, x_1) + s^2\beta^{n+1} d(x_0, x_1) + \cdots + s^k \beta^{n+k-1} d(x_0, x_1) \\ &= \left[1 + (s\beta) + (s\beta)^2 + \cdots + (s\beta)^{k-1}\right] s\beta^n d(x_0, x_1) \\ &= \left[\frac{1-(s\beta)^k}{1-s\beta}\right] s\beta^n d(x_0, x_1). \end{aligned}$$

Since $\beta < 1$, then, we get $\lim_{n\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is dqb-Cauchy sequence. Since (X, d, s) is complete, then sequence $\{x_n\}$ dqb-converges to some $x \in X$. We note that $\{x_{2n}\}$ is a sequence in A and $\{x_{2n-1}\}$ is a sequence in B in a way that both sequences tend to the same limit x. Since A and B are closed, we have $x \in A \cap B$. Next, we show that $x \in Tx$. Consider

$$\begin{aligned} d(x, Tx) &\leq sd(x, x_{n+1}) + sd(x_{n+1}, Tx) \\ &\leq sd(x, x_{n+1}) + sH(Tx_n, Tx) \\ &\leq sd(x, x_{n+1}) + sb\left[d(x_n, Tx_n) + d(x, Tx)\right] \\ &\leq sd(x, x_{n+1}) + sbd(x_n, x_{n+1}) + sbd(x, Tx) \\ &\leq sd(x, x_{n+1}) + sb\beta^n d(x_0, x_1) + sbd(x, Tx). \end{aligned}$$

So

$$d(x, Tx) \leq \frac{s}{1-sb} d(x, x_{n+1}) + \frac{sb}{1-sb}\beta^n d(x_0, x_1).$$

Since sequence $\{x_n\}$ dqb-converges to x and $\beta < 1$, then, we get $\lim_{n\to\infty} d(x, Tx) = 0$. Hence $d(x, Tx) = 0$. Since Tx is closed and by Lemma 18, we have $x \in Tx$. Therefore T has at least one fixed point in $A \cap B$ as required. □

The following corollary can be considered as a particular case of Theorem 21 if we take $s = 1$:

Corollary 23. *Let A and B be non-empty closed subsets of a complete dqb-metric space $(X, d, 1)$ and let T be a cyclic multi-valued mapping with closed and bounded value i.e., there exists a constant $a \in [0, 1)$ such that*

$$H(Tx, Ty) \leq ad(x, y),$$

for all $x \in A, y \in B$. Then T has at least one fixed point in $A \cap B$.

Notice that Corollary 23 is in fact the previous result on complete metric space (X, d) proved by Kaewkhao and Neammanee [13].

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