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COLUMN MEAN VANISHING MATRICES

Myungsup Kim[1], Do Y. Kwak[2 §]

[1]Tunitel Co, Kwan Pyong Dong
Yusung-Ku, Daejeon, 34017, KOREA

[2]Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Daejeon, 34141, KOREA

Abstract: In this paper, we study some properties of a special class of matrices having orthonormal columns. These matrices appear in some applications, especially in wireless communications. We study the column property and spectral decomposition. Using these properties, we suggest a new method of generating such matrices. For N even, the new method gives rise to a matrix which is more efficient. Numerical examples to compare two methods are included.

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1. Introduction

In this paper, we study some properties of a special class of rectangular matrices having orthonormal columns. These matrices appear in some applications, especially, in OFDM-wireless communications [2],[3],[7],[5]. In OFDM, the signals must not have large jumps near the boundary of frequencies. Otherwise, the spectrum spread would interfere with neighboring channels. This kind of matrices can be used to design a technology to carry multiple signals without interference between channels [6]. We will study some properties and

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[§]Correspondence author

and a spectral decomposition of them. Using these properties, we suggest a new method of generating such matrices. We use techniques of singular value decomposition(SVD) and nearest orthogonal matrix generation [1],[8],[9]. Numerical examples to compare two methods are included. For N even, the new method gives rise to a matrix which is more efficient.

2. Generation of a Matrix Having Orthonormal, Column Mean Vanishing Property

We start a notion about the matrices.

Definition 2.1. We say a matrix A has a *column mean vanishing (CMV) property* if the sum of each column is zero.

Let F and F^{-1} be the matrix representation of Fast Fourier Transform(FFT) and inverse Fast Fourier Transform (IFFT): Given any $n \times 1$ vector $\mathbf{x}$, its FFT and IFFT are given by

$$\hat{\mathbf{x}} = F\mathbf{x} \text{ and } \check{\mathbf{x}} = F^{-1}\mathbf{x}, \tag{2.1}$$

where $F = (f_{jk})$ and $F^{-1} = (f'_{jk})$ are

$$f_{jk} = e^{-\theta_j k}, \; f'_{jk} = \frac{1}{n} e^{\theta_j k}, \text{ with } \theta_j = \frac{2\pi i j}{n}, \; i = \sqrt{-1}. \tag{2.2}$$

Let $L = n \gg m$ and $N \ll LM = N - 1$. Using an initial $N \times (N-1)$ matrix, IFFT, zero padding, removing jumps, and truncation, FFT, we will generate a new matrix having the desired properties. The following scheme is suggested in [6].

Algorithm Orth-CMV

1. Given a $N \times (N-1)$ initial matrix K with orthonormal columns.
2. Multiply by $n \times N$ matrix $\mathbf{P}$ obtaining $A = \mathbf{P}K$.
3. Perform IFFT to obtain $\mathbf{F}^{-1}(\mathbf{P}K)$.
4. Subtract the first row from all the rows, the result is $\mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K)$.
5. Perform FFT to get $\mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K)$.
6. Multiply $\mathbf{P}^T$ to obtain $\hat{K} := \mathbf{P}^T \circ \mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K)$.
7. Normalize each column of $\hat{K}$, call it by $\hat{K}_0$.

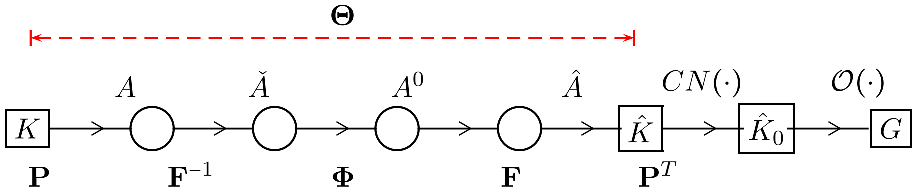

Figure 1: Signal flow diagram for matrix generation. $\mathbf{\Theta}$ is a jump removing operator in frequency domain, $CN(\cdot)$ and $\mathcal{O}(\cdot)$ are normalization and orthogonalization operator, resp.

8. Let $G = UV^H$ where $U\Sigma V^H$ is the SVD of $\hat{K}_0$.

In the next, we list some matrix notations:

Matrices	K	Initial matrix to generate the matrix G
	A	Permuted and zero padded matrices of K
	$\check{A}$	IFFT performed matrix of A
	A^0	Internal jump removed matrix
	$\hat{A}$	FFT performed matrix of A
	$\hat{K}$	Jump removed matrix from K
	$\hat{K}_0$	Normalized matrix after jump removing
	G	matrix with orthonormal columns
Operators	$\mathbf{P}$	Permutation and zero padding matrix
	$\mathbf{F}^{-1}$	IFFT matrix
	$\mathbf{\Phi}$	Jump removing matrix
	$\mathbf{F}$	FFT matrix
	$\mathbf{P}^T$	Permuting and truncating matrix
	$\mathbf{\Theta}$	Jump removing matrix in frequency domain

Now we will explain more details of the algorithm: We first assume $N = 2m+1$. Let the initial matrix K of size $N \times (N-1)$ be given by

$$K = \frac{1}{\sqrt{2}} \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \vdots & & & & \iddots & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \hline \vdots & & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{2.3}$$

Step (2). Permute and Pad Zeros

Starting from K, we construct an $L \times M$ matrix as follows: Move the last $m+1$ rows of K to the first $m+1$ rows of K. Next fill it with pad with $L-M$ zero rows (called zero padding). This process can be expressed as $\mathbf{P}K$ where

$$\mathbf{P} = \left[\begin{array}{c|c} \underline{0}_{(m+1)\times m} & \mathbf{I}_{m+1} \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline \mathbf{I}_m & \underline{0}_{m\times(m+1)} \end{array}\right]. \tag{2.4}$$

Here $\mathbf{I}_m, \mathbf{I}_{m+1}$ are identity matrices of size m and $m+1$.

Steps (3) and (4) : IFFT Followed by Subtraction of the First Row

Let us use the notation $K = (k_{ij})$ and $K_1 = (k_{ij}^1) := \mathbf{P}K$. Let $\check{K}_1 = \mathbf{F}^{-1}(\mathbf{P}K)$ be the inverse FFT of $\mathbf{P}K$. By definition of IFFT (2.1), the first row of $\check{K}_1$ is

$$\check{\mathbf{k}}_1 = \left[\check{k}_{11}, \check{k}_{12}, \cdots, \check{k}_{1M}\right] = \frac{1}{n}\left[\sum_{i=0}^{n-1} k_{i1}^1, \sum_{i=0}^{n-1} k_{i2}^1, \cdots, \sum_{i=0}^{n-1} k_{iM}^1\right]. \tag{2.5}$$

The process of IFFT of permuting the rows and eliminating first row is described by

$$\check{K}_1{}' = \begin{bmatrix} \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \check{k}_{21} & \check{k}_{22} & \cdots & \check{k}_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ \check{k}_{n,1} & \check{k}_{n,2} & \cdots & \check{k}_{n,M} \end{bmatrix} - \begin{bmatrix} \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \end{bmatrix} \equiv \check{K}_1 - \check{K}_1^*. \tag{2.6}$$

Here $\check{K}_1^*$ is the matrix all of whose rows are the vector $\check{\mathbf{k}}_1$. Let $\mathbf{\Phi}$ be the operator involved in the elimination of first row in step (4) of the algorithm. Then

$$\mathbf{\Phi} \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{2.7}$$

Step (5) and (6). FFT, Truncation and Band Limit

The step (5) is FFT and the process in step (6) corresponds to the permutation and truncation.

Lemma 2.1. *The result of step (5) is*

$$\mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K) = \mathbf{P}K - \mathbf{F}(\check{K}_1^*).$$

Hence after step (6) we obtain the matrix

$$\hat{K} := K - \mathbf{P}^T \circ \mathbf{F}(\check{K}_1^*). \tag{2.8}$$

Proof. From (2.6) we see

$$\begin{aligned} \mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K) &= \mathbf{F} \circ F^{-1}(\mathbf{P}K) - \mathbf{F}(\check{K}_1^*) \\ &= \mathbf{P}K - \mathbf{F}(\check{K}_1^*), \end{aligned}$$

and hence

$$\begin{aligned} \mathbf{P}^T \circ \mathbf{F} \circ \mathbf{\Phi} \circ F^{-1}(\mathbf{P}K) &= \mathbf{P}^T\mathbf{P}K - \mathbf{P}^T \circ \mathbf{F}(\check{K}_1^*) \\ &= K - \mathbf{P}^T \circ \mathbf{F}(\check{K}_1^*) := \hat{K}. \end{aligned}$$

□

Now we compute the matrix $\mathbf{P}^T \circ F(\check{K}_1^*)$. Since the FFT of the vector $[1, \cdots, 1]^T$ is $[n, 0, \cdots, 0]^T$, we see the FFT of $\check{K}_1^*$

$$\mathbf{F}(K_1^*) = n \begin{bmatrix} \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.9}$$

Here the first row is

$$n\left[\check{k}_{11}, \cdots, \check{k}_{1M}\right] = \left[\sum_{i=1}^{n} k_{i1}, \cdots, \sum_{i=1}^{n} k_{iM}\right] \equiv \left[x_{11}, x_{12}, \cdots, x_{1M}\right] \equiv \mathbf{x}. \tag{2.10}$$

By multiplying by $\mathbf{P}^T$, we obtain $N \times M$ matrix

$$\mathbf{P}^T \circ \mathbf{F}(\check{K}_1^*) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -x_{11} & -x_{12} & \cdots & -x_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.11}$$

Now as the result of step (6), we obtain

$$\hat{K} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ k_{m+1,1} - x_{11} & k_{m+1,2} - x_{12} & \cdots & k_{m+1,M} - x_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ k_{N1} & k_{N2} & \cdots & k_{NM} \end{bmatrix} = K - \mathbf{P}^T \circ F(\check{K}_1^*). \quad (2.12)$$

Lemma 2.2. *The sum of all entries of each column of the matrix $\hat{K}$ or $\hat{K}_0$ is zero.*

Proof. Clear from (2.10) and (2.12). □

Nearest Orthogonal Matrix

The step (8) can be described by another way: Let

$$\hat{K}_0 = U\Sigma V^H \quad (2.13)$$

be the singular value decomposition (SVD) of $\hat{K}_0$. Then the matrix $G = UV^H$ is the same as the polar decomposition [4]:

Lemma 2.3.

$$UV^H = \hat{K}_0 \left(\hat{K}_0^H \hat{K}_0\right)^{-1/2}. \quad (2.14)$$

It is also well-known that G is the nearest matrix to $\hat{K}_0$ having orthonormal columns. (Corollary 2.3 of [4] and the remark following it.)

Theorem 2.1. *The matrix G obtained in step (8) satisfies CMV property:*

Proof. Let $\vec{\mathbf{1}} = [1, \cdots, 1]$. Then by Lemma 2.2, we have

$$\vec{\mathbf{1}} \cdot \hat{K}_0 = [0, 0, \cdots, 0].$$

Hence by (2.14) we see

$$\vec{\mathbf{1}} \cdot G = \vec{\mathbf{1}} \cdot \hat{K}_0 \left(\hat{K}_0^H \hat{K}_0\right)^{-1/2} = [0, 0, \cdots, 0].$$

□

2.1. The Case N Even

Now we investigate the even case.

Steps (1), (2)

The initial matrix is different from the odd case. Let $N = 2m$ and let the initial matrix K of size $N \times (N-1)$ be given by

$$K_e^{(0)} = \frac{1}{\sqrt{2}} \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 1 & 1 \\ \vdots & & & \iddots & 0 & 0 \\ \hline 0 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \hline \vdots & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \tag{2.15}$$

This is obtained by removing the central row and first column from the odd case (2.3). Using this matrix we will generate a matrix having the desired properties. Starting from $K_e^{(0)}$, we proceed similarly to the odd case. The matrix $\mathbf{P}$ (resp. $\mathbf{P}^T$) involved in zero padding (resp. truncation) is the following

$$\mathbf{P} = \left[\begin{array}{c|c} \underline{0}_{m\times m} & \mathbf{I}_m \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline \mathbf{I}_m & \underline{0}_{m\times m} \end{array}\right], \quad \mathbf{P}^T = \left[\begin{array}{c|ccc|c} \underline{0}_{m\times m} & 0 & \cdots & 0 & \mathbf{I}_m \\ \hline \mathbf{I}_m & 0 & \cdots & 0 & \underline{0}_{m\times m} \end{array}\right]. \tag{2.16}$$

Steps (3) through (8) are the Same

3. Some Spectral Analysis - Property of Even (Odd) Columns

Let $\mathbf{k}_i$ and $\mathbf{g}_i$ and denote the i-th column of the matrix K and G respectively.

Lemma 3.1. *Then we have the following.*

1. *For odd N, the even columns of the matrix K are the same as those of K*

2. *For even N, the odd columns of the matrix K are the same as those of K.*

For example, if $N = 5$, then

$$\mathbf{k}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \mathbf{g}_2, \quad \mathbf{k}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \mathbf{g}_4.$$

Proof. We explain the case when $N = 5$, the general case is exactly the same. We see from (2.12) that $\hat{K}$ in step (6) has changed only in the third row. In fact, $\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 & 2 & 0 \end{bmatrix}$. Hence we have

$$\hat{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} - x_{11} & k_{32} & k_{33} - x_{13} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \\ k_{51} & k_{52} & k_{53} & k_{54} \end{bmatrix} \tag{3.1}$$

Hence the even columns of $\hat{K}$ or the normalized matrix $\hat{K}_0$ are equal to the corresponding even columns of K. □

Lemma 3.2. *Then we have the following.*

1. *Let N be odd. The even columns of $\hat{K}_0$ are orthogonal to all other columns of $\hat{K}_0$. As a consequence, for all even j, $\mathbf{e}_j$ is an eigenvector of $\hat{K}_0^H \hat{K}_0$ corresponding to eigenvalue* 1.
2. *Let N be even. The odd columns of $\hat{K}_0$ are orthogonal to all other columns of $\hat{K}_0$. As a consequence, for all odd j, $\mathbf{e}_j$ is an eigenvector of $\hat{K}_0^H \hat{K}_0$ corresponding to eigenvalue* 1.

Proof. Assume N is odd. The proof of even case is exactly the same. Let $\hat{\mathbf{k}}_{0,i}$ be the i-th column of $\hat{K}_0$. Then $\hat{\mathbf{k}}_{0,i} = \mathbf{k}_i$ for i even and $\hat{\mathbf{k}}_{0,i} = \mathbf{k}_i - \mathbf{x}$ for i odd. Since the $m+1$-st entry of even columns is zero, the subtraction of $\mathbf{x}$ from the third row in (3.1) does not affect the orthogonality. Hence when j is even

$$\hat{\mathbf{k}}_{0,i}^T \cdot \hat{\mathbf{k}}_{0,j} = \begin{cases} (\mathbf{k}_i - \mathbf{x})^T \cdot \mathbf{k}_j = \delta_{ij} & \text{if } i \text{ is odd} \\ \mathbf{k}_i^T \cdot \mathbf{k}_j = \delta_{ij} & \text{if } i \text{ is even} \end{cases}.$$

Hence the j -th column of $\hat{K}_0^H \hat{K}_0$ satisfies

$$\hat{K}_0^H \hat{K}_0 \mathbf{e}_j = \hat{K}_0^H \hat{\mathbf{k}}_{0,j} = \begin{bmatrix} \hat{\mathbf{k}}_{0,1}^T \cdot \hat{\mathbf{k}}_{0,j} \\ \hat{\mathbf{k}}_{0,2}^T \cdot \hat{\mathbf{k}}_{0,j} \\ \vdots \\ \hat{\mathbf{k}}_{0,N}^T \cdot \hat{\mathbf{k}}_{0,j} \end{bmatrix} = \mathbf{e}_j. \tag{3.2}$$

This means that when j is even, the j-th columns of $\hat{K}_0$ are orthogonal to all other columns of $\hat{K}_0$. Clearly (3.2) implies the second assertion of the lemma. □

Example 3.1. For $N = 5$ we see

$$\hat{K}_0^H \hat{K}_0 = \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & \boxed{0} & -1 & \boxed{0} \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & |0| & 1 & |0| \\ 0 & |2| & 0 & |0| \\ 1 & |0| & 3 & |0| \\ 0 & |0| & 0 & |2| \end{bmatrix} \quad (3.3)$$

The zeros in the box keep the even columns of $\hat{K}_0$ orthogonal to other columns. In view of (3.2), $\hat{K}_0^H \hat{K}_0$ has two eigenvectors $\mathbf{e}_j, j = 2, 4$ corresponding to the eigenvalue 1.

Theorem 3.2. *We have the following result.*

1. *For N odd, G has eigenvector $\mathbf{e}_j$ for all j even with the corresponding eigenvalue* 1. *The even columns of $G = \hat{K}_0(\hat{K}_0^H \hat{K}_0)^{-1/2}$ are the same as those of K.*

2. *For N even, G has eigenvector $\mathbf{e}_j$ for all j odd with the corresponding eigenvalue* 1. *The odd columns of $G = \hat{K}_0(\hat{K}_0^H \hat{K}_0)^{-1/2}$ are the same as those of K.*

Proof. Since $\hat{K}_0 = U\Sigma V^H$ from (2.13), we have the spectral decomposition of $\hat{K}_0^H \hat{K}_0$:

$$\hat{K}_0^H \hat{K}_0 = V\Sigma^H \Sigma V^H := V\Lambda V^H (V^H = V^{-1}), \quad (3.4)$$

where by (3.2) Λ and V have the following form:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \ (N \text{ odd}) \ \Lambda = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} (N \text{ even}). \quad (3.5)$$

The eigenvector corresponding to the eigenvalue 1 is $\mathbf{e}_j$. Hence when N is odd, $V\mathbf{e}_j = \mathbf{e}_j$ for j even and so $V^{-1}\mathbf{e}_j = V^{-1}V\mathbf{e}_j = \mathbf{e}_j$. Hence for each even j,

$$\begin{aligned} \hat{K}_0(\hat{K}_0^H \hat{K}_0)^{-1/2}\mathbf{e}_j &= \hat{K}_0 V\Lambda^{-1/2}V^{-1}\mathbf{e}_j \\ &= \hat{K}_0 V\Lambda^{-1/2}\mathbf{e}_j \end{aligned}$$

$$= \hat{K}_0 V \mathbf{e}_j$$
$$= \hat{K}_0 \mathbf{e}_j.$$

In view of (3.1), this is the same as j-th column of K (normalization does not change even columns). While when N is even, the same conclusion holds for j odd. □

4. Numerical Example

In all of the computations, we used the Matlab.

Example 4.1. When $N = 5$ and $M = 4$, the initial matrix is

$$K = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

With this we get

$$G = \begin{bmatrix} -0.1954 & -0.0000 & 0.5117 & 0.7071 \\ 0.5117 & 0.7071 & -0.1954 & -0.0000 \\ -0.6325 & -0.0000 & -0.6325 & 0.0000 \\ 0.5117 & -0.7071 & -0.1954 & -0.0000 \\ -0.1954 & 0.0000 & 0.5117 & -0.7071 \end{bmatrix}$$

Example 4.2. When $N = 7, M = 6$, we get

$$G = \begin{bmatrix} -0.1466 & -0.0000 & -0.1466 & -0.0000 & 0.5605 & 0.7071 \\ -0.1466 & -0.0000 & 0.5605 & 0.7071 & -0.1466 & 0.0000 \\ 0.5605 & 0.7071 & -0.1466 & 0.0000 & -0.1466 & 0.0000 \\ -0.5345 & 0.0000 & -0.5345 & -0.0000 & -0.5345 & 0.0000 \\ 0.5605 & -0.7071 & -0.1466 & 0.0000 & -0.1466 & 0.0000 \\ -0.1466 & 0.0000 & 0.5605 & -0.7071 & -0.1466 & 0.0000 \\ -0.1466 & 0.0000 & -0.1466 & -0.0000 & 0.5605 & 0.7071 \end{bmatrix}$$

where the following initial matrix was used.

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Example 4.3 (N even). When $N = 6, M = 5$, we get

$$G = \begin{bmatrix} -0.1327 & -0.1723 & -0.0000 & 0.5348 & 0.7071 \\ -0.1327 & 0.5348 & 0.7071 & -0.1723 & 0.0000 \\ 0.8934 & -0.1327 & 0.0000 & -0.1327 & -0.0000 \\ -0.3626 & -0.5924 & 0.0000 & -0.5924 & -0.0000 \\ -0.1327 & 0.5348 & -0.7071 & -0.1723 & 0.0000 \\ -0.1327 & -0.1723 & -0.0000 & 0.5348 & -0.7071 \end{bmatrix}$$

5. A Direct Generation of G

In this section we introduce a method of generating G without using FFT and SVD. To do that, we first observe the following fact:

- If a matrix has a CMV property, then the Step (4) of the algorithm is not necessary.

Hence the operator $\mathbf{\Phi}$ in the Figure **??** becomes identity and we have

$$\mathbf{P}^T\mathbf{F}\mathbf{\Phi}\mathbf{F}^{-1}\mathbf{P} = \mathbf{P}^T\mathbf{F}\mathbf{I}_L\mathbf{F}^{-1}\mathbf{P} = \mathbf{P}^T\mathbf{P} = \mathbf{I}_N.$$

Here $\mathbf{I}_L$ and $\mathbf{I}_N$ are identity operators in $\mathbb{R}^{L\times L}$ and $\mathbb{R}^{N\times N}$ respectively. Hence the whole process reduces to finding the nearest orthogonal matrix only. (step (8)) Using this fact, we suggest a simple method to generate such a matrix. The minimal requirements are

1. Every column of the matrix G is a unit vector.
2. All the columns of the matrix G are orthogonal to each other
3. The sum of each columns of the matrix G is zero.

So we need at least three variables to design a matrix. In fact, three variables are enough for N odd. For even N, it seems four variables are needed. We use an example to explain. Let $N = 5, M = 4$. From Theorem 3.2, we know the even columns of G are the same as those of K. We set

$$G = \begin{bmatrix} c & 0 & b & \frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{2}} & c & 0 \\ a & 0 & a & 0 \\ b & -\frac{1}{\sqrt{2}} & c & 0 \\ c & 0 & b & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and impose the orthonormality and CMV condition:

$$a^2 + 2(b^2 + c^2) = 1 \tag{5.1}$$

$$a^2 + 4bc = 0 \tag{5.2}$$

$$a + 2b + 2c = 0. \tag{5.3}$$

Solving (5.1)-(5.2) we get $2c^2 + 2b^2 - 4cb = 2(b - c)^2 = 1$ and together with (5.3) we get

$$a = -0.6325,\ b = 0.5117,\ c = -0.1954.$$

These values gives the same G as Example 4.1. The solution is not unique and we see another solution

$$a = -0.5345,\ b = 0.5345,\ c = -0.2673.$$

Example 5.1. For $N = 7$ we assume the matrix of the following form:

$$G = \begin{bmatrix} c & 0.0 & c & 0.0 & b & \frac{1}{\sqrt{2}} \\ c & 0.0 & b & \frac{1}{\sqrt{2}} & c & 0.0 \\ b & \frac{1}{\sqrt{2}} & c & 0.0 & c & 0.0 \\ a & 0.0 & a & 0.0 & a & 0.0 \\ b & -\frac{1}{\sqrt{2}} & c & 0.0 & c & 0.0 \\ c & 0.0 & b & -\frac{1}{\sqrt{2}} & c & 0.0 \\ c & 0.0 & c & 0.0 & b & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

We impose orthonormality conditions and CMV condition as before, to have

$$a^2 + 2(b^2 + 2c^2) = 1 \tag{5.4}$$

$$a^2 + 4bc + 2c^2 = 0 \tag{5.5}$$

$$a + 2b + 4c = 0. \tag{5.6}$$

Solving this algebraic system we get the following values

$$a = -0.5345,\ \ b = 0.5605,\ \ c = -0.1466.$$

The corresponding matrix is

$$G = \begin{bmatrix} -0.1466 & -0.0000 & -0.1466 & -0.0000 & 0.5605 \\ -0.1466 & -0.0000 & 0.5605 & 0.7071 & -0.1466 \\ 0.5605 & 0.7071 & -0.1466 & 0.0000 & -0.1466 \\ -0.5345 & 0.0000 & -0.5345 & -0.0000 & -0.5345 \\ 0.5605 & -0.7071 & -0.1466 & 0.0000 & -0.1466 \\ -0.1466 & 0.0000 & 0.5605 & -0.7071 & -0.1466 \\ -0.1466 & 0.0000 & -0.1466 & -0.0000 & 0.5605 \end{bmatrix}$$

which is the same as Example 4.2. Another solution is

$$a = 0.5345,\ \ b = 0.3823,\ \ c = -0.3248.$$

More generally, we can construct any size of K by assuming the even columns are

$$\begin{aligned} &[0, 0, \cdots, 0, \sqrt{2}/2, 0, -\sqrt{2}/2, 0, 0, \cdots, 0]^T, \\ &[0, \cdots, 0, \sqrt{2}/2, 0, 0, 0, -\sqrt{2}/2, 0, \cdots, 0]^T, \\ &\cdots\cdots\cdot \\ &[\sqrt{2}/2, 0, 0, \cdots, 0, 0, 0, 0, \cdots, 0, -\sqrt{2}/2]^T \end{aligned}$$

while the odd columns are of the form

$$\begin{aligned} &[c, c, \cdots, c, c, b, a, b, c, c, \cdots, c, c]^T, \\ &[c, c, c, \cdots, b, c, a, c, b, c, \cdots, c, c]^T, \\ &[c, c, \cdots, b, c, c, a, c, c, b, \cdots, c, c]^T, \\ &\cdots\cdots\cdot \\ &[b, c, c, c, \cdots, c, c, a, c, c, \cdots, c, b]^T \end{aligned}$$

Now impose the following conditions: for $j = 2, 3, \cdots,$

$$a^2 + 2(b^2 + (j-1)c^2) = 1 \tag{5.7}$$

$$a^2 + 4bc + 2(j-2)c^2 = 0 \tag{5.8}$$

$$a + 2b + 2(j-1)c = 0. \tag{5.9}$$

By solving this simple algebraic system by Newton's method with certain initial values, we can find a desired matrix G of any size.

Example 5.2. $[9 \times 9]$ Assume

$$G = \begin{bmatrix} c & 0 & c & 0 & c & 0 & b & \frac{1}{\sqrt{2}} \\ c & 0 & c & 0 & b & \frac{1}{\sqrt{2}} & c & 0 \\ c & 0 & b & \frac{1}{\sqrt{2}} & c & 0 & c & 0 \\ b & \frac{1}{\sqrt{2}} & c & 0 & c & 0 & c & 0 \\ a & 0 & a & 0 & a & 0 & a & 0 \\ b & -\frac{1}{\sqrt{2}} & c & 0 & c & 0 & c & 0 \\ c & 0 & b & -\frac{1}{\sqrt{2}} & c & 0 & c & 0 \\ c & 0 & c & 0 & b & -\frac{1}{\sqrt{2}} & c & 0 \\ c & 0 & c & 0 & c & 0 & b & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

We need to solve

$$a^2 + 2(b^2 + 3c^2) = 1 \tag{5.10}$$

$$4c^2 + 4bc + a^2 = 0 \tag{5.11}$$

$$a + 2b + 6c = 0. \tag{5.12}$$

For example, with initial guess $[a, b, c] = [-0.6, -0.7, 0.3]$, we obtain

$$a = -0.4714, \ b = -0.4714, \ c = 0.2357.$$

But with different initial $[-0.6, 40.7, 0.3]$, we obtain

$$a = -0.4714, \ b = 0.5893, \ c = -0.1179.$$

Example 5.3. $[11 \times 11]$ Let

$$G = \begin{bmatrix} c & -0.0 & c & -0.0 & c & -0.0 & c & -0.0 & b & \frac{1}{\sqrt{2}} \\ c & -0.0 & c & -0.0 & c & -0.0 & b & \frac{1}{\sqrt{2}} & c & 0.0 \\ c & -0.0 & c & -0.0 & b & \frac{1}{\sqrt{2}} & c & -0.0 & c & 0.0 \\ c & -0.0 & b & \frac{1}{\sqrt{2}} & c & -0.0 & c & -0.0 & c & 0.0 \\ b & \frac{1}{\sqrt{2}} & c & 0.0 & c & 0.0 & c & -0.0 & c & 0.0 \\ a & 0.0 & a & 0.0 & a & 0.0 & a & 0.0 & a & 0.0 \\ b & -\frac{1}{\sqrt{2}} & c & 0.0 & c & 0.0 & c & -0.0 & c & 0.0 \\ c & 0.0 & b & -\frac{1}{\sqrt{2}} & c & 0.0 & c & 0.0 & c & 0.0 \\ c & 0.0 & c & 0.0 & b & -\frac{1}{\sqrt{2}} & c & 0.0 & c & 0.0 \\ c & -0.0 & c & -0.0 & c & -0.0 & b & -\frac{1}{\sqrt{2}} & c & 0.0 \\ c & -0.0 & c & -0.0 & c & -0.0 & c & -0.0 & b & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

For this case, we need to solve

$$a^2 + 2(b^2 + 4c^2) = 1 \tag{5.13}$$
$$6c^2 + 4bc + a^2 = 0 \tag{5.14}$$
$$a + 2b + 8c = 0. \tag{5.15}$$

With initial value $[-0.4264, 0.6083, -0.0988]$, we get

$$a = -0.4264, \;\; b = 0.6083, \;\; c = -0.0988.$$

while with initial value $[1.0, 0.3, -1.6]$, we get

$$a = 0.4264, \;\; b = 0.5230, \;\; c = -0.1841.$$

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