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RIGHTIDEALS IN Γ-SEMIGROUPS

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Abstract: In This article to prove an equivalent of the krull- intersection theorem $\cap_{m=1}^{\infty} M^m = 0$ in 2.11 and thus it is shown that the Γ - semigroup with the krull - intersection property is nearly a principal Γ - rightideal Γ - semigroup. Every principal Γ - rightideal Γ - semigroup need not have the krull - intersection property and so it is researched what type of principal Γ - rightideal Γ - semigroups, not vitally containing an identity, have this intersection property. It is initiate in 2.11, 2.13 and 3.1, that the equivalent of the krull - intersection theorem is true under some additional conditions unlike in ring theory.

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1. Introduction

A Γ -semigroup S is called a principal Γ-rightideal R is principal. Naturally one may ask whether it is possible to determine the structure of S by specializing some property of M. By virtue of the existence of the unique maximal Γ-rightideal. The influenced the author to prove an analogue of the krull inter-

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section theorem ($\cap_{m=1}^{\infty} M^m = 0$) and thus it is shown that the Γ-semigroup with the krull intersection property is nearly a principal Γ-rightideal Γ-semigroup.

2. Main Results

Definition 2.1. A Γ - semigroup S is called a principal Γ - rightideal if every right Γ - ideal R is principal. i.e $R = f \cup f\Gamma s$ for some $f \in S$. Since the Γ -rightideals in a principal Γ - rightideal of S form a chain under set inclusion, S is either right simple or has a unique maximal proper Γ - right ideal M.

Theorem 2.2. If S is a principal Γ - rightideal in Γ- semigroup, then the Γ- rightideals form a chain under set inclusion.

Proof. If P and Q are any two Γ - rightideals then $P \cup Q = e \cup e\Gamma S$ for some $e \in S$.

Since $e \in P$ or $e \in Q$, we must have $P \subseteq Q$ or $Q \subseteq P$.

Remark 2.3. The above theorem is not true. Suppose that S is not principal.

Example 2.4. Let us assume that $S = \{0, p, q\}$ subject to the condition $p^2 = p$; $q^2 = p\alpha q = q\alpha p = 0$; $\alpha \in \Gamma$ and $0\beta s = s\beta 0 = 0$ for each $s \in S$, $\beta \in \Gamma$.

Clearly $(0, a) \cap (0, b) = 0$ and excepts S every other Γ- ideal is principal.

We shall now indicate the nature of elements of S, if S is a right Noetherian Γ- semigroup fulfills the property of being right uniform, and derive from this the nature of elements of principal rightideals in Γ-semigroups.

Definition 2.5. A Γ- semigroup S is called a Γ-right Noetherian if S fulfills ascending chain conditions on Γ-rightideals.

Definition 2.6. A Γ- semigroup S is called a Γ-rightuniform if every Γ - rightideal has nonempty intersection with every other Γ- right ideal.

If S has zero, this intersection should be nonzero. For any subsets B of a Γ - semigroup S we denote $B^R = \{x \in S / B\Gamma x = 0\}$.

Lemma 2.7. Let S be a Γ - right Noetherian semigroup including zero. If $b^R \neq 0$, then either b is nilpotent or $\exists$ a positive integer $k \ni (b^k)^R \cap (b^k \Gamma S \cup b^k) = 0$.

Proof. By Γ - right Noetherian condition , the ascending chain of Γ - rightideals $b^R \subset (b^2)^R \subset \cdots$ teriminates. $\exists$ a positive integer $k \ni (b^k)^R = (b^{k+1})^R = \cdots$.

Let $x \in (b^k)^R \cap (b^k \Gamma S \cup b^k)$.

If $x = b^k$, $b^{2k} = 0$ and hence b is nilpotent.
If $x \neq b^k$; $x = b^k \alpha y$ and also $b^k \alpha x = 0$; $\alpha \in \Gamma$.
This implies that $b^{2k}\alpha y = 0$ and so $y \in (b^{2k})^R = (b^k)^R$. Thus $b^k \alpha y = x = 0$; $\alpha \in \Gamma$. As an easy consequence of 2.2 and 2.6, we receive the next result.

Theorem 2.8. Let S be a Γ-semigroup including zero. Then , for any $b \in S$, either $b^R = 0$ or b is nilpotent if S fulfills either of the following conditions:

a) S is a Γ-right Noetherian and Γ-right uniform.

b) S is a principal Γ-right ideal Γ-semigroup.

Remark 2.9. The theorem 2.8 fails if we leave the uniform condition from (a). Also 2.8 is not true if we suppose that every Γ-rightideal exclude S is principal in the condition (b). Both these facts can be shown by the example 2.4.

Let S be a principally generated Γ-semigroup; i.e S is of the form $e \cup e \cup \Gamma S$ where $e \in S$. Since S is not right simple, therefore S has a unique maximal proper Γ-rightideal M, which is the union of all Γ-rightideals of $S \nsubseteq e$. If $S\Gamma M \neq S$, then $S\Gamma M \subseteq M$ and hence M is a Γ-ideal. In this case $M = b \cup b\Gamma S$, then it is simple to show that $M^n = b^n \cup b^n \Gamma S$ for each positive integer n. However $\cap_{n=1}^{\infty} M^n$ need not be empty. If $S\Gamma M = S$ and $M = b \cup b\Gamma S$, then it can be simply verified that $M^n = b\Gamma S$ for each positive integer $n > 1$, which implies that $\cap_{n=1}^{\infty} M^n \neq \emptyset$. But if $\cap_{n=1}^{\infty} M^n = 0$ need not be empty. If $S\Gamma M = S$ and $M = b \cup b\Gamma S$, then it can be simply proved that $M^n = b\Gamma S$ for every positive integer $n > 1$ which implies that $\cap_{n=1}^{\infty} M^n \neq \phi$. But if $\cap_{n=1}^{\infty} M^n = \phi$ it will be observed below that S is nearly principal Γ - right ideal Γ- semigroup.

Notation 2.10. Let (S, Γ, M) assigned a Γ- semigroup with a unique maximal right ideal M. Throughout this paper we suppose that M exists.

Theorem 2.11. Let (S, Γ, M) be a Γ - semigroup such that $S = e \cup e\Gamma S$ and $M = b \cup b\Gamma S$. Then $\cap_{m=1}^{\infty} M^m = \phi$ if and only if:

i) every Γ-rightideal is either $b^m \Gamma S$ or $(b^m \Gamma S \cup b^n)$;

ii) $M^p \neq M^q$ for any pair of natural numbers p and q and $p \neq q$;

iii) $S\Gamma M \neq S$.

If S has zero and M^r is zero then $\cap_{m=1}^{\infty} M^m = 0$ and this Γ- idealstructure of S is as represented above.

Proof. Let B be a proper Γ- rightideal. Then $B \subseteq M$.

Assume that $\cap_{m=1}^{\infty} M^m = \emptyset$. Then there exist a least natural number r such that $B \subseteq M^r$ and $B \nsubseteq M^{r+1}$, i.e. $\exists$ an $y \in B \ni y = b^r$ or $y = b^r \alpha x$ where $x \notin M$; $\alpha \in \Gamma$. If $y = b^r$, then $B = b^r \cup b^r \alpha S$.

In the second case $x \notin M$ implies $x \cup x\Gamma S = S$. Now $y = b^r \alpha x$ and so $y\Gamma S = b^r \alpha x \Gamma S = b^r \Gamma (S \backslash x) = b^r \Gamma S \backslash b^r \alpha x$. Since $y\Gamma S \subseteq B$, $b^r \Gamma S \backslash b^r \alpha x \subseteq B$. But $b^r \alpha x \in B$. Hence $b^r \Gamma S \subseteq B \subseteq b^r \Gamma S \cup b^r$. Thus $B = b^r \Gamma S$ or $b^r \Gamma S \cup b^r$.

If $\cap_{m=1}^{\infty} M^m = \phi$, then the condition (ii) is obviously satisfied. Further more, from the introduction of this section, the condition (iii) is clear. Conversely , let $\cap_{m=1}^{\infty} M^m$ be non-empty.

If $\cap_{m=1}^{\infty} M^m = b^m \Gamma S \cup b^m$, then $M^m = M^{m+1}$, which is a contradiction.

If $\cap_{m=1}^{\infty} M^m = b^m \Gamma S$, then $M^{m+1} = b^m \Gamma S$. Hence $M^{m+1} = M^{m+2}$ which is a contradiction.

The remainder of the proof follows from the first part.

Corollary 2.12. If (S, Γ, M) is a Γ- semigroup with identity and $M = b\Gamma S$, then $\cap_{m=1}^{\infty} M^m$ is empty if and only if:

(i) every Γ-rightideal is of the form M^r.

(ii) $M^p \neq M^q$ for natural number p and q and $p \neq q$.

(iii) $S\Gamma M \neq S$.

Not every principal Γ- right ideal Γ - semigroup (S, Γ, M) satisfies the property that $\cap_{m=1}^{\infty} M^m$ is empty. For, let $S = \{b, b^2, b^3 = b^4\}$ or $S = \{1, b, b^2 = b^3\}$.

The following theorem gives a class of principal Γ- rightideal Γ- semigroups with the above conditions.

Theorem 2.13. Let (S, Γ, M) be a left Noetherian Γ- semigroup without identity such that $S\Gamma M \neq S$, $S = e \cup e\Gamma S$ and $M = b \cup b\Gamma S$. Then $\cap_{m=1}^{\infty} M^m = \phi$ (and hence the only Γ-right ideals of S are either M^r or $b^r \Gamma S$) if either of the following condition is fullfilled:

(i) there exists no p and q in S such that $p = q\alpha p$ or $p = p\alpha q$.

(ii) S is Γ- cancellative and S has no Γ- idempotents.

Proof. Since $S\Gamma M \neq S$, M is an ideal and so $M^m = b^m \cup b^m \Gamma S$. Assume that $\cap_{m=1}^{\infty} M^m \neq \phi$. Then $p = b\alpha t_1 = b^2 \alpha t_2 = \cdots$. By virtue of the condition (i) and (ii) it is clear that $t_i \neq t_j$. Then by Γ-left Noetherian condition the chain $s_1 \cup S\Gamma t_1 \subset s_2 \cup S\Gamma t_2 \subset \cdots$ teriminates. So $p = b^n \alpha t_n$, where $t_n \notin M$. Then $t_n = t_n \cup t_n \Gamma S = S$ which implies that $p\Gamma S = b^n \alpha t_n \Gamma S = b^n \alpha (S \backslash t_n) = b^r \alpha S \backslash b^n \alpha t_n$. Hence $p \cup p\Gamma S = b^n \Gamma S$. Since $b^{2n} \in M^{2n+1}$. Hence $b^{2n} = b^{2n+1}$ or $b^{2n} = b^{2n+1} \alpha s$, which implies that the condition (i) can not fulfilled.

Assume condition (ii),evidently $b^{2n} \neq b^{2n+1}$. Then $b^{2n} = b^{2n+1}\alpha s$ implies that $b = b^2\alpha s$ by the cancellative condition. Now $a \in M$ implies $a = b^t\alpha q$ where $q \notin M$, by the Γ - left Noetherian condition (otherwise $a = b\alpha q, q \in M$ then $q = ab\alpha q_1 \cdots$ and so we have an ascending chain of Γ-left ideals $q \cup S\Gamma q \subset q_1 \subset S\Gamma q_1 \subset \cdots$) clearly $b^t\alpha q = b = b^{t-1}\alpha(b^2\beta s)rq = b^t\alpha(b\beta srq)$. Since $q \notin M$, $q \cup q \cup \Gamma S = S$. Then $b\alpha s\beta q = q$ or $b\alpha s\beta q = q\alpha m$ where $m \in S$; $\alpha, \beta \in \Gamma$. The first equality is not possible, since otherwise $q \in M$, a contradiction. Thus we have $b^t\alpha q = b^t\alpha q\beta m$. Then by the cancellative condition, $q = q\alpha m$ and so $q\alpha m = q\alpha m^2$. Again by the cancellative condition, $m = m^2$, which is a contradiction by hypothesis.

Remark 2.14 The theorem 2.13 fails if we drop the condition that S is cancellative in (i). Note the example where $S = \{b, b^2, b^3 = b^4\}$.

Note 2.15. In 2.13 condition (ii) always implies condition (i). The author does't know whether these two condition are equivalent.

3. Principal Γ-Rightideal Γ-Semigroups with Identity

In 2.12 we considered conditions under which $\cap_{m=1}^{\infty} M^m = \phi$, for a principal Γ-right ideal Γ-semigroup (S, Γ, M) with identity. In 2.13 we obtained a class of principal right ideal Γ-semigroups without identity having the krull- intersection property.

The purpose of this section is to show that theorem 2.13 can be proved with a weaker hypothesis if the Γ- semigroup has identity.

Theorem 3.1. Let (S, Γ, M) be a left Noetherian Γ-semigroup with identity suppose $M = b\Gamma S$ and S satisfies either of the following conditions:

(i) S is concellative.

(ii) there exists no element q in S such that $b\alpha q = q$ and $M^x \neq M^y$ for all positive integer x and y and $x \neq y$.

Then every element $m \in M$ is of the form $b^x\alpha q$, x being a positive integer and q is a left union in S. Every Γ-rightideal is of the form $b^t\Gamma S$. If $S\Gamma M \neq S$, then every $m \in M$ is of the form $b^x\alpha q$ where q is a unit of S. Also every Γ-rightideal of S is of the form M^t and $\cap_{m=1}^{\infty} M^m = \emptyset$.

Proof. Let $m \in M$. Then $m = b\alpha t_1$. Evidently $t_1 \neq m$ by condition (i). Condition (ii) also implies $t_1 \neq m$, since otherwise $m = b\alpha m$ implies $1 = b$ by the cancellative property. If $t_1 \in M$, then $t_1 = b\alpha t_2$. Proceeding in this manner, we obtain an ascending chain of Γ - leftideals $S\Gamma t_1 \subset S\Gamma t_2 \subset \cdots$ by the left Noetherian condition this chain teriminates. Hence $m = b^x\alpha q$, $q \notin M$, $\alpha \in \Gamma$. Since every Γ-rightideal is included in the unique maximal Γ-rightideal M, we have $q\alpha t = 1$, i.e q is a Γ-left unit. Let A be a Γ-rightideal. If $r \in A$, $r = b^x\alpha q$ where $q\alpha t = 1$. Hence $b^x = r\alpha t \in A$. Let $k = min\{x/b^x \in A\}$. Then $A = b^k\Gamma S$.

If $S\Gamma M \neq S$, then M is an ideal and so $M^t = b^t\Gamma S$ for every t. Furthermore every $r \in M$ is of the form $b^x\alpha q$. q must be a unit. For since $q \notin M$ and so $S\Gamma q \not\subset M$. Hence $S\Gamma q = S$, which implies $t\alpha q = 1$.

If $\cap_{m=1}^{\infty} M^m \neq \emptyset$, then $\cap_{m=1}^{\infty} M^m = M^t$ and so $M^t = M^{t+1} = \cdots$ This leads to a contradiction by condition (ii). Also by condition (i) $M^t = M^{t+1} \Rightarrow b^t = b^{t+1}\alpha y \Rightarrow 1 = b\alpha y \Rightarrow M = S$ a contradiction.

Remark 3.2. The hypothesis that there exist no r such that $b\alpha r = r$ is essential in 3.1.

Example 3.3. For, condition the Γ - semigroup $\{a, x, y, 1\}$ with the multiplication table

$\bullet$	a	x	y	1
a	a	a	a	a
x	a	x	x	x
y	a	x	x	y
1	a	x	y	1

In this Γ- semigroup the rightideals are $y\Gamma S$, $y^2\Gamma S$ and $a\Gamma S$ and the unique maximal rightideal is $y\Gamma S$; $y\alpha a = a$ and $a \neq y^n$ for any n.

Remark 3.3. Let us give an example [1; 109] illustrating the theorem for the Γ- semigroup (S, Γ, M) such that $S\Gamma M = S$. Let $S = \{1, x, y\}$ subject to the condition $x\alpha y = 1$. Then $M = y\Gamma S$ and there exist no b in S such that $y\alpha b = b$. Also S is cancellative. Every element of $S$is of the form $y^i x^i$ where $i, j = 0, 1, - - -$; $x^0 = y^0 = 1$ and every Γ-rightideal is of the form $y^i\Gamma S$. However $M^t = M = x\Gamma S$ forevery positive integer t.

Theorem 3.4. Let M be a nilpotent and $M = a\Gamma S$ in a Γ- semigroup (S, Γ, M) with zero and identity. Then $\cap_{n=1}^{\infty} M^n = 0$. If $S\Gamma M = S$, then S is right simple, while if $S\Gamma M \neq S$, then $S = G \cup M$, where G is the group of units of S and $M = \{a^r\alpha g : 1 \leq r \leq t; g \in G$ and t is the index of nilpotency of $M\}$

Proof. Clearly $\cap_{m=1}^{\infty} M^m = 0$. If $S\Gamma M = S$, then $0 = M^t = M$ and hence S is right simple.

To prove the last part: $x \neq 0$; $x \neq a^r$ and $x \in M \Rightarrow x = a\alpha t$. $t \neq x$ since otherwise, $x = a\alpha x \Rightarrow a^{t-1}\alpha x = 0 \Rightarrow a^{t-2}\alpha x = 0 \Rightarrow - - - \Rightarrow a\alpha x = 0 \Rightarrow$

$x = 0$. If t is not a unit , $t = a\alpha S$ and thus $x = a^2\alpha S$. By proceeding in this manner, since $a^t = 0$, we must have $x = a^r\alpha y$, where y is a unit.

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