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THE POINT WISE BEHAVIOR OF 2-DIMENSIONAL WAVELET EXPANSIONS IN $L^p(R^2)$

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Abstract: We show that the two dimensional wavelet expansion of $L^p(R^2)$ function for $1 < p < \infty$ converges pointwise almost everywhere under wavelet projection operator. This convergence can be established by assuming some minimal regularity to get the rapidly decreasing for two dimensional wavelet ψ_{j_1,j_2,k_1,k_2}. The Kernel function of the wavelet projection operator in two dimension converges absolutely, distributionally and is bounded. Also the wavelet expansions in two dimension are controlled in a magnitude by the maximal function operator. All these conditions can be utilized to achieve the convergence almost everywhere.

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1. Introduction

In this work, the two dimension wavelet expansion are used under wavelet projections operator to expand the $L^p(R^2)$ functions f in two dimension, for $1 < p < \infty$. This expansion generated by two dimensional orthogonal wavelets basis functions $\{\psi_{\mathbf{j},\mathbf{k}}\} = \{2^{-j_1/2}2^{-j_2/2}\psi(2^{j_1}\cdot - k_1)\psi(2^{j_2}\cdot - k_2)\}$ for $\{\mathbf{j} \le \mathbf{J}\}$ and $\mathbf{j}, \mathbf{k} \in Z^2$ with 0-regularity, which form an orthonormal basis for $L^2(R^2)$. The wavelet expansion of f under wavelet projection operator in two dimension is given by the following formula:

$$P_{\mathbf{J}}f = \sum_{\mathbf{j}<\mathbf{J}} \sum_{\mathbf{k}} a_{\mathbf{j},\mathbf{k}} \psi_{\mathbf{j},\mathbf{k}}(x_1, x_2), \tag{1}$$

where

$$a_{\mathbf{j},\mathbf{k}} = \int_R \int_R f(y_1, y_2)\psi_{\mathbf{j},\mathbf{k}}(y_1, y_2)dy_1 dy_2 \tag{2}$$

for $\mathbf{j} = (j_1, j_2)$, $\mathbf{k} = (k_1, k_2)$ and $(x_1, x_2) \in R^2$ is a Lebesgue point of f.

The above expression lead us to define the two dimension kernel function $K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)$, of the wavelet projection operator P as follow:

$$P_{J_1,J_2}f = \int_R \int_R \sum_{j_1<J_1} \sum_{j_2<J_2} \sum_{k_1} \sum_{k2} \psi_{j_1,j_2,k_1,k_2}(x_1, x_2)\psi_{j_1,j_2,k_1,k_2}(y_1, y_2)$$

$$\times f(y_1, y_2)dy_1 dy_2.$$

$$= \int_R \int_R \sum_{j_1<J_1} \sum_{j_2<J_2} \sum_{k_1} \sum_{k2} 2^{j_1}2^{j_2}\psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2)\psi(2^{j_1}y_1 - k_1)$$

$$\times \psi(2^{j_2}y_2 - k_2)f(y_1, y_2)dy_1 dy_2.$$

Hence, the kernel function is

$$K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2) = \sum_{j_1\le J_1} \sum_{j_2\le J_2} \sum_{k_1} \sum_{k_2} \psi(2^{j_1}x_1 - k_1) \tag{3}$$

$$\times \psi(2^{j_2}x_2 - k_2)\psi(2^{j_1}y_1 - k_1)\psi(2^{j_2}y_2 - k_2),$$

for $\{(J_1, J_2) : J_1 \ge j_1, J_2 \ge j_2\}$ and $(J_1, J_2) \in Z^2$.

From the definition of multi-resolution analysis of $L^2(R^2)$ space, we can find that at $J_1 = 0 = J_2$ the projection operator $P_{0,0}$ onto $V_{(0,0)}$ space come to be as:

$$P_{0,0}f = \int\limits_R \int\limits_R K_{0,0}(x_1, x_2, y_1, y_2) f(y_1, y_2) dy_1 dy_2,$$

where $K_{0,0}(x_1, x_2, y_1, y_2) = \sum\limits_{k_1} \sum\limits_{k_2} \phi(x_1 - k_1)\phi(x_2 - k_2)\phi(y_1 - k_1)$

$$\times \phi(y_2 - k_2).$$

But when $f(x_1, x_2) \in V_{(J_1,J_2)}$ then $f(2^{-j_1}x_1, 2^{-j_2}x_2) \in V_{(0,0)}$, this is lead to

$$P_{0,0}f = \int\limits_R \int\limits_R 2^{J_1}2^{J_2} K(2^{-J_1}x_1, 2^{-J_2}x_2, 2^{-J_1}y_1, 2^{-J_2}y_2) f(y_1, y_2) dy_1 dy_2,$$

where

$$K(2^{-J_1}x_1, 2^{-J_2}x_2, 2^{-J_1}y_1, 2^{-J_2}y_2) = \sum_{j_1<0} \sum_{j_2<0} \sum_{k_1} \sum_{k_2} \psi(2^{-j_1}x_1 - k_1)$$
$$\times \psi(2^{-j_2}x_2 - k_2)\psi(2^{-j_1}y_1 - k_1)\psi(2^{-j_2}y_2 - k_2).$$

As a consequence of these above observations, we apply the following theorem:

Theorem 1. *Consider $\psi_{\mathbf{j},\mathbf{k}}(x_1, x_2)$ is the wavelet basis function for $L^2(R^2)$ space such that $\mathbf{j} = (j_1, j_2)$, $\mathbf{k} = (k_1, k_2)$ and $\mathbf{j}, \mathbf{k} \in Z^2$. Then the kernel function K_{J_1,J_2} of the wavelet projections operator $P_{J_1,J_2}f$ can be defined almost everywhere by one of the following:*

$$K(2^{-J_1}x_1, 2^{-J_2}x_2, 2^{-J_1}y_1, 2^{-J_2}y_2) = \sum_{j_1<0} \sum_{j_2<0} \sum_{k_1} \sum_{k_2} \psi_{j_1,j_2,k_1,k_2} \tag{4}$$

$$(2^{-j_1}x_1 - k_1, 2^{-j_2}x_2 - k_2) \times \psi_{j_1,j_2,k_1,k_2}(2^{-j_1}y_1 - k_1, 2^{-j_2}y_2 - k_2).$$

$$K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2) = -\sum_{j_1\geq 0} \sum_{j_2\geq 0} \sum_{k_1} \sum_{k_2} \psi_{j_1,j_2,k_1,k_2} \tag{5}$$

$$(2^{j_1}x_1 - k_1, 2^{j_2}x_2 - k_2) \times \psi_{j_1,j_2,k_1,k_2}(2^{j_1}y_1 - k_1, 2^{j_2}y_2 - k_2).$$

Proof. We used the Dirac delta distribution with the space of test functions $\{\psi_{j_1,j_2,k_1,k_2}\}$ which is defined by $\langle \delta, \psi_{j_1,j_2,k_1,k_2}\rangle = \psi_{j_1,j_2,k_1,k_2}(0,0)$. So that, by applying the Dirac delta distribution on the Kernel function in Equation (3) at $(x_1, x_2) \neq (y_1, y_2)$, we get

$$\left\langle \delta, \sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2} \psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2) \right\rangle$$
$$= \sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2} \psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2),$$

for $x_1 = 0 \neq y_1$, $x_2 = 0 \neq y_2$ and $j_1, j_2 \in Z$, $k_1, k_2 \in Z$.

Estimate the above term by integrating it

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2)dx_1dx_2dy_1dy_2.$$

Since the compact supported property of wavelet function give us

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_{j_1,j_2,k_1,k_2}(y_1,y_2)dy_1dy_2 = 0.$$

This lead

$$\int_{-\infty}^{x}\int_{-\infty}^{x} \psi_{j_1,j_2,k_1,k_2}(y_1,y_2)dy_1dy_2 = -\int_{x}^{\infty}\int_{x}^{\infty} \psi_{j_1,j_2,k_1,k_2}(y_1,y_2)dy_1dy_2.$$

Thus,

$$\sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2} \psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2) = 0,$$

and

$$\sum_{j_1<0}\sum_{j_2<0}\sum_{k_1}\sum_{k_2} \psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2)$$
$$= -\sum_{j_1\geq 0}\sum_{j_2\geq 0}\sum_{k_1}\sum_{k_2} \psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2).$$

The previous observations give us the following:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x_1,x_2,y_1,y_2)dx_1dx_2dy_1dy_2 = 0,$$

and $K(x_1, x_2, y_1, y_2)$ can be represented by one of the two formulas:

$K(2^{-J_1}x_1, 2^{-J_2}x_2, 2^{-J_1}y_1, 2^{-J_2}y_2)$, at $(j_1, j_2) < (0, 0)$

$= -K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)$, at $(j_1, j_2) \geq (0, 0)$.

□

Meyer was the first who introduced the concept of the unconditional convergence of wavelet expansion in multiresolution analysis see [9]. After that, the problem of the convergence of wavelet expansions has been studied by many researchers. In [5], [6] Kelly et al explained the convergence of wavelet expansions of $L^p(R^n)$ functions for $(1 \leq p < \infty)$ at every Lebesgue point of f by using the radial decreasing and partial continuous wavelet functions. Walter in [3] expand the distribution by using regular orthogonal wavelets, this expansion was converged pointwise to the value of the distribution. Tao in [8] noted that when one consider some minimal regularity for ψ, the point wise convergence of the wavelet projections operator can get on the entire Lebesgue set of f. Zhao et al. in [4] used the approximation method to converge the wavelet expansions of $L^2(R)$ function to the mean value of its both sides limits at a generalized continuous point. By using the biorthogonal B-spline wavelets, Junjian in [10] investigate the convergence property of wavelet expansion which is divergence-free and non divergence-free wavelet expansion by using the characterization of vector- valued Besov spaces function. By using the wavelet expansions, the pointwise behavior of Schwartz distributions in several variables is studied in [7], also the characterizations of the quasi asymptotic behavior of distributions at finite points are provided and the connections with $\alpha-$density points of measures are discussed. In [1] Singh study the point wise convergence of wavelet expansions of $L^2(R)$ functions, by using prolate spheroidal wavelet. Many works dealing with pointwise convergence of wavelet expansions have been produced, we linked some of the above results to develop the pointwise convergence almost everywhere by expanding the dimension of wavelet expansions of $L^p(R^2)$ functions and finding some new properties of wavelet basis functions in two dimension.

For that, in this work the pointwise behavior of the series in two dimension wavelet expansion has been studied, we show that the wavelet projection operator convergence almost everywhere in L^p norm to f. We restrict our concern to the two dimension wavelets and discuss the rapidity of the decrease in the magnitude of the wavelet functions and the bounds of wavelet expansions and Kernel function of wavelet projection operator in case of two dimension. For that, these properties are explained in the following main results.

2. Main Results

Theorem 2. *Consider* $f \in L^p(R^2)$, *for* $1 < p < \infty$ *and* $\psi_{\boldsymbol{j},\boldsymbol{k}}(x_1, x_2)$ *is the wavelet basis function for wavelet expansion then for almost everywhere* $(x_1, x_2) \in R^2$ *one has:*

$$\lim_{\boldsymbol{J}\to\infty} P_{\boldsymbol{J}} f(x_1, x_2) = f(x_1, x_2), \tag{6}$$

for $\boldsymbol{j} = (j_1, j_2)$, $\boldsymbol{k} = (k_1, k_2)$ *and* $\boldsymbol{j}, \boldsymbol{k} \in Z^2$. *If the following conditions are hold:*

1. $\psi_{\mathbf{j},\mathbf{k}}(x_1, x_2)$ *is rapidly decreasing.*
2. *The summation in* $P_{\boldsymbol{J}} f(x_1, x_2)$ *converge absolutely and distributionally.*
3. $a_{\mathbf{j},\boldsymbol{k}}$ *and* $P_{\boldsymbol{J}} f(x_1, x_2)$ *are bounded.*

Proof. To prove this theorem, we discussed all the above conditions in the next theorems.

To satisfy the first condition, we need to invite the following theorem to drive the rapidly decreasing property of wavelet function in two dimension:

Theorem 3. *Consider* $\psi_{\boldsymbol{j},\boldsymbol{k}}(x_1, x_2)$ *is the wavelet basis function for* $L^2(R^2)$ *space such that* $\boldsymbol{j} = (j_1, j_2)$, $\boldsymbol{k} = (k_1, k_2)$ *and* $\boldsymbol{j}, \boldsymbol{k} \in Z^2$ *then:*

$$|\psi_{\boldsymbol{j},\boldsymbol{k}}(x_1, x_2)| \leq \frac{2^{-j_1/2} 2^{-j_2/2} c_N}{[1 + |2^{j_1} x_1 - 2^{j_2} x_2|]^N}. \tag{7}$$

Proof. Since $\psi_{\mathbf{j},\mathbf{k}}(x_1, x_2)$ is defined in two dimension

$$\psi_{\mathbf{j},\mathbf{k}}(x_1, x_2) = \psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2).$$

$$= 2^{-j_1/2} 2^{-j_2/2} \psi(2^{j_1} x_1 - k_1)\psi(2^{j_2} x_2 - k_2).$$

Since $|\psi_{j,k}(x)| \leq \frac{2^{-j/2} c_N}{[1+|2^j x - k|]^N}$, for $(j, k) \in Z$.

$$|\psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2)| \leq \frac{2^{-j_1/2} 2^{-j_2/2} c_N}{[1 + |2^{j_1} x_1 - k_1|]^N [1 + |2^{j_2} x_2 - k_2|]^N}. \tag{8}$$

$$\leq \frac{2^{-j_1/2} 2^{-j_2/2} c_N}{[1 + |2^{j_1} x_1 - k_1| + |2^{j_2} x_2 - k_2| + |2^{j_1} x_1 - k_1|\, |2^{j_2} x_2 - k_2|]^N},$$

if we take the term

$$\left[1+\left|2^{j_1}x_1-k_1\right|+\left|2^{j_2}x_2-k_2\right|+\left|2^{j_1}x_1-k_1\right|\left|2^{j_2}x_2-k_2\right|\right]$$
$$\geq 1+\left|2^{j_1}x_1-k_2\right|+\left|k_2-2^{j_2}x_2\right|+\left|k_2-k_1\right|+\left|2^{j_1}x_1-k_1\right|\left|2^{j_2}x_2-k_2\right|.$$
$$\geq 1+\left|2^{j_1}x_1-2^{j_2}x_2\right|.$$

Hence the Equation(8) becomes

$$|\psi_{\mathbf{j},\mathbf{k}}(x_1,x_2)|=|\psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2)|\leq\frac{2^{-j_1/2}2^{-j_2/2}c_N}{\left[1+|2^{j_1}x_1-2^{j_2}x_2|\right]^N}.$$

This is the proof of Theorem 3 □

The second condition hold by proving both the absolutely and distributionally convergence almost everywhere in the following theorem:

Theorem 4. *Consider $\psi_{\mathbf{j},\mathbf{k}}(x_1,x_2)$ is the wavelet basis function for $L^2(R^2)$ space such that $\boldsymbol{j}=(j_1,j_2)$, $\boldsymbol{k}=(k_1,k_2)$ and $\boldsymbol{j},\boldsymbol{k}\in Z^2$ then:*

1. The summation in $P_{\mathbf{J}}f(x_1,x_2)$ converges absolutely, if the following fact satisfied almost everywhere for $(x_1,x_2),(y_1,y_2)\in R^2$ and for all $N>0$:

$$|\psi_{\mathbf{j},\mathbf{k}}(x_1,x_2)\psi_{\mathbf{j},\mathbf{k}}(y_1,y_2)|\leq\frac{2^{j_1}2^{j_2}c_N}{\left[1+2^{j_1}|x_1-y_1|+2^{j_2}|x_2-y_2|\right]^N}. \tag{9}$$

2- The summation in $P_{\mathbf{J}}f(x_1,x_2)$ converges distributionally, if the following fact satisfied almost everywhere for $(x_1,x_2),(y_1,y_2)\in R^2$:

$$\sum_{\mathbf{j}}\sum_{\mathbf{k}}\psi_{\mathbf{j},\mathbf{k}}(x_1,x_2)\psi_{\mathbf{j},\mathbf{k}}(y_1,y_2)=\delta\left((x_1,x_2)-(y_1,y_2)\right). \tag{10}$$

Proof. From the Equation (7), we find that

$$|\psi_{\mathbf{j},\mathbf{k}}(x_1,x_2)\psi_{\mathbf{j},\mathbf{k}}(y_1,y_2)|\leq\frac{2^{j_1}2^{j_2}c_N}{\left[1+|2^{j_1}x_1-2^{j_2}x_2|\right]^N\left[1+|2^{j_1}y_1-2^{j_2}y_2|\right]^N}.$$

If we take the term

$$\left[1+\left|2^{j_1}x_1-2^{j_2}x_2\right|\right]^N\left[1+\left|2^{j_1}y_1-2^{j_2}y_2\right|\right]^N$$
$$=\left[1+\left|2^{j_1}y_1-2^{j_2}y_2\right|+\left|2^{j_1}x_1-2^{j_2}x_2\right|+\left|2^{j_1}x_1-2^{j_2}x_2\right|\left|2^{j_1}y_1-2^{j_2}y_2\right|\right]^N$$
$$\geq\left[1+2^{j_1}\left|x_1-y_1\right|+2^{j_2}\left|x_2-y_2\right|\right]^N.$$

Hence, we got

$$|\psi_{\mathbf{j},\mathbf{k}}(x_1,x_2)\psi_{\mathbf{j},\mathbf{k}}(y_1,y_2)| \leq \frac{2^{j_1}2^{j_2}c_N}{\left[1+2^{j_1}|x_1-y_1|+2^{j_2}|x_2-y_2|\right]^N}.$$

By proving the Equation (9), we show that the sum in $P_{\mathbf{J}}$ converges absolutely.

Now, we come to verify the second fact in Equation (10). Since ψ_{j_1,j_2,k_1,k_2} form an orthonormal basis for $L^2(R^2)$, so that the summation in Equation (3) converges distributional sense to $\delta\left((x_1,x_2)-(y_1,y_2)\right)$. To explain that, we apply the Dirac delta distribution on the summation in Equation (3)

In case $(x_1,x_2)\neq(y_1,y_2)$,

$$\left\langle \delta, \sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2}\psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2)\right\rangle$$
$$= \delta\left(\sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2}\psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2)\right)$$
$$= \sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2}\psi_{j_1,j_2,k_1,k_2}(0,0)\psi_{j_1,j_2,k_1,k_2}(0+t_1,0+t_2) = 0,$$

for $t_1\neq 0\neq t_2$.

In case $(x_1,x_2)=(y_1,y_2)$,

$$\delta\left(\sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2}\psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\psi_{j_1,j_2,k_1,k_2}(y_1,y_2)\right)$$
$$= \sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2}\psi_{j_1,j_2,k_1,k_2}(0,0)\psi_{j_1,j_2,k_1,k_2}(0+t_1,0+t_2) = 1,$$

for $t_1 = 0 = t_2$.

By proving the Equation (10), we show that the sum in $P_{\mathbf{J}}$ converges distributionally.

This is the end of the proof □

Now we show that, the bounds of two dimensional wavelet expansions are explained in the following theorem:

Theorem 5. *Consider* $\psi_{\mathbf{j},\mathbf{k}}(x_1,x_2)$ *is the wavelet basis function for* $L^2(R^2)$ *space such that* $\boldsymbol{j}=(j_1,j_2)$, $\boldsymbol{k}=(k_1,k_2)$ *and* $\boldsymbol{j},\boldsymbol{k}\in Z^2$ *and for each* $f\in L^p(R^2)$, *for* $1<p<\infty$ *one has:* a_{j_1,j_2,k_1,k_2} *is bounded by* $2^{-j_1/2}2^{-j_2/2}2^{j_1/p}2^{j_2/p}\|f(x_1,x_2)\|_{L^p}$.

Proof. From Equation (2), we have

$$a_{\mathbf{j},\mathbf{k}} = \int_R \int_R f(y_1, y_2)\psi_{\mathbf{j},\mathbf{k}}(y_1, y_2)dy_1dy_2$$

by applying Holder inequality, we have

$$|a_{\mathbf{j},\mathbf{k}}| \leq \left(\int_R \int_R |f(y_1, y_2)|^p dy_1dy_2\right)^{1/p} \left(\int_R \int_R |\psi_{\mathbf{j},\mathbf{k}}(y_1, y_2)|^q dy_1dy_2\right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.
Since ψ is rapidly decreasing in two dimension and by using Equation (7)

$$\leq \|f(x_1, x_2)\|_{L^p} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{2^{-j_1/2}2^{-j_2/2}c_N}{[1 + |2^{j_1}y_1 - 2^{j_2}y_2|]^N}\right)^q dy_1dy_2\right)^{1/q}.$$

To simplify the above integral, let $2^{j_1}y_1 = t_1$ and $2^{j_2}y_2 = t_2$ and get

$$\leq 2^{-j_1/2}2^{-j_2/2}c_N\|f\|_{L^p} \left(4\int_0^{\infty} \int_0^{\infty} \frac{1}{(1 + (t_1 - t_2))^{Nq}} 2^{-j_1}dt_1 2^{-j_2}dt_2\right)^{1/q}.$$

$$\leq 2^{-j_1/2}2^{-j_2/2}2^{-j_1/q}2^{-j_2/q}2^{2/q}c_N\|f\|_{L^p} \left(\int_0^{\infty} \int_0^{\infty} \frac{1}{(1 + (t_1 - t_2))^{Nq}} dt_1dt_2\right)^{1/q}.$$

$$\leq 2^{-j_1/2}2^{-j_2/2}2^{j_1/p}2^{j_2/p}c_N \left(\frac{1}{(Nq-1)(Nq-2)}\right)^{1/q} \|f\|_{L^p}.$$

Since the quantity in the term $\left(\frac{c_N}{(Nq-1)(Nq-2)}\right)^{1/q}$ is decreasing, this led us

$$|a_{j_1,j_2,k_1,k_2}| = o\left(2^{-j_1/2}2^{-j_2/2}2^{j_1/p}2^{j_2/p}\|f\|_{L^p}\right).$$

The proof end here □

To show that P_{J_1,J_2} is bounded, we verify that it can be limited in magnitude by the maximal function which defined in the Hardy- Littlewood maximal operator M as:

$Mf(x_1,x_2) = \sup_{r_1,r_2>0} |A_{r_1,r_2}(f)(x_1,x_2)|$, where A_{r_1,r_2} is a maximal function $A_{r_1,r_2}(f)(x_1,x_2) = \frac{1}{|B(x_1,x_2,r_1,r_2)|} \int_{B(x_1,r_1)} \int_{B(x_2,r_2)} f(y_1,y_2)dy_1dy_2$, and the measure of ball B
$|B(x_1,x_2,r_1,r_2)| = \sqrt{x_1^2+x_2^2} \leq 2r^2$, for $(x_1,x_2) \in R^2$ and the center point $(0,0)$. For more details, we refer the reader to see [2].
The following theorem explained the above issue:

Theorem 6. *Consider $f \in L^p(R^2)$ for $1 < p < \infty$ and under the assumption that the wavelet function ψ is rapidly decreasing, then for almost everywhere (x_1,x_2) is a Lebesgue point of f one has*

$$\sup_{\boldsymbol{J}} |P_{\boldsymbol{J}} f(x_1,x_2)| \leq cMf(x_1,x_2)$$

where Mf is the maximal operator of f and $\mathbf{J} = \{(J_1,J_2) : J_1,J_2 \in Z\}$.

Proof. From Equation (5), we find that

$$|P_{J_1,J_2}f| = \left| \int_R \int_R 2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)f(y_1,y_2)dy_1dy_2 \right|.$$

The integral isolated in two terms by depending on the values of R

$$\left| \int_{|x_1-y_1|\leq\varepsilon_1} \int_{|x_2-y_2|\leq\varepsilon_2} 2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)f(y_1,y_2)dy_1dy_2 \right| \tag{11}$$

$$+ \left| \int_{|x_1-y_1|>\varepsilon_1} \int_{|x_2-y_2|>\varepsilon_2} 2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)f(y_1,y_2)dy_1dy_2 \right|.$$

Consider that $\varepsilon_1 = 2^{-J_1}$, $\varepsilon_2 = 2^{-J_2}$ and $(\varepsilon_1,\varepsilon_2) \to (0,0)$.
Two cases have been discussed:
I- In case that $|x_1-y_1| \leq \varepsilon_1$ and $|x_2-y_2| \leq \varepsilon_2$.
To simplify the first part of Equation(11), let us consider

$$\Lambda = \left| \int_{|x_1-y_1|\leq\varepsilon_1} \int_{|x_2-y_2|\leq\varepsilon_2} 2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)f(y_1,y_2)dy_1dy_2 \right|.$$

This term

$$\leq \left(\int\limits_{|x_1-y_1|\leq\varepsilon_1} \int\limits_{|x_2-y_2|\leq\varepsilon_2} \left|2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)\right|^p dy_1dy_2 \right)^{1/p}$$

$$\times \left(\int\limits_{|x_1-y_1|\leq\varepsilon_1} \int\limits_{|x_2-y_2|\leq\varepsilon_2} |f(y_1,y_2)|^q dy_1dy_2 \right)^{1/q}.$$

If we take the term $\left|2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)\right|^p$

$$= \left|2^{J_1}2^{J_2} \sum_{j_1\geq 0}\sum_{j_2\geq 0}\sum_{k_1}\sum_{k_2} \psi(2^{j_1}x_1-k_1, 2^{j_2}x_2-k_2)\psi(2^{j_1}y_1-k_1, 2^{j_2}y_2-k_2)\right|^p.$$

$$\leq 2^{J_1}2^{J_2} \sum_{j_1\geq 0}\sum_{j_2\geq 0}\sum_{k_1}\sum_{k_2} \left|\psi(2^{j_1}x_1-k_1, 2^{j_2}x_2-k_2)\psi(2^{j_1}y_1-k_1, 2^{j_2}y_2-k_2)\right|^p.$$

Estimate the term to get

$$\leq 2^{J_1}2^{J_2}2^{j_1p}2^{j_2p} \int\limits_R\int\limits_R \frac{c_N}{\left[1+2^{j_1}|x_1-y_1|+2^{j_2}|x_2-y_2|\right]^{Np}} dy_1dy_2.$$

Let $|x_1-y_1| = \lambda_1$ this led to $dy_1 = -d\lambda_1$, and $|x_2-y_2| = \lambda_2$ this led to $dy_2 = -d\lambda_2$.

Hence, the above term becomes

$$\leq 2^{J_1}2^{J_2}2^{j_1p}2^{j_2p} \int\limits_R\int\limits_R \frac{c_N}{\left[1+2^{j_1}\lambda_1+2^{j_2}\lambda_2\right]^{Np}} d\lambda_1d\lambda_2.$$

$$\leq 2^{J_1}2^{J_2}2^{j_1p}2^{j_2p}2^2c_N \frac{2^{-j_1}}{(-Np+1)}\frac{2^{-j_2}}{(-Np+2)}.$$

$$\sup_{J_1,J_2}\Lambda \leq \sup_{J_1,J_2} \left(\int\limits_{|x_1-y_1|\leq\varepsilon_1} \int\limits_{|x_2-y_2|\leq\varepsilon_2} 2^{J_1}2^{J_2}2^{j_1p}2^{j_2p}2^2c_N \frac{2^{-j_1}}{(-Np+1)}\frac{2^{-j_2}}{(-Np+2)} dy_1dy_2 \right)^{1/p}$$

$$\times \left(\int\limits_{|x_1-y_1|\leq\varepsilon_1} \int\limits_{|x_2-y_2|\leq\varepsilon_2} |f(y_1,y_2)|^q dy_1dy_2 \right)^{1/q}.$$

Hence, this case led us to $\sup_{J_1,J_2} \Lambda \leq c_N Mf(x_1,x_2)$.

In this part, the second case has been discussed
II- In case that $|x_1 - y_1| > \varepsilon_1$ and $|x_2 - y_2| > \varepsilon_2$.
By taking the second part of Equation(11), let consider

$$\beta = \left| \int\limits_{|x_1-y_1|>\varepsilon_1} \int\limits_{|x_2-y_2|>\varepsilon_2} 2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2) f(y_1,y_2)dy_1dy_2 \right|.$$

This term

$$\leq \left(\int\limits_{|x_1-y_1|>\varepsilon_1} \int\limits_{|x_2-y_2|>\varepsilon_2} \left|2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)\right|^p dy_1dy_2 \right)^{1/p}$$

$$\times \left(\int\limits_{|x_1-y_1|>\varepsilon_1} \int\limits_{|x_2-y_2|>\varepsilon_2} |f(y_1,y_2)|^q dy_1dy_2 \right)^{1/q}.$$

The term $\left|2^{J_1}2^{J_2}K(2^{J_1}x_1, 2^{J_2}x_2, 2^{J_1}y_1, 2^{J_2}y_2)\right|^p$

$$= \left| 2^{J_1}2^{J_2} \sum_{j_1\geq J_1} \sum_{j_2\geq J_2} \sum_{k_1} \sum_{k_2} \psi(2^{j_1}x_1 - k_1, 2^{j_2}x_2 - k_2)\psi(2^{j_1}y_1 - k_1, 2^{j_2}y_2 - k_2) \right|^p.$$

$$\leq 2^{J_1}2^{J_2} \sum_{j_1\geq J_1} \sum_{j_2\geq J_2} \sum_{k_1} \sum_{k_2} \left|\psi(2^{j_1}x_1 - k_1, 2^{j_2}x_2 - k_2)\psi(2^{j_1}y_1 - k_1, 2^{j_2}y_2 - k_2)\right|^p.$$

By applying the Equation(9) from Theorem 4, we have

$$\leq 2^{J_1}2^{J_2} \sum_{j_1\geq J_1} \sum_{j_2\geq J_2} \sum_{k_1} \sum_{k_2} \frac{2^{pj_1}2^{pj_2}c_N}{\left[1 + 2^{j_1}|x_1 - y_1| + 2^{j_2}|x_2 - y_2|\right]^{Np}}.$$

$$\leq 2^{J_1}2^{J_2} \sum_{j_1\geq J_1} \sum_{j_2\geq J_2} \sum_{k_1} \sum_{k_2} \frac{2^{pj_1}2^{pj_2}c_N}{\left[1 + 2^{j_1}\varepsilon_1 + 2^{j_2}\varepsilon_2\right]^{Np}}.$$

Since the term

$$\sum_{j_1\geq J_1} \sum_{j_2\geq J_2} \sum_{k_1} \sum_{k_2} \frac{2^{pj_1}2^{pj_2}c_N}{\left[1 + 2^{j_1}\varepsilon_1 + 2^{j_2}\varepsilon_2\right]^{Np}} \leq \sum_{j_1\geq J_1} \sum_{j_2\geq J_2} \sum_{k_1} \sum_{k_2} \frac{2^{pj_1}2^{pj_2}c_N}{\left[2^{j_1}\varepsilon_1 + 2^{j_2}\varepsilon_2\right]^{Np}},$$

and

$$\leq \sum_{j_1 \geq J_1} \sum_{j_2 \geq J_2} \sum_{k_1} \sum_{k_2} \frac{2^{pj_1} 2^{pj_2} c_N}{[2^{j_1}\varepsilon_1]^{Np}}.$$

Since, $\lim_{(j_1,j_2)\to(\infty,\infty)} \frac{2^{-j_1 p(N-1)} 2^{pj_2} c_N}{[\varepsilon_1]^{Np}} = 0.$

This lead, at $(j_1, j_2) \to (\infty, \infty)$

$$\sum_{j_1 \geq J_1} \sum_{j_2 \geq J_2} \sum_{k_1} \sum_{k_2} \frac{2^{pj_1} 2^{pj_2} c_N}{[1 + 2^{j_1} |x_1 - y_1| + 2^{j_2} |x_2 - y_2|]^{Np}} \to 0.$$

Hence $\beta = 0$

So that $\sup_{J_1,J_2} |P_{J_1,J_2} f| \leq \sup_{J_1,J_2} \Lambda + \sup_{J_1,J_2} \beta = c_N M f(x_1, x_2) + 0.$

The proof end here □

We note that since all the conditions in Theorem 2 are satisfied by proving the Theorems 3, 4, 5 and 6, the wavelet projection operator in Equation (6) converges pointwise almost everywhere.

The proof of Theorem 2 end here □

3. Conclusion

The 2-dimensional wavelet expansions of $L^p(R^2)$ functions converged pointwise almost everywhere. The work with this convergence investigated by depending on some properties like the rapidly decreasing and the bounds of two dimensional wavelets. The representation of Kernel function be used to simplify the convergence process due to its absolutely and distributionally convergence properties. All these matters make us determine the behavior of wavelet expansion under wavelet projection operator at all Lebesgue set $(x_1, x_2) \in R^2$.

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